Multiquark functions in effective models

R. G. Jafarov[a*], V. E. Rochev[b†]

[a] *Institute for Physical Problems, Baku State University*
P.O. Box AZ 1148, Z. Khalilov, 23, Baku, Azerbaijan

[b] *Theoretical Division, Institute of High Energy Physics*
P.O. Box Protvino, Moscow Region, Russia

Abstract

In this paper we present our results of the investigation of multi-quark equations in the Nambu–Jona-Lasinio model with chiral symmetry of $SU(2)-$group in the mean-field expansion. To formulate the mean-field expansion we have used an iteration scheme of solution of the Schwinger-Dyson equations with the fermion bilocal source. We have considered the equations for Green functions of the Nambu–Jona-Lasinio model up to third step for this iteration scheme.

To calculate the high-order corrections to the mean-field approximation, we propose the method of the Legendre transformation with respect to the bilocal source, which allows effectively to take into account the symmetry constraints related with the chiral Ward identity.

We discuss also the problem of calculating the multiquark functions in the mean-field expansion for Nambu-Jona-Lasinio-type models with other types of the multifermion sources.

1 Introduction and Summary

The fundamental theory of strongly interacting particles (hadrons) and the processes of their collisions is the quantum chromodynamics – the theory of interactions of quarks and gluons. The self-action of gluons leads to the important physical consequences, in particular, to the property of asymptotical freedom, i.e. to reducing the effective coupling constant of quarks and gluons at very small distances, which plays an exclusive role in a description of the processes with the high momentum transfers and facilitates a problem of calculation of such processes with the modified perturbation theory. But in the region of intermediate and large distances, which is very important from the physical point of view, being the region of hadronization and confinement, the possibilities of the adequate mathematical description of the quantum-chromodynamical equations are very few. Therefore for the description of variety of the important physical phenomena the simplified effective models, which reflect some separate important aspects of the strong interactions physics, are necessary.

A number of perspective physical applications of the effective models are connected with multi-quark functions, which are the subject of present work. The basic method of calculations is a formalism of multilocal sources [1], [2].

As an object of application of the method we choose Nambu - Jona-Lasinio (NJL) model [3]. This model is one of the most successful effective models of quantum chromodynamics for the light hadrons (for review see [4]-[6]). The NJL model is widely used also in various aspects of the physics of light nuclei. It is necessary to note, that this method has been successfully applied for the other field-theoretic models (see [7]) and can be applied also for analogous calculations in other similar effective models.

*e-mail: jafarov@hotmail.com
†e-mail: Vladimir.Rochev@ihep.ru

The multi-quark functions arise in higher orders of the mean-field expansions (MFE) for the NJL model. To formulate the MFE we have used an iteration scheme of solution of the Schwinger-Dyson equation with the fermion bilocal source, which has been developed in works [7]. We have considered the equations for Green functions of the NJL model up to the third order of the MFE. The leading approximation and first order of the MFE maintains equations for the quark propagator and the two-quark function and also the next-to-leading (NLO) correction to the quark propagator. The second order of MFE includes the equations for the four-quark and the three-quark functions and also the equations for the NLO two-quark function and next-to-next-to-leading order (NNLO) quark propagator.

Furthermore we have considered the generalization of the method in the framework of the NJL-type models, which includes the other multilocal sources (specifically, the diquark and three-quark sources).

We have found a solution of the four-quark and three-quark equations. The solution of the four-quark function is a disconnected combination of the leading-order functions and, consequently, the corresponding physical effects (i.e., pion-pion scattering) are suppressed in this order of the MFE. Therefore, we also investigate the third step of iterations, which gives us the equations for the six-quark and five-quark functions and the equations for the NLO four-quark and three-quark functions. The solution of the six-quark functions equation has the disconnected form, which is similar to the solution for the four-quark function of the preceding step.

The solution of the second-step four-quark equation gives us a possibility to close the equation for the three-quark function. The solution of three-quark equation contains both disconnected and connected parts. The connected part of three-quark function includes the two-meson and three-meson contributions.

The paper is organized as follows. In Section 2 we describe the method of construction of the MFE with the fermion bilocal source for the NJL model with the $SU_V(2) \times SU_A(2)$–symmetric four-quark interaction and, for the sake of completeness, consider the well-known leading approximation results of this model. Also in this section we investigate the first-after-leading step of the iteration scheme, which gives us the equations for the leading order two-particle Green function and NLO correction to the propagator of quarks.

In Section 3 we describe the second step of the iteration scheme. As a result we obtain the equations for four-quark Green function and for the three-quark Green function. We also obtain in this step the equations for NLO two-quark function and NNLO correction to quark propagator. We discuss the structure of second step equations and obtain the solutions of four-quark and three-quark equations.

The main problem of the calculation of corrections to the leading approximation in this method is the appearance of terms, which break the symmetry constraints following from the chiral Ward identity. To solve this problem we use in Section 4 the Legendre transformation method with respect to a bilocal source (see also [8] and the Refs. therein). We introduce the generating functional of the Legendre transformation, formulate the chiral Ward identity for this generating functional, and determine the mean-field approximation and first-order corrections in the NJL model in the language of the Legendre transformation.

In Section 5 we describe the third step of iteration scheme. As a result we obtain the equations for six-quark Green function and for the five-quark Green function, and, the NLO equations for four-quark and three-quark Green functions. We also obtain in this step the equations for NNLO two-particle function and NNNLO correction to quark propagator.

In Section 6 the modification of the MFE for the NJL model in the formalism with the multilocal diquark and triple-quark sources is briefly discussed.

In Appendices A, B and C the formulae for multi-quark functions are listed.

2 The mean-field expansion in the bilocal source formalism. Leading and first step equations

As an example of the physically interesting model we consider the chiral-symmetric NJL model. The model contains up and down quarks fields $\psi(x)$, each with n_c colors. The Lagrangian of the two-flavor NJL model may be written in the well-known form

$$L = \bar{\psi} i\hat{\partial}\psi + \frac{g}{2}\Big[(\bar{\psi}\psi)^2 + (\bar{\psi}i\gamma_5\tau^a\psi)^2\Big], \tag{1}$$

where we omit the small bare masses of the u and d quarks, i.e. take the chiral limit. Here g is the coupling constant with dimension of m^{-2}. The Lagrangian (1) is $SU_V(2) \times SU_A(2)$-invariant. In Eq. (1) we have used denotations $\psi \equiv \psi_j^{\alpha c}$, where $\alpha = 1,2,3,4$ is Dirac spinor index, $c = 1, ..., n_c$ is color index, $j = 1,2$ is isotopic (flavor) index, $\hat{\partial} \equiv \gamma^\mu \partial_\mu$ and the $SU(2)$-group generators $\tau^a \equiv (\tau_1, \tau_2, \tau_3)$ (Pauli matrices) are normalized as $tr(\tau^a\tau^b) = 2\delta^{ab}$.

To construct the MFE we use an iteration scheme of the solution of functional-differential Schwinger-Dyson equation (SDE)

$$\delta(x-y)G + i\hat{\partial}_x\frac{\delta G}{\delta\eta(y,x)} + ig\Big\{\frac{\delta}{\delta\eta(y,x)}tr\Big[\frac{\delta G}{\delta\eta(x,x)}\Big] - \gamma_5\tau^a\frac{\delta}{\delta\eta(y,x)}tr\Big[\gamma_5\tau^a\frac{\delta G}{\delta\eta(x,x)}\Big]\Big\} =$$

$$= \int dx_1 \eta(x,x_1)\frac{\delta G}{\delta\eta(y,x_1)} \tag{2}$$

for the generating functional G of Green functions (vacuum expectation values of T-products of fields). The generating functional G can be represented as the functional integral with bilocal fermion source $\eta(y,x)$:

$$G(\eta) = \int D(\psi,\bar{\psi}) \exp i\Big\{\int dxL - \int dxdy\bar{\psi}(y)\eta(y,x)\psi(x)\Big\}. \tag{3}$$

The n-th functional derivative of G over source η is the n-particle (2n-point) Green function:

$$\frac{\delta^n G}{\delta\eta(y_1,x_1)\cdots\delta\eta(y_n,x_n)}\Big|_{\eta=0} = i^n < 0 \mid T\Big\{\psi(x_1)\bar{\psi}(y_1)\cdots\psi(x_n)\bar{\psi}(y_n)\Big\} \mid 0 >\equiv S_n\begin{pmatrix} x_1 & y_1 \\ \cdots & \cdots \\ x_n & y_n\end{pmatrix}.$$

We shall solve Eq. (2) employing the method which proposed in work [9]: we shall consider as a leading approximation the equation

$$G^{(0)} + i\hat{\partial}_x\frac{\delta G^{(0)}}{\delta\eta} + ig\Big\{\frac{\delta}{\delta\eta}tr\Big[\frac{\delta G^{(0)}}{\delta\eta}\Big] - \gamma_5\tau^a\frac{\delta}{\delta\eta}tr\Big[\gamma_5\tau^a\frac{\delta G^{(0)}}{\delta\eta}\Big]\Big\} = 0,$$

i.e., the functional-differential SDE (2) with zero r.h.s. The solution of this equation is

$$G^{(0)} = \exp\Big\{\mathrm{Tr}\Big(S * \eta\Big)\Big\}$$

(here Tr is a trace in the operator sense, and $*$ is the operator multiplication), where a single Green function S is solution of equation

$$\delta(x) + i\hat{\partial}S(x) + ig\Big\{S(x)tr[S(0)] - \gamma_5\tau^a S(x)tr[\gamma_5\tau^a S(0)]\Big\} = 0. \tag{4}$$

The Green functions are the derivatives of $G(\eta)$ at zero and only the behavior of G near $\eta = 0$ is essential, therefore such an approximation seems to be acceptable. The leading approximation generates the linear iteration scheme, i.e. the term omitted, which contains the source

η manifestly, should be treated as a perturbation [7]. Hence, the iteration procedure for the generating functional

$$G = G^{(0)} + G^{(1)} + \cdots + G^{(n)} + \cdots ,$$

consists in the step-by-step solutions of the equations

$$G^{(n)} + i\hat{\partial}_x \frac{\delta G^{(n)}}{\delta\eta} + ig\Big\{ \frac{\delta}{\delta\eta} tr\left[\frac{\delta G^{(n)}}{\delta\eta}\right] - \gamma_5\tau^a \frac{\delta}{\delta\eta} tr\left[\gamma_5\tau^{\mathbf{a}} \frac{\delta G^{(n)}}{\delta\eta}\right]\Big\} = \eta * \frac{\delta G^{(n-1)}}{\delta\eta}. \tag{5}$$

Functional $G^{(n)}$ is

$$G^{(n)} = P^{(n)} G^{(0)},$$

where $P^{(n)}$ is a polynomial of $2n$ -th degree on the bilocal source η.

The unique connected Green function of the leading approximation S is the quark propagator. A solution of Eq. (4) is

$$S_1^{(0)}(p) \equiv S(p) = (m - \hat{p})^{-1}, \tag{6}$$

where m is the dynamical quark mass, which is a solution of the gap equation of the NJL model in the chiral limit

$$m = -8igmn_c \int \frac{d\tilde{q}}{m^2 - q^2}. \tag{7}$$

Here and below we include a phase factor in a integration over momentum space $d\tilde{p} \equiv d^4q/(2\pi)^4$. Eq. (7) is the main equation, which describes DCSB – the principal phenomenon of NJL model. The divergent integral in r.h.s. of Eq. (7) should be treated as some regularization (for more details about regularization procedure in NJL model see [9]-[10]). The chiral-symmetric trivial solution $m = 0$, undoubtedly always exists. Energetically preferable physical solution corresponds to $m \neq 0$, i.e. DCSB. Below we shall consider just such a solution. Note that, other connected Green functions, including two-particle functions, appear in the following steps of iterations.

The functional of the first step of iteration scheme can be represented as

$$G^{(1)} = \Big\{ \frac{1}{2}\mathrm{Tr}\Big(S_2 * \eta^2\Big) + \mathrm{Tr}\Big(S^{(1)} * \eta\Big)\Big\} G^{(0)}. \tag{8}$$

In this step of iteration the equation for two-particle function S_2 and the equation for NLO quark propagators correction $S^{(1)}$ arise.

Using representation (8) for the first step functional in iteration-scheme equation (5), and according to Eq. (4) we go to equations for two-particle function S_2 and NLO correction to quark propagator $S^{(1)}$, which have the following forms:

$$\begin{aligned} S_2\begin{pmatrix} x & y \\ x' & y' \end{pmatrix} &= -S(x-y')S(x'-y) + \\ +ig\int dx_1 \Big\{ \Big(S(x-x_1)S(x_1-y)\Big) & tr_u\left[S_2\begin{pmatrix} x_1 & x_1 \\ x' & y' \end{pmatrix}\right] - \\ -\Big(S(x-x_1)\gamma_5\tau^a S(x_1-y)\Big) & tr_u\left[\gamma_5\tau^a S_2\begin{pmatrix} x_1 & x_1 \\ x' & y' \end{pmatrix}\right]\Big\}, \end{aligned} \tag{9}$$

(here and below tr_u denotes the trace, which includes the upper line of functions S_2) and

$$\begin{aligned} S^{(1)}(x-y) = ig\int dx_1 S(x-x_1)\Big\{ S_2\begin{pmatrix} x_1 & y \\ x_1 & x_1 \end{pmatrix} - \gamma_5\tau^a S_2\begin{pmatrix} x_1 & y \\ x_1 & x_1 \end{pmatrix}\gamma_5\tau^a\Big\} + \\ +ig\int dx_1 S(x-x_1)S(x_1-y)tr[S^{(1)}](0). \end{aligned} \tag{10}$$

Let pass to the amputated two-particle function

$$F_2 = S^{-1} * S^{-1} * S_2 * S^{-1} * S^{-1}, \tag{11}$$

and extract the connected part of F_2

$$F_2^c \begin{pmatrix} x & y \\ x' & y' \end{pmatrix} = F_2 \begin{pmatrix} x & y \\ x' & y' \end{pmatrix} + S^{-1}(x-y')S^{-1}(x'-y). \tag{12}$$

Then for the two-quark function F_2^c we obtain the equation

$$F_2^c \begin{pmatrix} x & y \\ x' & y' \end{pmatrix} = -ig\delta(x-y)\delta(x-y')\delta(x'-y)\{1\cdot 1 - \gamma_5\tau^a\cdot\gamma_5\tau^a\}+$$

$$+ig\delta(x-y)\int dx_1dy_1\Big\{tr_u\Big[S(x-x_1)F_2^c \begin{pmatrix} x_1 & y_1 \\ x' & y' \end{pmatrix} S(y_1-y)\Big]-$$

$$-\gamma_5\tau^a tr_u\Big[\gamma_5\tau^a S(x-x_1)F_2^c \begin{pmatrix} x_1 & y_1 \\ x' & y' \end{pmatrix} S(y_1-y)\Big]\Big\}. \tag{13}$$

Taking into account the color, flavor and Lorentz structure of the inhomogeneous term of the equation we obtain the following general form of solution

$$F_2^c \begin{pmatrix} x & y \\ x' & y' \end{pmatrix} = \delta(x-y)\delta(x'-y')\{1\cdot 1A_\sigma(x-x') + \gamma_5\tau^a\cdot\gamma_5\tau^a A_\pi(x-x')\}, \tag{14}$$

where A_σ is the scalar (sigma-meson) amplitude and A_π is the pseudoscalar (pion) amplitude, which are the solutions of following equations

$$A_\sigma(x) = -ig\delta(x) + ig\int dx_1 tr[S(x-x_1)S(x_1-x)]A_\sigma(x_1), \tag{15}$$

$$A_\pi(x) = ig\delta(x) - ig\int dx_1 tr[\gamma_5 S(x-x_1)\gamma_5 S(x_1-x)]A_\pi(x_1). \tag{16}$$

As a result of standard operation in Eqs. (15)-(16) and taking into account gap equation (7) it is easy to obtain for the sigma-meson and pion amplitudes the following representations:

$$A_\sigma(p) = \frac{1}{4n_c(4m^2-p^2)I_0(p)} \tag{17}$$

and

$$A_\pi(p) = \frac{1}{4n_c p^2 I_0(p)}. \tag{18}$$

Here

$$I_0(p) = \int d\tilde{q}\frac{1}{(m^2-(p+q)^2)(q^2-m^2)} \tag{19}$$

is the divergent single-loop integral which should be dealt with by some regularization. Since the NJL model is based on the unrenormalized four-fermion interaction, the choice of the regularization is a significant moment in the practical use of the NJL model in hadron physics, although it does not lead to any appreciable principal difference in the description of the main effects. The integral I_0 takes finite values for $P^2 = 4m^2$ and $P^2 = 0$ in all regularization schemes that are most often used in the NJL models, such as a regularization by the four-dimensional cutoff in Euclidean momentum space, a regularization by the three-dimensional cutoff, the Fock-Schwinger "proper time" regularization, the Pauli-Villars regularization. A most popular gauge-invariant dimensional regularization should be used in the NJL model with some special treatment (see [9], [10], [11], [12] and Refs. therein).

Therefore, the scalar and pseudoscalar amplitudes, as can be seen from Eqs. (17) and (18), have the simple poles corresponding to the scalar particle with mass $2m$ (the sigma meson) and to the massless pseudoscalar pion. The appearance of the pseudoscalar massless (Goldstone) particle in the spectrum of the model with spontaneously broken chiral symmetry corresponds to the general Nambu-Goldstone-Bogolyubov (NGB) theorem.

Eq. (10) for NLO quark propagator $S^{(1)}$ gives us a possibility for define the meson corrections to quark mass. This equation reduces in the momentum space to a system of simple algebraic forms. Introducing the NLO mass operator $\Sigma^{(1)} = S^{(-1)} * S^{(1)} * S^{(-1)}$, we obtain from Eq. (10):

$$\Sigma^{(1)}(x) = S(x)A_\sigma(x) + 3S(-x)A_\pi(x) + ig\delta(x)tr[S^{(1)}(0)]. \tag{20}$$

3 Second step equations. Solutions for four-quark and three-quark functions

The second-step generating functional is

$$G^{(2)} = \left\{\frac{1}{4!}\mathrm{Tr}\Big(S_4 * \eta^4\Big) + \frac{1}{3!}\mathrm{Tr}\Big(S_3 * \eta^3\Big) + \frac{1}{2}\mathrm{Tr}\Big(S_2^{(1)} * \eta^2\Big) + \mathrm{Tr}\Big(S^{(2)} * \eta\Big)\right\}G^{(0)}, \tag{21}$$

i.e., the second step procedure of the considering iteration scheme gives us four-quark function S_4, three-quark function S_3, and also NLO two-quark function $S_2^{(1)}$ and NNLO quark propagator $S^{(2)}$. For these four functions we have a system of four integral equations. All these equations (and all equations of following steps of the iteration scheme) possess the structure, which is similar to the structure of Eq. (9) (see also [13]):

$$\begin{aligned} S_n\begin{pmatrix} x_1 & y_1 \\ ... & ... \\ x_n & y_n \end{pmatrix} &= S_n^0\begin{pmatrix} x_1 & y_1 \\ ... & ... \\ x_n & y_n \end{pmatrix} + \\ &+ig\int dx_1'\Bigg\{\Big(S(x_1-x_1')S(x_1'-y_1)\Big)tr\left[S_n\begin{pmatrix} x_1' & x_1' \\ ... & ... \\ x_n & y_n \end{pmatrix}\right] - \\ &-\Big(S(x_1-x_1')\gamma_5\tau^a S(x_1'-y_1)\Big)tr\left[\gamma_5\tau^a S_n\begin{pmatrix} x_1' & x_1' \\ ... & ... \\ x_n & y_n \end{pmatrix}\right]\Bigg\}, \end{aligned} \tag{22}$$

and differ from each other by the structure of inhomogeneous terms S_n^0. The inhomogeneous term in the equation for four-quark function S_4 is

$$S_4^0\begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_4 & y_4 \end{pmatrix} = -\left\{S(x_1-y_2)S(x_2-y_1)S_2\begin{pmatrix} x_3 & y_3 \\ x_4 & y_4 \end{pmatrix}\right\} - \Big\{2\leftrightarrow 3\Big\} - \Big\{2\leftrightarrow 4\Big\}, \tag{23}$$

where S_2 is defined in preceding section by Eq. (9), i.e. the equation for S_4 does not include another three second-step functions.

The inhomogeneous term in the equation for three-quark function S_3 include four-quark function S_4, two-quark function S_2, NLO quark propagator $S^{(1)}$ and leading order quark propagator S. This inhomogeneous term has the form:

$$S_3^0\left(\begin{array}{cc} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \end{array}\right) =$$

$$-\Big\{S(x_1-y_2)S(x_2-y_1)S^{(1)}(x_3-y_3)\Big\} - \Big\{2\leftrightarrow 3\Big\} - \Big[S(x_1-y_2)S_2\left(\begin{array}{cc} x_2 & y_1 \\ x_3 & y_3 \end{array}\right)\Big] - \Big[2\leftrightarrow 3\Big]$$

$$+ig\int dx_1' S(x_1-x_1')\Big\{tr_u\Big[S_4\left(\begin{array}{cc} x_1' & x_1' \\ x_1' & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \end{array}\right)\Big] - \gamma_5\tau tr_u\Big[\gamma_5\tau S_4\left(\begin{array}{cc} x_1' & x_1' \\ x_1' & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \end{array}\right)\Big]\Big\}. \qquad (24)$$

The explicit form of inhomogeneous terms of other second-step equations ($(S_2^{(1)})^0$ and $(S^{(2)})^0$) see in Appendix A.

Due to such structure of the system its solution should be started with the equation for four-quark function S_4, then should be solved the equation for three-quark function S_3, etc.

The equation for four-quark Green function S_4 with inhomogeneous term (23) has following simple solution (see also [13], [14])

$$S_4\left(\begin{array}{cc} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_4 & y_4 \end{array}\right) = \Big\{S_2\left(\begin{array}{cc} x_1 & y_1 \\ x_2 & y_2 \end{array}\right) S_2\left(\begin{array}{cc} x_3 & y_3 \\ x_4 & y_4 \end{array}\right)\Big\} + \Big\{2\leftrightarrow 3\Big\} + \Big\{2\leftrightarrow 4\Big\}. \qquad (25)$$

This solution is disconnected, and it means the absence of physical effects due to four-particle functions in the given order of MFE. Particularly, the pion-pion scattering can not describe in the given order and will be appear only in next step of iteration scheme (third order of MFE), in principle.

Solution (25) of the equation for the four-quark function gives us the closed equation of type (22) for three-quark function S_3. The inhomogeneous term of this equation is

$$S_3^0\left(\begin{array}{cc} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \end{array}\right) =$$

$$-\Big\{S(x_1-y_2)S(x_2-y_1)S^{(1)}(x_3-y_3)\Big\} - \Big\{2\leftrightarrow 3\Big\} - \Big[S(x_1-y_2)S_2\left(\begin{array}{cc} x_2 & y_1 \\ x_3 & y_3 \end{array}\right)\Big] - \Big[2\leftrightarrow 3\Big]$$

$$+ig\int dx_1'\Big\{tr\Big[S(x_1-x_1')S_2\left(\begin{array}{cc} x_1' & x_1' \\ x_1' & y_1 \end{array}\right)\cdot S_2\left(\begin{array}{cc} x_2 & y_2 \\ x_3 & y_3 \end{array}\right)\Big]$$

$$+tr_u\Big[S(x_1-x_1')S_2\left(\begin{array}{cc} x_1' & y_1 \\ x_2 & y_2 \end{array}\right)\cdot S_2\left(\begin{array}{cc} x_1' & x_1' \\ x_3 & y_3 \end{array}\right)\Big] + tr_u\Big[2\leftrightarrow 3\Big]\Big\}-$$

$$-ig\int dx_1'\Big\{tr\Big[S(x_1-x_1')\gamma_5\tau^{\mathbf{a}}S_2\left(\begin{array}{cc} x_1' & x_1' \\ x_1' & y_1 \end{array}\right)\cdot\gamma_5\tau^{\mathbf{a_1}}S_2\left(\begin{array}{cc} x_2 & y_2 \\ x_3 & y_3 \end{array}\right)\Big]$$

$$+tr_u\Big[S(x_1-x_1')\gamma_5\tau^{\mathbf{a}}S_2\left(\begin{array}{cc} x_1' & y_1 \\ x_2 & y_2 \end{array}\right)\cdot\gamma_5\tau^{\mathbf{a_1}}S_2\left(\begin{array}{cc} x_1' & x_1' \\ x_3 & y_3 \end{array}\right)\Big] + tr_u\Big[2\leftrightarrow 3\Big]\Big\}.$$

The solution can be obtained likewise to solving of Eq. (9) for the two-quark function. We go to the amputated function

$$F_3 = S^{-1}*S^{-1}*S^{-1}*S_3*S^{-1}*S^{-1}*S^{-1}, \qquad (26)$$

and then separate the functions with the algebraic structures, which are reproduced by iterations. The connected part of the amputated three-quark function possesses two-meson and three-meson contributions. The explicit form of this function in momentum space see in Appendix B.

4 The corrections to the two-quark function and the Legendre transformation in mean-field expansion

In this Section we use the method of the Legendre transformation with respect to a bilocal source [1], [2] to determine corrections the two-quark function in the NJL model (see [8]). As we note above, the first iteration step contains the leading-order equation for the two-quark function and also the equation for the NLO correction to the propagator (see Section 2), and the next step contains the equations for the four- and three-quark functions and also the equations for the NLO corrections to the two-quark function and NNLO (second order) corrections to the quark propagator. This iterative scheme is attractive because the equations of any order have a simple analytic structure. Solving the equations of any order is actually an algebraic problem, which is a reminiscence of constructing a perturbation series. The reverse side of this simplicity is problems arising related with symmetry constraints and its physical consequences. The main problem of such kind arises in calculating corrections to the two-quark function.

The two-quark function of the leading approximation in the NJL model contains two single-pole terms corresponding to composite mesons, namely, scalar and pseudoscalar mesons. The presence of these mesons in the spectrum of the NJL model is one of the main effects of this model. The scalar meson (the sigma meson) is massive, while the pseudoscalar meson (the pion) is massless in the chiral limit according to the general NGB theorem for systems with spontaneously broken symmetry. But in the process of calculating the corrections to the two-quark function in the framework of the above-mentioned method the terms containing second-order poles in the quark-antiquark channel appear. It is very difficult to coordinate these terms, which are artifacts of the computational scheme, with the widely accepted interpretation of results in terms of particles. Another problem arises in attempting to interpret them as the first terms of some expansions: the pion acquires mass in the chiral limit, which contradicts the NGB theorem. A way out of this dilemma is to modify the calculation method in such a manner that the pion masslessness will be preserved in the chiral limit in the theory with higher-order corrections taken into account. The method for such a modification is to pass from the generating functional G to the generating functional of the Legendre transformation with respect to the bilocal source η. Such a passage in terms of perturbation theory diagrams means that only those diagrams that are two-quark irreducible (in a given channel) are taken into account. The other diagrams, just as in the usual perturbation theory, are effectively taken into account by the functional combinatorics of the Legendre transformation.

The generating functional of the Legendre transformation is defined after passing to the logarithm $Z = \frac{1}{i}\log G$ by introducing the complete quark propagator

$$S(x,y|\eta) = i\frac{\delta Z[\eta]}{\delta\eta(x,y)} \tag{27}$$

as a new functional variable [8]. We regard relation (27) as an equation for η . Solving this equation, we obtain η as a functional of S: $\eta = \eta[S]$. Further, we introduce the generating functional of the Legendre transformation depending on the functional variable S:

$$W[S] = Z[\eta[S]] + iTr[S \star \eta[S]]. \tag{28}$$

The main property of the Legendre transformation is the relation

$$\frac{1}{i}\frac{\delta W}{\delta S(x,y)} = \eta(x,y), \tag{29}$$

which follows from definitions (27), (28). In view of these relations, the SDE for the generating functional of the Legendre transformation becomes

$$\frac{1}{i}\frac{\delta W}{\delta S(x,y)} = S^{-1}(x,y) + i\hat{\partial}\delta(x-y) + ig\delta(x-y)\{tr[S(x,y)] + i\gamma_5\tau^a\ tr[i\gamma_5\tau^a S(x,y)]\}+$$

$$+ig\int\left\{S_2\left(\begin{array}{cc} x & y_1 \\ x & x\end{array}\right)S^{-1}(y_1,y) + i\gamma_5\tau^a S_2\left(\begin{array}{cc} x & y_1 \\ x & x\end{array}\right)i\gamma_5\tau^a S^{-1}(y_1,y)\right\}dy_1, \tag{30}$$

where

$$S_2\left(\begin{array}{cc} x & y \\ x' & y'\end{array}\right) = i\frac{\delta^2 Z}{\delta\eta(y,x)\delta\eta(y',x')} = \frac{\delta S(x,y)}{\delta\eta(y',x')} \tag{31}$$

is two-quark function.

The connected part of two-quark function (31) together with quark propagator (27) is the main object in the NJL-model calculations. Precisely this function is used to determine observed quantities, such as the masses of mesons, their decay constants, form factors, etc. By definition (31), the two-quark function, as a functional of S, is defined by the relation $S_2 = [\delta\eta/\delta S]^{-1}$ or, in the expanded form,

$$\int S_2\left(\begin{array}{cc} x & y \\ y_1 & x_1\end{array}\right)\frac{\delta\eta(x_1,y_1)}{\delta S(y',x')}dx_1dy_1 = \int S_2\left(\begin{array}{cc} x & y \\ y_1 & x_1\end{array}\right)\frac{1}{i}\frac{\delta^2 W}{\delta S(y',x')\delta S(y_1,x_1)}dx_1dy_1 =$$

$$= \delta(x-y')\delta(x'-y). \tag{32}$$

Relation (32) is essentially the equation for the two-quark function (Bethe-Salpeter equation) in the formalism of the Legendre transformation with respect to the bilocal source η. The kernel K of the Bethe-Salpeter equation (BSE) is the connected part of the second derivative of the generating functional of the Legendre transformation and is defined by the relation

$$\frac{\delta\eta(x,y)}{\delta S(y',x')} = -S^{-1}(x,y')S^{-1}(x',y) + K\left(\begin{array}{cc} x & y \\ x' & y'\end{array}\right). \tag{33}$$

Let us go to the amputated two-particle function (11) and define the connected part as in Eq. (12). As a result we give the equation for F_2^c, which is the connected part of the amputated two-quark function (the two-quark amplitude)

$$F_2^c\left(\begin{array}{cc} x & y \\ x' & y'\end{array}\right) = -K\left(\begin{array}{cc} x & y \\ x' & y'\end{array}\right)+$$

$$+\int S(y_1,x_2)F_2^c\left(\begin{array}{cc} x & y \\ x_2 & y_2\end{array}\right)S(y_2,x_1)K\left(\begin{array}{cc} x_1 & y_1 \\ x' & y'\end{array}\right)dx_1dx_2dy_1dy_2. \tag{34}$$

Calculating the generating functional of the Legendre transformation (more precisely, its derivatives) thus yields the equation for the quark propagator (this is Eq. (30) with the switched-off source η) and the kernel of the BSE for the two-particle function.

An important element of a field theory with spontaneous chiral symmetry breaking is the chiral Ward identity (see, e.g., [15], [16]). This identity relates the axial vector part of the two-quark function to the propagator and has the form

$$tr_u\left[\tau^a\gamma_5 i\hat{\partial}_z S_2\left(\begin{array}{cc} z & z \\ x & y\end{array}\right)\right] = \delta(z-y)S(x-z)\gamma_5\tau^a + \delta(z-x)\tau^a\gamma_5 S(z-y) \tag{35}$$

in the chiral limit. One of most important consequences of this identity is that the function S_2 must have a simple pole of the form $1/P^2$ in the quark-antiquark channel (here $P = p_x + p_y$ is the total momentum) in the case where the chiral symmetry is broken spontaneously, i.e., where the initially massless quarks acquire dynamical mass. This pole means that a massless pseudoscalar particle (the pion) appears in the spectrum of the theory in complete accordance with the NGB theorem.

As a matter of fact, the chiral Ward identity ensures the validity of the NGB theorem in a theory with spontaneously broken chiral symmetry. We note that this identity is a model-independent relation for any local theory with chiral symmetry and must hold in both the NJL model and QCD-type fundamental theory. In terms of the generating functional of the Legendre transformation, this identity has the form

$$\tau^a\gamma_5 i\hat{\partial}_z(\delta(x-z)\delta(z-y)) = \int tr_u\left[\frac{\delta\eta(z,x_1)}{\delta S(y,x)}S(x_1,z)\gamma_5\tau^a + \tau^a\gamma_5 S(z,x_1)\frac{\delta\eta(x_1,z)}{\delta S(y,x)}\right]dx_1 \quad (36)$$

in the chiral limit. We use relation (32) in passing from (35) to (36). Formula (36) shows that to verify the chiral Ward identity in the Legendre transformation formalism, there is no need to calculate the two-quark function, and it suffices to verify it for the kernel of the BSE.

The mean-field approximation for the generating functional of the Legendre transformation is SDE (30) without the last two terms:

$$\eta_0(x,y) = \frac{1}{i}\frac{\delta W_0}{\delta S(y,x)} =$$

$$S^{-1}(x,y) + i\hat{\partial}\delta(x-y) + ig\delta(x-y)\{tr[S(x,y)] + i\gamma_5\tau^a tr[i\gamma_5\tau^a S(x,y)]\}. \quad (37)$$

With the switched-off source, Eq. (37) is the equation for the quark propagator in the leading approximation (see Eq. (4)), whose solution is Eq. (6).

Differentiating leading-approximation equation (37) with respect to S, we obtain

$$\frac{1}{i}\frac{\delta^2 W_0}{\delta S(y',x')\delta S(y,x)} = -S^{-1}(x,y')\cdot S^{-1}(x',y)+$$

$$ig\delta(x-y)\delta(x'-y')\delta(x-y')\{1\cdot 1 + i\gamma_5\tau^a\cdot i\gamma_5\tau^a\}. \quad (38)$$

The connected part of this expression (i.e., the second term in the right-hand side) is the kernel of the BSE for two-quark amplitude (34) in the leading order of the MFE for the NJL model. By a direct verification, we can easily determine that chiral Ward identity (36) holds in the leading approximation, and this guarantees the existence of a massless pion in the given approximation. Before calculating the two-quark amplitude in the leading approximation, we can thus be assured that the mean-field approximation in the Legendre transformation formalism satisfies the main symmetry requirement of the NJL model in the chiral limit, namely, the NGB theorem.

With the switched-off source, the solution of the equation for two-quark amplitude has the form (14).

In the leading order, there are no differences in the descriptions of physical effects between the Legendre transformation formalism and the mean-field expansion considered in Section 2. Significant differences appear in studying corrections to the leading approximation. To take these corrections into account, we approximate the exact two-particle function (as the functional of S) in Eq. (31) with the two-particle function of the leading approximation. Then the equation for the generating functional generates a closed system of equations for the Green functions with corrections taken into account.

The equation for the generating functional of the Legendre transformation with the corrections included has the form

$$\eta(x,y) = S^{-1}(x,y) + i\hat{\partial}\delta(x-y) + ig\delta(x-y)\{tr[S(x,y)] + i\gamma_5\tau^a tr[i\gamma_5\tau^a S(x,y)]\} + \frac{1}{i}\frac{\delta W_1}{\delta S(y,x)}, \quad (39)$$

where

$$\frac{1}{i}\frac{\delta W_1}{\delta S(y,x)} = ig\int\{S(x,x_1)tr_d[S(x,x_2)F_0\left(\begin{array}{cc} x_1 & y \\ x_2 & y_2 \end{array}\right)S(y_2,x)] +$$

$$+\Big(i\gamma_5\tau^a S(x,x_1)\Big)tr_d[i\gamma_5\tau^a S(x,x_2)F_0\left(\begin{array}{cc} x_1 & y \\ x_2 & y_2 \end{array}\right)S(y_2,x)]\}dx_1dx_2dy_2. \quad (40)$$

Using the equation for the two-particle function in the leading order, we can reduce this equation to the more compact form

$$\frac{1}{i}\frac{\delta W_1}{\delta S(y,x)} = \int S(y_1,x_1)F_0^c\left(\begin{array}{cc} x_1 & y \\ x & y_1 \end{array}\right)dx_1dy_1 \quad (41)$$

In Eq. (41) F_0^c is the functional of S, whose derivatives can be calculated using relation (32). The derivative of Eq. (41) with respect to the functional variable S yields the correction to the kernel of the BSE for the two-particle function:

$$K_1\left(\begin{array}{cc} x & y \\ x' & y' \end{array}\right) = F_0^c\left(\begin{array}{cc} x' & y \\ x & y' \end{array}\right) + \int S(y_1,x_1)\frac{\delta F_0^c\left(\begin{array}{cc} x_1 & y \\ x & y_1 \end{array}\right)}{\delta S(y',x')}dx_1dy_1 \quad (42)$$

The result of the calculation is the expression

$$\begin{aligned}\frac{\delta F_0^c\left(\begin{array}{cc} x & y \\ x'' & y'' \end{array}\right)}{\delta S(y',x')} = -\delta(x''-y'')\Big\{A_\sigma(x''-y')\int F_0^c\left(\begin{array}{cc} x & y \\ x' & y_1 \end{array}\right)S(y_1-y')dy_1 + \\ +A_\sigma(x'-y'')\int S(x'-x_1)F_0^c\left(\begin{array}{cc} x & y \\ x_1 & y' \end{array}\right)dx_1 + \\ +i\gamma_5\tau^a\Big[A_\pi(x''-y')\int F_0^c\left(\begin{array}{cc} x & y \\ x' & y_1 \end{array}\right)S(y_1-y')i\gamma_5\tau^a\,dy_1 + \\ +A_\pi(x'-y'')\int i\gamma_5\tau^a S(x'-x_1)F_0^c\left(\begin{array}{cc} x & y \\ x_1 & y' \end{array}\right)dx_1\Big]\Big\} \quad (43)\end{aligned}$$

Formula (43) together with (42) yields the correction to the kernel of the BSE for the two-particle function. As can be seen, this correction consists of the one-meson and two-meson contributions.

The main problem in calculating the corrections to the two-particle amplitude is the requirement that these corrections correspond to the NGB theorem. The chiral Ward identity holds in the determinations of these corrections by the Legendre transformation method, which ensures the validity of the NGB theorem in this approach. Verifying the chiral Ward identity for the kernel of the BSE including corrections (42) and (43) is a less trivial procedure than similarly verifying the leading approximation, nevertheless the result is positive for this case too. This result shows that the approximation under consideration is physically acceptable, i.e. it is a symmetry-preserving approximation for calculating the corrections to the two-particle function. (For the more details of this calculation see also [8].)

5 Structure of third step of iteration

As we have showed above the equation for the four-quark function S_4 has a simple exact solution which is the product of first-order two-quark functions (see Eq. (25)). As it seen from this solution, the pion-pion scattering in NJL model is suppressed, i.e. in the second order of MFE this scattering is absent. This process arises in the third order of our iterative scheme, i.e. in NLO four-quark function $S_4^{(1)}$.

The third-step generating functional is

$$G^{(3)}\left[\eta\right]=\left\{\frac{1}{6!}\mathrm{Tr}\Big(S_6*\eta^6\Big)+\frac{1}{5!}\mathrm{Tr}\Big(S_5*\eta^5\Big)+\frac{1}{4!}\mathrm{Tr}\Big(S_4^{(1)}*\eta^4\Big)+\right.$$

$$\left.\frac{1}{3!}\mathrm{Tr}\Big(S_3^{(1)}*\eta^3\Big)+\frac{1}{2}\mathrm{Tr}\Big(S_2^{(2)}*\eta^2\Big)+\mathrm{Tr}\Big(S^{(3)}*\eta\Big)\right\}G^{(0)}.$$

After standard operations we obtain the equations for six-quark function S_6 and for five-quark function S_5. Inhomogeneous terms are following:

$$S_6^0\left(\begin{array}{cc} x_1 & y_1\\ \dots & \dots\\ x_6 & y_6\end{array}\right)=$$

$$\left\{-S(x_1-y_2)S(x_2-y_1)S_4\left(\begin{array}{cc} x_3 & y_3\\ x_4 & y_4\\ x_5 & y_5\\ x_6 & y_6\end{array}\right)\right\}+\Big\{2\leftrightarrow 3\Big\}+\Big\{2\leftrightarrow 4\Big\}+\Big\{2\leftrightarrow 5\Big\}+\Big\{2\leftrightarrow 6\Big\} \quad (44)$$

and

$$S_5^0\left(\begin{array}{cc} x_1 & y_1\\ \dots & \dots\\ x_5 & y_5\end{array}\right)=$$

$$\left\{-S(x_1-y_2)S(x_2-y_1)S_3\left(\begin{array}{cc} x_3 & y_3\\ x_4 & y_4\\ x_5 & y_5\end{array}\right)\right\}+\Big\{2\leftrightarrow 3\Big\}+\Big\{2\leftrightarrow 4\Big\}+\Big\{2\leftrightarrow 5\Big\}+$$

$$\left[-S(x_1-y_2)S_4\left(\begin{array}{cc} x_2 & y_1\\ x_3 & y_3\\ x_4 & y_4\\ x_5 & y_5\end{array}\right)\right]+\Big[2\leftrightarrow 3\Big]+\Big[2\leftrightarrow 4\Big]+\Big[2\leftrightarrow 5\Big]+$$

$$ig\int dx_1'\left\{tr_u\left[S(x_1-x_1')S_6\left(\begin{array}{cc} x_1' & y_1\\ x_1' & x_1'\\ x_2 & y_2\\ \dots & \dots\\ x_6 & y_6\end{array}\right)\right]-tr_u\left[S(x_1-x_1')\gamma_5\tau^a S_6\left(\begin{array}{cc} x_1' & y_1\\ x_1' & x_1'\\ x_2 & y_2\\ \dots & \dots\\ x_6 & y_6\end{array}\right)\gamma_5\tau^a\right]\right\}, \quad (45)$$

accordingly.

The equations for six-quark function and for the five-quark function with inhomogeneous term (44) and (45) in our iteration scheme are new. The third step of iterative scheme gives us the equation for four-quark function ($S_4^{(1)}$). As we note above the structure of this equation have are the form (22) with following inhomogeneous term

$$(S_4^{(1)})^0\left(\begin{array}{cc} x_1 & y_1\\ \dots & \dots\\ x_4 & y_4\end{array}\right)=$$

$$\left\{-S(x_1-y_2)S(x_2-y_1)S_2^{(1)}\left(\begin{array}{cc} x_3 & y_3\\ x_4 & y_4\end{array}\right)\right\}+\Big\{2\leftrightarrow 3\Big\}+\Big\{2\leftrightarrow 4\Big\}+$$

$$\left[-S(x_1-y_2)S_3\left(\begin{array}{cc} x_2 & y_1\\ x_3 & y_3\\ x_4 & y_4\end{array}\right)\right]+\Big[2\leftrightarrow 3\Big]+\Big[2\leftrightarrow 4\Big]+$$

$$ig\int dx_1'\Big\{tr_u\Big[S(x_1-x_1')S_5\begin{pmatrix} x_1' & y_1\\ x_1' & x_1'\\ x_2 & y_2\\ x_3 & y_3\\ x_4 & y_4\end{pmatrix}\Big] - tr_u\Big[S(x_1-x_1')\gamma_5\tau^a S_5\begin{pmatrix} x_1' & y\\ x_1' & x_1'\\ x_2 & y_2\\ x_3 & y_3\\ x_4 & y_4\end{pmatrix}\gamma_5\tau^a\Big]\Big\}. \quad (46)$$

The equation for NLO four-quark function $S_4^{(1)}$ gives us possibility to describe the pion-pion scattering in quark fields context. The inhomogeneous term (46) of equations for four-quark function $S_4^{(1)}$ contains five-quark function S_5, three-quark function S_3 and two-quark function $S_2^{(1)}$. The inhomogeneous term (45) for five-quark equation include the six-quark function S_6, four-quark function S_4 and three-quark function S_3. Before the investigation of four-quark function $S_4^{(1)}$ it is necessary to find the solution of equation for six-quark function S_6, because the inhomogeneous part (45) includes function S_6. Also it is necessary to find a solution of equation for NLO two-quark function $S_2^{(1)}$ (see Section 4).

The solution of six-quark equation is the sum of products of two-quark functions S_2 and four-quark functions S_4:

$$S_6\begin{pmatrix} x_1 & y_1\\ ... & ...\\ x_6 & y_6\end{pmatrix} =$$

$$\Big\{S_2\begin{pmatrix} x_1 & y_1\\ x_2 & y_2\end{pmatrix}\cdot S_4\begin{pmatrix} x_3 & y_3\\ x_4 & y_4\\ x_5 & y_5\\ x_6 & y_6\end{pmatrix}\Big\} + \Big\{2\leftrightarrow 3\Big\} + \Big\{2\leftrightarrow 4\Big\} + \Big\{2\leftrightarrow 5\Big\} + \Big\{2\leftrightarrow 6\Big\} \quad (47)$$

In this step we obtain also the equations for NLO three-quark function $S_3^{(1)}$, NNLO two-quark function $S_2^{(2)}$ and the equation for NNNLO correction to the quark propagator $S^{(3)}$, which matter the forms (44), at $n=3$, $n=2$, $n=1$, accordingly. The explicit forms for inhomogeneous terms of this equations see in Appendix C.

6 The generalization of the method for other types of multi-quark sources

In this last Section we consider the generalization of MFE of Section 2, which includes other types of multi-quark sources except of bilocal source η. Such generalization can be useful for the description of baryons in the framework of MFE.

Firstly, consider the formalism with diquark sources. For this purpose, we add two diquark-source terms ξ and $\bar{\xi}$ in the exponent of Eq. (3) for generating functional G:

$$G(\eta,\xi,\bar{\xi}) = \int D(\psi,\bar{\psi})\exp i\Big\{\int dxL - \int dxdy\bar{\psi}(y)\eta(y,x)\psi(x) +$$
$$+\int dx_1dx_2\bar{\psi}(x_1)\bar{\psi}(x_2)\xi(x_1,x_2) + \int dx_1dx_2\bar{\xi}(x_1,x_2)\psi(x_1)\psi(x_2)\Big\}. \quad (48)$$

With these sources SDE (2) is modified as follows:

$$\delta(x-y)G + i\hat{\partial}_x\frac{\delta G}{\delta\eta(y,x)} + ig\Big\{\frac{\delta}{\delta\eta(y,x)}tr\Big[\frac{\delta G}{\delta\eta(x,x)}\Big] - \gamma_5\tau^a\frac{\delta}{\delta\eta(y,x)}tr\Big[\gamma_5\tau^a\frac{\delta G}{\delta\eta(x,x)}\Big]\Big\} =$$
$$= \int dx_1\eta(x,x_1)\frac{\delta G}{\delta\eta(y,x_1)} + 2\int dx_1\frac{\delta G}{\delta\xi(y,x_1)}\,\xi(x,x_1). \quad (49)$$

We have, apart from SDE (49), the additional SDE, which generates by new sources:

$$i\hat{\partial}_{x_1}\frac{\delta G}{\delta\bar{\xi}(x_2,x_1)}+ig\Big\{\frac{\delta}{\delta\bar{\xi}(x_2,x_1)}tr\left[\frac{\delta G}{\delta\eta(x_1,x_1)}\right]-\gamma_5\tau^a\frac{\delta}{\delta\bar{\xi}(x_2,x_1)}tr\left[\gamma_5\tau^a\frac{\delta G}{\delta\eta(x_1,x_1)}\right]\Big\}=$$

$$=\int dy_1\eta(x_1,y_1)\frac{\delta G}{\delta\bar{\xi}(x_2,y_1)}-2\int dy_1\bar{\xi}(x_1,y_1)\frac{\delta G}{\delta\eta(y_1,x_2)}. \qquad (50)$$

It should be noted, that the presence of the new diquark source leads to the connection condition for derivatives of generating functional:

$$\frac{\delta^2 G}{\delta\bar{\xi}(x_2,x_1)\delta\eta(y,x)}=-\frac{\delta^2 G}{\delta\bar{\xi}(x_1,x)\delta\eta(y,x_2)}. \qquad (51)$$

Due to this connection condition SDE (50) can be rewritten in the alternative forms. These alternative forms, being fully equivalent from the point of view of an exact solution of SDE's, can lead to different approximations in the MFE. The choice of the suitable forms for the construction of MFE in the case should be made with an assistance of corresponding physical reasons.

In the very similar manner one can introduce three-quark, or baryon sources. These sources can be used for the direct description of nucleons and other baryons omitting the intermediate diquark modelling.

The generating functional with anti-commutative three-quark sources ζ and $\bar{\zeta}$ is

$$G(\eta,\zeta,\bar{\zeta})=\int D(\psi,\bar{\psi})\exp i\Big\{\int dxL-\int dxdy\bar{\psi}(y)\eta(y,x)\psi(x)+$$

$$+\int dx_1dx_2dx_3\bar{\psi}(x_1)\bar{\psi}(x_2)\bar{\psi}(x_3)\zeta(x_1,x_2,x_3)+\int dx_1dx_2dx_3\bar{\zeta}(x_1,x_2,x_3)\psi(x_1)\psi(x_2)\psi(x_3)\Big\}. (52)$$

SDE (2) with three-quark sources is modified as follows:

$$\delta(x-y)G+i\hat{\partial}_x\frac{\delta G}{\delta\eta(y,x)}+ig\Big\{\frac{\delta}{\delta\eta(y,x)}tr\left[\frac{\delta G}{\delta\eta(x,x)}\right]-\gamma_5\tau^a\frac{\delta}{\delta\eta(y,x)}tr\left[\gamma_5\tau^a\frac{\delta G}{\delta\eta(x,x)}\right]\Big\}=$$

$$=\int dx_1\eta(x,x_1)\frac{\delta G}{\delta\eta(y,x_1)}-3\int dx_1dx_2\frac{\delta G}{\delta\xi(y,x_1,x_2)}\,\zeta(x,x_1,x_2). \qquad (53)$$

As above, apart from SDE (53), the additional SDE exists, which generates by the three-quark sources:

$$i\hat{\partial}_{x_1}\frac{\delta G}{\delta\bar{\zeta}(x_3,x_2,x_1)}+ig\Big\{\frac{\delta}{\delta\bar{\zeta}(x_3,x_2,x_1)}tr\left[\frac{\delta G}{\delta\eta(x_1,x_1)}\right]-\gamma_5\tau^a\frac{\delta}{\delta\bar{\zeta}(x_3,x_2,x_1)}tr\left[\gamma_5\tau^a\frac{\delta G}{\delta\eta(x_1,x_1)}\right]\Big\}=$$

$$=\int dy_1\eta(x_1,y_1)\frac{\delta G}{\delta\bar{\zeta}(x_3,x_2,y_1)}+3i\int dy_2dy_3\frac{\delta^2 G}{\delta\eta(y_3,x_3)\delta\eta(y_2,x_2)}\bar{\zeta}(x_1,y_2,y_3). \qquad (54)$$

The connection condition for the derivatives of the generating functional, which is very similar to the condition (51), also exists in the three-quark-source formalism, and also leads to alternative forms of SDE (54).

The method of the construction of MFE for these system of equations is similar to that of Section 2. An elaboration of this construction is the object of future investigations.

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Appendix A

The inhomogeneous term of the equation for the second-step two-quark function is

$$(S_2^{(1)})^0 \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix} = -S(x_1-y_2)S^{(1)}(x_2-y_1)+$$

$$ig\int dx_1'\Big\{tr_u\Big[S(x_1-x_1')S_3\begin{pmatrix} x_1' & y_1 \\ x_1' & x_1' \\ x_2 & y_2 \end{pmatrix}\Big] - tr_u\Big[S(x_1-x_1')\gamma_5\tau^a S_3\begin{pmatrix} x_1' & y \\ x_1' & x_1' \\ x_2 & y_2 \end{pmatrix}\gamma_5\tau^a\Big]\Big\}.$$

It contains three-quark function S_3 and NLO single-quark function $S^{(1)}$. The second step gives us also to the equation for NNLO correction to quark propagator $S^{(2)}$. The inhomogeneous term is:

$$(S^{(2)})^0 = ig\int dx_1'\Big\{tr_u\Big[S(x_1-x_1')S_2^{(1)}\begin{pmatrix} x_1' & y_1 \\ x_1' & x_1' \end{pmatrix}\Big] - tr_u\Big[S(x_1-x_1')\gamma_5\tau^a S_2^{(1)}\begin{pmatrix} x_1' & y \\ x_1' & x_1' \end{pmatrix}\gamma_5\tau^a\Big]\Big\}.$$

This term contains NLO two-quark function $S_2^{(1)}$.

Appendix B

The connected part of amputated three-quark function is

$$F_3^c = F_3^{two-meson} + F_3^{three-meson},$$

where

$$\begin{aligned}
F_3^{two-meson} &= -(S(p+P')+S(p+P''))\cdot \mathrm{A}_\sigma(P')\cdot \mathrm{A}_\sigma(P'')- \\
&-\mathrm{A}_\sigma(P)\cdot(S(p'+P'')+S(p'+P))\cdot \mathrm{A}_\sigma(P'')- \\
&-\mathrm{A}_\sigma(P)\cdot \mathrm{A}_\sigma(P')\cdot(S(p''+P)+S(p''+P'))+ \\
&+(\gamma_5 S(p+P')\gamma_5+\gamma_5 S(p+P'')\gamma_5)\cdot \mathbf{A}_\pi(P')\cdot \mathbf{A}_\pi(P'')+ \\
&+\mathrm{A}_\sigma(P)\cdot(\gamma_5 S(p'+P'')+S(p'+P)\gamma_5)\tau\cdot \mathbf{A}_\pi(P'')+ \\
&+\mathrm{A}_\sigma(P)\cdot \mathbf{A}_\pi(P')\cdot\tau(S(p''+P)\gamma_5+\gamma_5 S(p''+P'))+ \\
&+(S(p+P')\gamma_5+\gamma_5 S(p+P''))\tau\cdot \mathrm{A}_\sigma(P')\cdot \mathbf{A}_\pi(P'')+ \\
&+\mathbf{A}_\pi(P)\cdot(\gamma_5 S(p'+P'')\gamma_5+\gamma_5 S(p'+P)\gamma_5)\cdot \mathbf{A}_\pi(P'')+ \\
&+\mathbf{A}_\pi(P)\cdot \mathrm{A}_\sigma(P')\cdot\tau(\gamma_5 S(p''+P)+S(p''+P')\gamma_5)+ \\
&+(\gamma_5 S(p+P')+S(p+P'')\gamma_5)\tau\cdot \mathbf{A}_\pi(P')\cdot \mathrm{A}_\sigma(P'')+ \\
&+\mathbf{A}_\pi(P)\cdot\tau(S(p'+P'')\gamma_5+\gamma_5 S(p'+P))\cdot \mathrm{A}_\sigma(P'')+ \\
&+\mathbf{A}_\pi(P)\cdot \mathbf{A}_\pi(P')\cdot(\gamma_5 S(p''+P)\gamma_5+\gamma_5 S(p''+P')\gamma_5)+ \\
&+i(\gamma_5 S(p+P')\gamma_5-\gamma_5 S(p+P'')\gamma_5)\tau\cdot[\mathbf{A}_\pi(P')\times \mathbf{A}_\pi(P'')]+ \\
&+i\mathbf{A}_\pi(P)\cdot[(\gamma_5 S(p'+P'')\gamma_5-\gamma_5 S(p'+P)\gamma_5)\tau\times \mathbf{A}_\pi(P'')]+ \\
&+i\mathbf{A}_\pi(P)\cdot[\mathbf{A}_\pi(P')\times\tau(\gamma_5 S(p''+P)\gamma_5-\gamma_5 S(p''+P')\gamma_5)]
\end{aligned}$$

$$\begin{aligned}
F_3^{three-meson} &= 2n_c \mathrm{A}_\sigma(P)\cdot \mathrm{A}_\sigma(P')\cdot \mathrm{A}_\sigma(P'')[\Delta_{sss}(P;P+P')+\Delta_{sss}(P;P+P'')]- \\
&-2n_c \mathrm{A}_\sigma(P)\cdot \mathbf{A}_\pi(P')\cdot \mathbf{A}_\pi(P'')[\Delta_{spp}(P;P+P')+\Delta_{spp}(P;P+P'')]- \\
&-2n_c \mathbf{A}_\pi(P)\cdot \mathbf{A}_\pi(P')\cdot \mathrm{A}_\sigma(P'')[\Delta_{spp}(P'';P'+P'')+\Delta_{spp}(P'';P+P'')]-
\end{aligned}$$

$$-2n_c\mathbf{A}_\pi(P)\cdot \mathrm{A}_\sigma(P')\cdot \mathbf{A}_\pi(P'')[\Delta_{spp}(P';P'+P'')+\Delta_{spp}(P';P+P')].$$

Here $p=p_{x_1}$, $P=p_{x_1}+p_{y_1}$, $p'=p_{x_2}$, $P'=p_{x_2}+p_{y_2}$, $p''=p_{x_3}$, $P''=p_{x_3}+p_{y_3}$, where p_{x_i},p_{y_i} are quark momenta and $P+P'+P''=0$. We also use notations $A_\sigma=1\circ A_\sigma$ for the scalar-singlet function, $\mathbf{A}_\pi=\gamma_5\tau\circ A_\pi$ for the pseudoscalar-isovector function see (17), (18)) and

$$\Delta_{sss}(P;Q)=\int d\tilde{q}tr[S(P+q)S(Q+q)S(Q)], \Delta_{spp}(P;Q)=\int d\tilde{q}tr[S(P+q)\gamma_5 S(Q+q)\gamma_5 S(Q)]$$

for quark triangles. (In last formulae the traces are taken over Lorentz spinor indices only.)

Appendix C

The inhomogeneous term of NLO three-quark function $S_3^{(1)}$ is:

$$(S_3^{(1)})^0\begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \end{pmatrix}=$$

$$\Big\{-S(x_1-y_2)S(x_2-y_1)S^{(2)}(x_3-y_3)\Big\}+\Big\{2\leftrightarrow 3\Big\}+\Big[-S(x_1-y_2)S_2^{(1)}\begin{pmatrix} x_2 & y_1 \\ x_3 & y_3 \end{pmatrix}\Big]+\Big[2\leftrightarrow 3\Big]+$$

$$ig\int dx_1'\Big\{tr_u\Big[S(x_1-x_1')S_4^{(1)}\begin{pmatrix} x_1' & y_1 \\ x_1' & x_1' \\ x_2 & y_2 \\ x_3 & y_3 \end{pmatrix}\Big]-tr_u\Big[S(x_1-x_1')\gamma_5\tau^a S_4^{(1)}\begin{pmatrix} x_1' & y \\ x_1' & x_1' \\ x_2 & y_2 \\ x_3 & y_3 \end{pmatrix}\gamma_5\tau^a\Big]\Big\},$$

which enter: NLO four-quark function $S_4^{(1)}$, NLO two-quark function $S_2^{(1)}$ and NNLO single-quark function $S^{(2)}$.

The equation for NNLO two-quark function $S_2^{(2)}$ which kernel is same form as second- and first step equations (see Eq.(9), or the equation (22) in common presentation) with inhomogeneous term

$$(S_2^{(2)})^0\begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix}=-S(x_1-y_2)S^{(2)}(x_2-y_1)+$$

$$ig\int dx_1'\Big\{tr_u\Big[S(x_1-x_1')S_3^{(1)}\begin{pmatrix} x_1' & y_1 \\ x_1' & x_1' \\ x_2 & y_2 \end{pmatrix}\Big]-tr_u\Big[S(x_1-x_1')\gamma_5\tau^a S_3^{(1)}\begin{pmatrix} x_1' & y \\ x_1' & x_1' \\ x_2 & y_2 \end{pmatrix}\gamma_5\tau^a\Big]\Big\},$$

which contains NLO three-quark function $S_3^{(1)}$ and NNLO single-quark function $S^{(2)}$. The third step gives us also to the equation for NNNLO correction to quark propagator $S^{(3)}$. The explicit form of this equation is (22) at $n=1$ and it inhomogeneous term is:

$$(S^{(3)})^0=ig\int dx_1'\Big\{tr_u\Big[S(x_1-x_1')S_2^{(2)}\begin{pmatrix} x_1' & y_1 \\ x_1' & x_1' \end{pmatrix}\Big]-tr_u\Big[S(x_1-x_1')\gamma_5\tau^a S_2^{(2)}\begin{pmatrix} x_1' & y \\ x_1' & x_1' \end{pmatrix}\gamma_5\tau^a\Big]\Big\}.$$

This term contains NNLO two-quark function $S_2^{(2)}$ (note that the inhomogeneous term of analogous equation for NNLO quark propagator $S^{(2)}$ contain NLO two-quark function $S_2^{(1)}$ and the inhomogeneous term for first-step equation for NLO quark propagator $S^{(1)}$ maintain two-quark Green function S_2), and all equations for quark-propagators have a analogous form, except of the inhomogeneous term.

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