The $Z \to c\bar{c} \to \gamma\gamma^*$, $Z \to b\bar{b} \to \gamma\gamma^*$ triangle diagrams and the $Z \to \gamma\psi$, $Z \to \gamma\Upsilon$ decays

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Abstract

It is expounded the approach to the $Z \to \gamma\psi$ and $Z \to \gamma\Upsilon$ decay study, based on the sum rules for the $Z \to c\bar{c} \to \gamma\gamma^*$ and $Z \to b\bar{b} \to \gamma\gamma^*$ amplitudes and their derivatives. The branching ratios of the $Z \to \gamma\psi$ and $Z \to \gamma\Upsilon$ decays are calculated for different guesses as to saturation of the sum rules. The lower bounds of $\sum_\psi BR(Z \to \gamma\psi) = 1.95 \cdot 10^{-7}$ and $\sum_\Upsilon BR(Z \to \gamma\Upsilon) = 7.23 \cdot 10^{-7}$ are found. Deviations from the lower bounds are discussed, among them the possibility of $BR(Z \to \gamma J/\psi(1S)) \sim BR(Z \to \gamma\Upsilon(1S)) \sim 10^{-6}$, that could be probably measured in LHC. The angle distributions in the $Z \to \gamma\psi$ and $Z \to \gamma\Upsilon$ decays are calculated also.

1 Introduction

The purpose of this paper is to revise the study of the $Z \to \gamma\psi$ and $Z \to \gamma\Upsilon$ decays in a dispersion approach [1, 2] [1].

Section 2 is devoted to the $Z \to \gamma\psi$ and $Z \to \gamma\Upsilon$ decays. Here the invariant amplitudes of the triangle loop diagrams , Figure 1, describing the transition of the axial-vector current $\to q\bar{q} \to \gamma(k_1)\gamma(k_2)$ at $k_1^2 = 0$ and $k_2^2 \neq 0$, [2] are used to construct the sum rules for the $Z \to c\bar{c}$ (or $b\bar{b}$) $\to \gamma\gamma*$ amplitude and its derivative. Then the minima of the branching ratio sums ($\min \sum_V BR(Z \to \gamma V)$, where $V = \psi$ or Υ) are evaluated for different guesses as to saturation of the sum rules. Three assumption are investigated: i) the resonance saturation of the sum rule for the amplitude in Section 2.1, ii) the resonance saturation of the sum rule for the amplitude derivative in Section 2.2, iii) and the simultaneous resonance saturation of the amplitude and its derivative in Section 2.3. In Sections 2.1 and 2.2 it is shown that the resonance saturation of the sum rule for the amplitude derivative results in a reasonably small resonance contribution to the amplitude, whereas the resonance saturation of the sum rule for the amplitude results in an unacceptably large resonance contribution to the amplitude derivative. In Section 2.3 it is shown that the simultaneous resonance saturation of the amplitude and its derivative allows to conclude that the resonance saturation of the sum rule for the amplitude derivative results in the minimum of $\min \sum_V BR(Z \to \gamma V)$, which agrees reasonably with the quark model prediction [7]. Various deviations from this lower bound are considered.

In Section 3 there is a brief conclusion. Specifically, it is discussed the possibility of $BR(Z \to \gamma J/\psi(1S)) \sim BR(Z \to \gamma\Upsilon(1S)) \sim 10^{-6}$, that could be probably measured in LHC.

In Appendix the angle distributions in the $Z \to \gamma\psi$ and $Z \to \gamma\Upsilon$ decays are calculated.

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[1] Note that an analogous dispersion approach was used to investigate the decays involved the Higgs boson: $H \to \gamma\psi\,, \gamma\Upsilon$ and of the decays $\psi\,, \Upsilon \to \gamma H$(or axion) [3].

[2] These amplitudes are calculated in Ref. [2] and can be found also in Refs. [4, 5]. These calculations made it possible to show [5, 6] that in the chiral limit there is the massless particle-like pole in the transverse part of the axial-vector channel of the $axial - vector\ current \to q\bar{q} \to vector\ current\ (k_1) \times vector\ current\ (k_2)$ amplitude at $k_1^2 = 0$ and $k_2^2 \neq 0$ in parallel with the massless particle-like pole in the longitudinal one, generally accepted at that time.

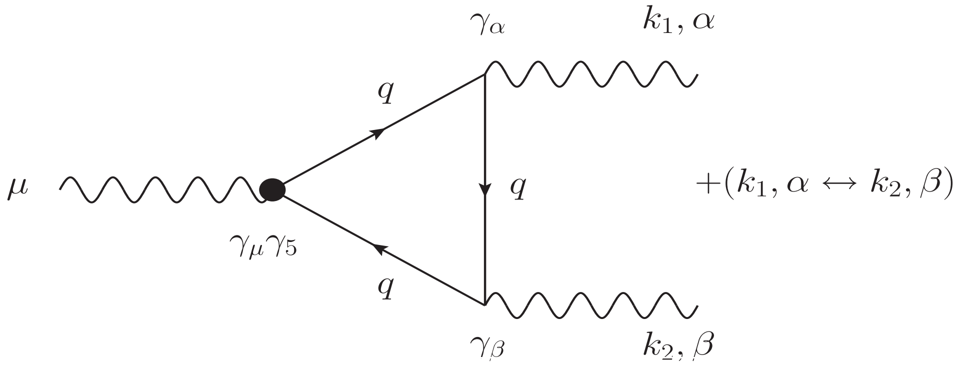

Figure 1: The triangle diagrams

2 Decays $Z \to \gamma\psi$ and $Z \to \gamma\Upsilon$ in dispersion approach

As is generally known [8, 9] the axial-vector vertex, resulted from the triangle diagrams

$$\begin{aligned} T_{\alpha\beta\mu} &= A_1 k_1^\sigma \epsilon_{\sigma\alpha\beta\mu} + A_2 k_2^\sigma \epsilon_{\sigma\alpha\beta\mu} + A_3 k_{1\beta} k_1^\delta k_2^\sigma \epsilon_{\delta\sigma\alpha\mu} + \\ &\quad + A_4 k_{2\beta} k_1^\delta k_2^\sigma \epsilon_{\delta\sigma\alpha\mu} + A_5 k_{1\alpha} k_1^\delta k_2^\sigma \epsilon_{\delta\sigma\beta\mu} + A_6 k_{2\alpha} k_1^\delta k_2^\sigma \epsilon_{\delta\sigma\beta\mu}. \end{aligned} \tag{1}$$

The local gauge invariance

$$k_1^\alpha T_{\alpha\beta\mu} = k_2^\beta T_{\alpha\beta\mu} = 0 \tag{2}$$

is ensured by the next constraints:

$$A_1 = k_2^2 A_4 + (k_1 k_2) A_3, \qquad A_2 = k_1^2 A_5 + (k_1 k_2) A_6. \tag{3}$$

Besides that

$$A_3(k_1, k_2) = -A_6(k_2, k_1), \qquad A_4(k_1, k_2) = -A_5(k_2, k_1). \tag{4}$$

A_3, A_4, A_5 and A_6 are the invariant amplitudes free of kinematical singularities. They are well-defined and can be calculated in the analytic form if $k_1^2 = 0$ (or $k_2^2 = 0$). Let us consider the region $k_1^2 = 0,\ \ Q^2 = -k_2^2 = -E^2 > 0,\ \ W^2 = -M^2 = -(k_1 + k_2)^2 > 0$ which is suitable for the calculations with the help of the dispersion relations over M^2 (and over E^2). In the following, it is required only the A_4 and A_6 amplitudes:

$$\begin{aligned} A_4 &= -\frac{1}{2\pi^2} \cdot \frac{1}{Q^2 - W^2} L_1\,, \\ A_6 &= \frac{1}{2\pi^2} \cdot \frac{1}{Q^2 - W^2} \left\{ \frac{Q^2}{Q^2 - W^2} L_1 + \frac{m_q^2}{Q^2 - W^2} L_2 - 1 \right\}, \end{aligned} \tag{5}$$

where

$$\begin{aligned} L_1 &= -\rho \ln \frac{\rho + 1}{\rho - 1} + \beta \ln \frac{\beta + 1}{\beta - 1}\ , \\ L_2 &= -\ln^2 \frac{\rho + 1}{\rho - 1} + \ln^2 \frac{\beta + 1}{\beta - 1}\ , \\ \rho^2 &= 1 + \frac{4m_q^2}{W^2}, \qquad \beta^2 = 1 + \frac{4m_q^2}{Q^2}\ . \end{aligned} \tag{6}$$

In the other regions of $M^2 = -W^2$ and $E^2 = -Q^2$ the functions L_1 and L_2 are continued analytically in the following way [10] i)

$$
\begin{aligned}
0 < -W^2 \quad &= \quad M^2 < 4m_q^2 \quad : \\
\rho \to i\sqrt{-\rho^2}, \qquad & \frac{1}{2}\ln\frac{\rho+1}{\rho-1} \to -i\arctan\frac{1}{\sqrt{-\rho^2}} \quad , \\
2m_q < M \qquad & : \\
\sqrt{-\rho^2} \to -i\rho, \qquad & \arctan\frac{1}{\sqrt{-\rho^2}} \to \frac{\pi}{2} + \frac{i}{2}\ln\frac{1+\rho}{1-\rho}.
\end{aligned}
\tag{7}
$$

ii)

$$
\begin{aligned}
0 < -Q^2 \quad &= \quad E^2 < 4m_q^2 \quad : \\
\beta \to i\sqrt{-\beta^2}, \qquad & \frac{1}{2}\ln\frac{\beta+1}{\beta-1} \to -i\arctan\frac{1}{\sqrt{-\beta^2}} \quad , \\
2m_q < E \qquad & : \\
\sqrt{-\beta^2} \to -i\beta, \qquad & \arctan\frac{1}{\sqrt{-\beta^2}} \to \frac{\pi}{2} + \frac{i}{2}\ln\frac{1+\beta}{1-\beta}.
\end{aligned}
\tag{8}
$$

It is seen from Eqs.(5)-(8) that the amplitudes A_4 and A_6 contain no singularities (both for $Q^2 = -k_2^2 \neq 0$ and $k_2^2 = -Q^2 = 0$) except for the dynamical cuts over $4m_q^2 \leq M^2 < \infty$ and $4m_q^2 \leq E^2 < \infty$ resulted from the $q\bar{q}$ intermediate states.

Let us use Eqs.(1)-(8) to calculate the amplitude for $Z \to \gamma(k_1)\gamma^*(k_2)$ according to the triangle diagrams with the intermediate heavy quarks ($Z \to c\bar{c} \to \gamma\gamma^*$ or $Z \to b\bar{b} \to \gamma\gamma^*$) for $0 \leq k_2^2 = E^2 \leq 4m_q^2$ $(k_1^2 = 0)$ in the rest frame of the Z boson.

$$
\begin{aligned}
&T\Big(Z \to q\bar{q} \to \gamma\gamma^*\Big) = M^2 E\left(1 - E^2/M^2\right) t_q \times \\
&\times\Big\{\Big(\frac{E}{M}\Big)\Big(\mathbf{n}\cdot\mathbf{e}(Z)\Big)\Big(\mathbf{n}\cdot\Big[\mathbf{e}(\gamma^*)\times\mathbf{e}(\gamma)\Big]\Big) + \Big(\mathbf{n}\cdot\mathbf{e}(\gamma^*)\Big)\Big(\mathbf{n}\cdot\Big[\mathbf{e}(\gamma)\times\mathbf{e}(Z)\Big]\Big)\Big\},
\end{aligned}
\tag{9}
$$

where $M \equiv M_Z$ is the mass of the Z boson, $M^2 = (k_1 + k_2)^2$; $\mathbf{n} = \mathbf{k}_1/\Big|\mathbf{k}_1\Big|$; $\mathbf{e}(Z)$ and $\mathbf{e}(\gamma^*)$ are the polarization three-vectors of the Z boson and the γ^* quantum in their rest frames ; $\mathbf{e}(\gamma)$ is the polarization three-vector of the γ quantum. The amplitude t_q takes into account three identical loops corresponding to three colors.

$$
t_q = -\sigma_q\frac{3}{4}\cdot\frac{e^3 e_q^2}{\sin 2\Theta_W}\left(A_4 + A_6\right), \tag{10}
$$

where $\sigma_c = 1, \sigma_b = -1, e_c = 2/3, e_b = -1/3$.

It is seen from Eqs.(5)-(8) that t_q satisfies a dispersion relation without subtractions both in M^2 and in E^2. Consequently, t_q is the amplitude convenient for obtaining sum rules in the E^2 channel. Since at the present time it appears to test theoretically only the resonance saturation of the sum rules evaluated below, it is most convenient to derive them with the help of the following consideration.

The amplitude t_q describes the full amplitude for $Z \to q\bar{q} \to \gamma\gamma^*$ in the region $E^2 \leq 0$ accurate up to higher corrections in QCD and the standard electroweak theory, i.e., accurate up to corrections of order $\alpha_S(4m_q^2)/\pi$, $\alpha_S(M^2)/\pi$ and α/π. On the other hand, the full amplitude for $Z \to q\bar{q} \to \gamma\gamma^*$ can be represented with the help of the intermediate hadronic states in the E^2 channel as the sum of resonance contributions and a continuum spectrum contribution:

$$
\begin{aligned}
&T\Big(Z \to q\bar{q} \to \gamma\gamma^*\Big) = M^2 E\left(1 - E^2/M^2\right) t_h^q \times \\
&\times\Big\{\Big(\frac{E}{M}\Big)\Big(\mathbf{n}\cdot\mathbf{e}(Z)\Big)\Big(\mathbf{n}\cdot\Big[\mathbf{e}(\gamma^*)\times\mathbf{e}(\gamma)\Big]\Big) + \Big(\mathbf{n}\cdot\mathbf{e}(\gamma^*)\Big)\Big(\mathbf{n}\cdot\Big[\mathbf{e}(\gamma)\times\mathbf{e}(Z)\Big]\Big)\Big\},
\end{aligned}
\tag{11}
$$

where

$$t_h^q = \sum_V \frac{m_V^2}{m_V^2 - E^2} \cdot \frac{e}{f_V} T_V^q + eT_{cont}^q. \qquad (12)$$

Here V is a $(q\bar{q})$ vector quarkonium ; T_{cont}^q is the continuum contribution $(D\bar{D}, D^*\bar{D}, D\bar{D}^*, D^*\bar{D}^*, \cdots$ or $B\bar{B}, B^*\bar{B}, B\bar{B}^*, B^*\bar{B}^*, \cdots)$.

There is every reason to believe that where $E^2 \approx 0$

$$t_h^q \approx t_q \approx -\sigma_q \frac{3e\alpha e_q^2}{2\sin 2\Theta_W} \cdot \frac{1}{M^2}\left(i - \frac{2}{\pi}\ln\frac{M}{m_q} + \frac{1}{\pi} + \frac{\beta}{\pi}\ln\frac{\beta+1}{\beta-1}\right). \qquad (13)$$

Eq.(13) incorporates the fact that $2m_q/M \ll 1$.

Let us consider at $E^2 = 0$ the sum rule for the amplitude

$$t_q\Big|_{E^2=0} = t_h^q\Big|_{E^2=0} \qquad (14)$$

and its first derivative

$$\frac{d}{dE^2} t_q\Big|_{E^2=0} = \frac{d}{dE^2} t_h^q\Big|_{E^2=0}. \qquad (15)$$

It follows from Eqs.(12), (13) and (14) that

$$\sum_V \frac{1}{f_V} T_V^q + T_{cont}^q\Big|_{E^2=0} \equiv T_q(Res) + T_{cont}^q\Big|_{E^2=0} =$$
$$= T_q \equiv -\sigma_q \frac{3\alpha e_q^2}{2\sin 2\Theta_W} \cdot \frac{1}{M^2}\left(i - \frac{2}{\pi}\ln\frac{M}{m_q} + \frac{3}{\pi}\right). \qquad (16)$$

An unusual feature of this sum rule is the presence of an imaginary part on the right-hand side of Eq. (16) coming from the amplitude discontinuity due to the real $q\bar{q}$ intermediate states in the M^2 channel [3].

It follows from Eqs. (12), (13) and (15) that

$$\sum_V \frac{1}{f_V m_V^2} T_V^q + \frac{d}{dE^2} T_{cont}^q\Big|_{E^2=0} = D_q(Res) + \frac{d}{dE^2} T_{cont}^q\Big|_{E^2=0} =$$
$$= D_q \equiv \sigma_q \frac{\alpha e_q^2}{4\pi m_q^2 \sin 2\Theta_W} \cdot \frac{1}{M^2}. \qquad (17)$$

$ImD_q = 0$ for the approximation (13).

The width of the decay $Z \to \gamma V$ is

$$\Gamma(Z \to \gamma V) =$$
$$= \frac{1}{24\pi}(1 - m_V^2/M^2)^3(1 + m_V^2/M^2) M^3 m_V^2 \left|T_V^q\right|^2 \approx \frac{1}{24\pi} M^3 m_V^2 \left|T_V^q\right|^2. \qquad (18)$$

To determine $f_V^2/4\pi$, one uses the experimental data [11] on

$$\Gamma(V \to e^+e^-) = \frac{4\pi}{3}\frac{m_V}{f_V^2}\alpha^2 \qquad (19)$$

As a result one gets for the ψ family $c1 \equiv J/\psi(1S) \equiv \psi(3097),\ c2 \equiv \psi(3686),\ c3 \equiv \psi(3770),\ c4 \equiv \psi(4040),\ c5 \equiv \psi(4160),\ c6 \equiv \psi(4415)$,

$$f_{c1} : f_{c2} : f_{c3} : f_{c4} : f_{c5} : f_{c6} = 1 : 1.67 : 5.05 : 2.9 : 3 : 3.69\,, \qquad (20)$$

[3] The contribution of resonances in $Im(T_q)$ should be small for the vertex $q^*(\text{virtual})\bar{q} \to V$ (or $\bar{q}^*(\text{virtual})q \to V$) should be suppressed by the wave function of the quarkonium. This reason was missed in Refs. [1, 2].

and for the Υ family $b1 \equiv \Upsilon(9460)$, $b2 \equiv \Upsilon(10023)$, $b3 \equiv \Upsilon(10355)$, $b4 \equiv \Upsilon(10579)$, $b5 \equiv \Upsilon(10860)$, $b6 \equiv \Upsilon(11020)$,

$$f_{b1} : f_{b2} : f_{b3} : f_{b4} : f_{b5} : f_{b6} = 1 : 1.52 : 1.82 : 2.35 : 2.23 : 3.47\,, \tag{21}$$

where

$$\begin{aligned} f_{c1} &= 11.16, \quad f_{c1}^2/4\pi \equiv f_{J/\psi(1S)}^2/4\pi \equiv f_{\psi(3097)}^2/4\pi = 9.91\,, \\ f_{b1} &= 39.69\,, \quad f_{b1}^2/4\pi \equiv f_{\Upsilon(1S)}^2/4\pi \equiv f_{\Upsilon(9460)}^2/4\pi = 125.38\,. \end{aligned} \tag{22}$$

2.1 Sum rule for amplitude

Let us assume initially that the real part of the sum rule for the amplitude, Eq. (16), is saturated with a ground state, that is,

$$T_V^q = f_V Re(T_q)\,, \quad \text{where}\,, \; V = J/\psi(1S), \;\; \Upsilon(1S). \tag{23}$$

Using Eqs. (16), (18), (22), (23), $m_c = 1.27$ GeV, $m_b = 4.2$ GeV, $M = 91.19$ GeV, $\Gamma_Z = 2.5$ GeV, $\alpha = 1/137$, and $\sin 2\Theta_W = 0.84$, [11], we find

$$BR(Z \to \gamma J/\psi(1S)) = 7.2 \cdot 10^{-6}, \quad BR(Z \to \gamma\Upsilon(1S)) = 1.7 \cdot 10^{-5}, \tag{24}$$

which are two orders of magnitude higher than it is expected in the quark model [7].

Let us saturate now the real part of the sum rule for the amplitude Eq. (16) with the ψ and Υ families

$$\sum_V \frac{1}{f_V} T_V^q \equiv T_q(Res) = Re(T_q)\,. \tag{25}$$

Note that this takes partially into account the continuous spectrum since four members of the ψ family and three members of the Υ family lie in the continuous spectrum of $D\bar{D}, D\bar{D}^*, D^*\bar{D}, \bar{D}^*D^*$ and $B\bar{B}, B\bar{B}^*, B^*\bar{B}, \bar{B}^*B^*$, respectively.

Considering Eq. (25) as the constraint and using Eq. (18) one can find $\min \sum_V \Gamma(Z \to \gamma V)$ that is reached when

$$T_V^q = \frac{1}{a_q f_V m_V^2} Re(T_q)\,, \quad \text{where} \quad a_q = \sum_V \frac{1}{f_V^2 m_V^2}\,, \tag{26}$$

$$\Gamma(Z \to \gamma V) = \frac{1}{24\pi} M^3 \left(Re(T_q)\right)^2 \left(f_V m_V a_q\right)^{-2}\,, \tag{27}$$

and

$$\min \sum_V \Gamma(Z \to \gamma V) = \frac{1}{24\pi} M^3 \left(Re(T_q)\right)^2 a_q^{-1}\,. \tag{28}$$

For the Ψ family ($a_c = 1.2 \cdot 10^{-3}$ GeV^{-2})

$$\begin{aligned} &\min \sum_\Psi BR(Z \to \gamma\psi) = 5.05 \cdot 10^{-6}, \\ &BR(Z \to \gamma J/\psi(1S)) = 3.53 \cdot 10^{-6}. \end{aligned} \tag{29}$$

For the Υ family ($a_b = 1.43 \cdot 10^{-5}$ GeV^{-2})

$$\begin{aligned} &\min \sum_\Upsilon BR(Z \to \gamma\Upsilon) = 8.58 \cdot 10^{-6}, \\ &BR(Z \to \gamma\Upsilon(1S) = 4.25 \cdot 10^{-6}. \end{aligned} \tag{30}$$

When the amplitude is saturated with the ground state or the resonance family, it follows from Eqs. (23) or (25)

$$D_q(V) = \frac{1}{f_V m_V^2} T_V^q = \frac{1}{m_V^2} Re(T_q) \tag{31}$$

or

$$D_q(Res) = \sum_V \frac{1}{f_V m_V^2} T_V^q = \sum_V \frac{1}{f_V m_V^2} \cdot \frac{1}{a_q f_V m_V^2} Re(T_q) = \frac{d_q}{a_q} Re(T_q)\,. \tag{32}$$

Eq. (31) leads to

$$D_c(J/\psi(1S)) = 0.10 Re(T_c)\ \mathrm{GeV}^{-2} = 5.60 D_c \tag{33}$$

and

$$D_b(\Upsilon(1S)) = 0.01 Re(T_b)\ \mathrm{GeV}^{-2} = 3.73 D_b\,. \tag{34}$$

Eq. (32) leads to

$$D_c(Res) = 0.09 Re(T_c)\ \mathrm{GeV}^{-2} = 4.99 D_c \tag{35}$$

and

$$D_b(Res) = 0.01 Re(T_b)\ \mathrm{GeV}^{-2} = 3.42 D_b\,. \tag{36}$$

So, the saturation of the amplitudes with the ground states or the resonance families leads to the unacceptably large contributions of the resonances into the amplitude derivatives[4].

Using Eqs. (5)-(8) one can verify that the dispersion integral for T_q is determined by the region $2m_q \le E \sim M_Z$, which is hardly a low energy region. Consequently, it is reasonable to study the sum rule (17) for the amplitude derivative because the contribution of low-lying states in the dispersion integral for the amplitude derivative is significantly enhanced as compared to their contribution to the amplitude itself. Note that 90% of the dispersion integral for D_q is determined by the region of low energies $2m_q \le E \le 6m_q$.

2.2 Sum rule for the derivative of the amplitude

Let us assume initially that the sum rule for the amplitude derivative, Eq. (17), is saturated with a ground state, $V = J/\psi(1S),\ \ \Upsilon(1S)$,

$$T_V^q = f_V m_V^2 D_q\,, \tag{37}$$

then

$$\Gamma(Z \to \gamma V) = \frac{1}{24\pi} f_V^2 M^3 m_V^6 D_q^2\,, \tag{38}$$

resulting in

$$BR(Z \to \gamma J/\psi(1S)) = 2.31 \cdot 10^{-7}, \quad BR(Z \to \gamma \Upsilon(1S)) = 1.24 \cdot 10^{-6}. \tag{39}$$

As this takes place, the ground state contribution in T_q is

$$T_q(V) \equiv \frac{1}{f_V} T_V^q = m_V^2 D_q\,, \tag{40}$$

resulting in

$$T_c(J/\psi(1S)) = 9.59 D_c\ \mathrm{GeV}^2 = 0.18 Re(T_c) \tag{41}$$

and

$$T_b(\Upsilon(1S)) = 89.49 D_b\ \mathrm{GeV}^2 = 0.27 Re(T_b)\,. \tag{42}$$

[4]This crucial point was missed in Refs. [1, 2].

Now let us consider the saturation of of the sum rule for the amplitude derivative, Eq. (17), with the ψ and Υ families

$$\sum_V \frac{1}{f_V m_V^2} T_V^q \equiv D_q(Res) = D_q \tag{43}$$

Considering Eq. (43) as the constraint and using Eq. (18) one get that min $\sum_V \Gamma(Z \to \gamma V)$ is reached when

$$T_V^q = \frac{1}{g_q f_V m_V^4} D_q\,, \quad \text{where} \quad g_q = \sum_V \frac{1}{f_V^2 m_V^6}\,, \tag{44}$$

$$\Gamma(Z \to \gamma V) = \frac{1}{24\pi} M^3 D_q^2 \left(f_V m_V^3 g_q\right)^{-2}\,, \tag{45}$$

and

$$\min \sum_V \Gamma(Z \to \gamma V) = \frac{1}{24\pi} M^3 D_q^2 g_q^{-1}\,. \tag{46}$$

For the ψ family ($g_c = 1.08 \cdot 10^{-5}$ GeV^{-6})

$$\min \sum_\psi BR(Z \to \gamma\psi) = 1.95 \cdot 10^{-7} \tag{47}$$

and

$$\begin{aligned}
&BR(Z \to \gamma J/\psi(1S)) = 1.64 \cdot 10^{-7}\,, \quad BR(Z \to \gamma\psi(3686)) = 2.056 \cdot 10^{-8}\,,\\
&BR(Z \to \gamma\psi(3770)) = 2 \cdot 10^{-9}\,, \quad BR(Z \to \gamma\psi(4040)) = 4 \cdot 10^{-9}\,,\\
&BR(Z \to \gamma\psi(4160)) = 3 \cdot 10^{-9}\,, \quad BR(Z \to \gamma\psi(4415)) = 1.44 \cdot 10^{-9}\,.
\end{aligned} \tag{48}$$

For the Υ family ($g_b = 1.52 \cdot 10^{-9}$ GeV^{-6})

$$\min \sum_\Upsilon BR(Z \to \gamma\Upsilon) = 7.23 \cdot 10^{-7} \tag{49}$$

and

$$\begin{aligned}
&BR(Z \to \gamma\Upsilon(1S)) = 4.27 \cdot 10^{-7}\,, \quad BR(Z \to \gamma\Upsilon(10023)) = 1.31 \cdot 10^{-7}\,,\\
&BR(Z \to \gamma\Upsilon(10355)) = 7.5 \cdot 10^{-8}\,, \quad BR(Z \to \gamma\Upsilon(10579)) = 3.9 \cdot 10^{-8}\,,\\
&BR(Z \to \gamma\Upsilon(10860)) = 3.7 \cdot 10^{-8}\,, \quad BR(Z \to \gamma\Upsilon(11020)) = 1.4 \cdot 10^{-8}.
\end{aligned} \tag{50}$$

The branching ratios for the production of the ground states ($BR(Z \to \gamma J/\psi(1S))$ and $BR(Z \to \gamma\Upsilon(1S))$ in Eqs. (48) and (50)) more or less agree with the quark model predictions, $\sim 3.4 \cdot (10^{-8} - 10^{-7})$, [7].

It follows from Eq. (44) that

$$T_q(Res) \equiv \sum_V \frac{1}{f_V} T_V^q = \frac{d_q}{g_q} D_q(Res) = \frac{d_q}{g_q} D_q\,, \quad \text{where} \quad d_q = \sum_V \frac{1}{f_V^2 m_V^4}\,. \tag{51}$$

For the ψ family ($d_c = 1.12 \cdot 10^{-4}$ GeV^{-4})

$$T_c(Res) \equiv \sum_V \frac{1}{f_V} T_V^c = 10.19 D_c(Res) \text{ GeV}^2 = 10.19 D_c \text{ GeV}^2 = 0.19 Re(T_c)\,. \tag{52}$$

For the Υ family ($d_b = 1.47 \cdot 10^{-7}$ GeV^{-4})

$$T_b(Res) \equiv \sum_V \frac{1}{f_V} T_V^b = 96.71 D_b(Res) \text{ GeV}^2 = 96.71 D_b \text{ GeV}^2 = 0.29 Re(T_b)\,. \tag{53}$$

Eqs. (41), (42) and (52), (53) specify explicitly that the main body of T_c and T_b is saturated with the continuous spectrum, see Eq. (16).[5] In addition, Eqs. (41), (42) and (52), (53) corroborate the comment in the footnote 3.

2.3 Sum rules for the amplitude and its derivative

The simultaneous saturation of the amplitude and its derivative with the ground state is provided if only

$$Re(T_q)/D_q = m_V^2\,, \tag{54}$$

but in our case

$$\begin{aligned} &Re(T_c)/D_c = 53.688\ \mathrm{GeV}^2 \neq m^2_{J/\psi(1S)} = 9.59\ \mathrm{GeV}^2 \quad \text{and} \\ &Re(T_b)/D_b = 334\ \mathrm{GeV}^2 \neq m^2_{\Upsilon(1S)} = 89.49\ \mathrm{GeV}^2\,. \end{aligned} \tag{55}$$

As for the simultaneous saturation of the amplitude and its derivative with the resonance family, it's quite another matter.

Considering the resonance contributions in the sum rules for the amplitude, $T_q(Res)$, and its derivative, $D_q(Res)$, as the two constraints and using Eq. (18) we find

$$\begin{aligned} &\min \sum_V \Gamma(Z \to \gamma V) = \\ &= \frac{1}{24\pi} M^3 \cdot \frac{g_q T_q(Res^2)^2 + a_q D_q(Res)^2 - 2 d_q T_q(Res) D_q(Res)}{a_q g_q - d_q^2}\,. \end{aligned} \tag{56}$$

Eq. (56) takes place when

$$T_V^q = \frac{1}{f_V m_V^2} \cdot \frac{(g_q - d_q/m_V^2) T_q(Res) - (d_q - a_q/m_V^2) D_q(Res)}{a_q g_q - d_q^2}. \tag{57}$$

It is easy to verify that Eq. (57) is self-consistent for any $T_q(Res)$, $D_q(Res)$, and m_V^2.

$$\sum_V \frac{1}{f_V} T_V^q = T_q(Res)\,, \quad \sum_V \frac{1}{f_V m_V^2} T_V^q = D_q(Res)\,. \tag{58}$$

The minimum of Eq. (56) takes place when

$$T_q(Res) = \frac{d_q}{g_q} D_q(Res). \tag{59}$$

Setting $D_q(Res) = D_q$, we revert to the previous subsection, to the saturation of the amplitude derivative with the resonance family.

Let us consider the deviation from Eq. (59) [6]

$$T_q(Res) = \frac{d_q}{g_q} D_q(Res) \cdot (1 + x) = \frac{d_q}{g_q} D_q \cdot (1 + x)\,, \tag{60}$$

then

$$\min \sum_V \Gamma(Z \to \gamma V) = \frac{1}{24\pi} M^3 D_q^2 g_q^{-1} \left(1 + x^2 \cdot \frac{d_q^2}{\Delta_q} \right)\,, \tag{61}$$

[5] This point was considered in Refs. [1, 2] only partly.

[6] The deviations from the lower bounds of $\sum_\psi BR(Z \to \gamma\psi)$ and $\sum_\Upsilon BR(Z \to \gamma\Upsilon)$ were not studied in Refs. [1, 2] in the regular way.

where $\Delta_q = a_q g_q - d_q^2$,

$$T_V^q = \frac{1}{g_q f_V m_V^4} D_q \left[1 + x \cdot \frac{d_q \left(g_q m_V^2 - d_q \right)}{\Delta_q} \right] , \tag{62}$$

and

$$\Gamma(Z \to \gamma V) =$$

$$= \frac{1}{24\pi} M^3 D_q^2 \left(f_V m_V^3 g_q \right)^{-2} \left[1 + 2x \cdot \frac{d_q \left(g_q m_V^2 - d_q \right)}{\Delta_q} + x^2 \cdot \frac{d_q^2 \left(g_q m_V^2 - d_q \right)^2}{\Delta_q^2} \right] . \tag{63}$$

It is easy to verify that the term, proportional x in Eq. (63), vanishes in

$$\sum_V \Gamma(Z \to \gamma V) .$$

For the ψ family ($\Delta_c = 4.45 \cdot 10^{-10}$ GeV^{-8})

$$\min \sum_\psi BR(Z \to \gamma \psi) = 1.95 \cdot 10^{-7} \cdot (1 + x^2 \cdot 28.17) \tag{64}$$

and

$$\begin{aligned}
&BR(Z \to \gamma J/\psi(1S)) = 1.64 \cdot 10^{-7} \cdot (1 - x \cdot 4.29 + x^2 \cdot 4.60) , \\
&BR(Z \to \gamma \psi(3686)) = 2.056 \cdot 10^{-8} \cdot (1 + x \cdot 17.392 + x^2 \cdot 75.62) , \\
&BR(Z \to \gamma \psi(3770)) = 2 \cdot 10^{-9} \cdot (1 + x \cdot 20.79 + x^2 \cdot 108.07) , \\
&BR(Z \to \gamma \psi(4040)) = 4 \cdot 10^{-9} \cdot (1 + x \cdot 32.236 + x^2 \cdot 259.79) , \\
&BR(Z \to \gamma \psi(4160)) = 3 \cdot 10^{-9} \cdot (1 + x \cdot 37.58 + x^2 \cdot 353) , \\
&BR(Z \to \gamma \psi(4415)) = 1.44 \cdot 10^{-9} \cdot (1 + x \cdot 49.44 + x^2 \cdot 611.19) .
\end{aligned} \tag{65}$$

For the Υ family ($\Delta_b = 2.14 \cdot 10^{-16}$ GeV^{-8})

$$\min \sum_\Upsilon BR(Z \to \gamma \Upsilon) = 7.23 \cdot 10^{-7} \cdot (1 + x^2 \cdot 100.46) \tag{66}$$

and

$$\begin{aligned}
&BR(Z \to \gamma \Upsilon(1S)) = 4.27 \cdot 10^{-7} \cdot (1 - x \cdot 15.04 + x^2 \cdot 56.57) , \\
&BR(Z \to \gamma \Upsilon(10023)) = 1.31 \cdot 10^{-7} \cdot (1 + x \cdot 7.74 + x^2 \cdot 14.97) , \\
&BR(Z \to \gamma \Upsilon(10355)) = 7.5 \cdot 10^{-8} \cdot (1 + x \cdot 21.79 + x^2 \cdot 118.71) , \\
&BR(Z \to \gamma \Upsilon(10579)) = 3.9 \cdot 10^{-8} \cdot (1 + x \cdot 31.53 + x^2 \cdot 248.54) , \\
&BR(Z \to \gamma \Upsilon(10860)) = 3.7 \cdot 10^{-8} \cdot (1 + x \cdot 44.04 + x^2 \cdot 484.88) , \\
&BR(Z \to \gamma \Upsilon(11020)) = 1.4 \cdot 10^{-8} \cdot (1 + x \cdot 51.31 + x^2 \cdot 658.27).
\end{aligned} \tag{67}$$

When the resonances saturate T_q, $x = 4.26$ for the ψ family and $x = 2.45$ for the Υ one. As this takes place,

$$\begin{aligned}
&\sum_\psi BR(Z \to \gamma \psi) = 10^{-4} , \\
&BR(Z \to \gamma J/\psi(1S)) = 1.1 \cdot 10^{-5} , \quad BR(Z \to \gamma \psi(3686)) = 3 \cdot 10^{-5} , \\
&BR(Z \to \gamma \psi(3770)) = 4 \cdot 10^{-6} , \quad BR(Z \to \gamma \psi(4040)) = 1.9 \cdot 10^{-5} , \\
&BR(Z \to \gamma \psi(4160)) = 2 \cdot 10^{-5} , \quad BR(Z \to \gamma \psi(4415)) = 1.6 \cdot 10^{-5}
\end{aligned} \tag{68}$$

and

$$\sum_{\Upsilon} BR(Z \to \gamma\Upsilon) = 4.36 \cdot 10^{-4}\,,$$
$$BR(Z \to \gamma\Upsilon(1S)) = 1.28 \cdot 10^{-4}\,, \quad BR(Z \to \gamma\Upsilon(10023)) = 1.5 \cdot 10^{-5}\,,$$
$$BR(Z \to \gamma\Upsilon(10355)) = 5.9 \cdot 10^{-5}\,, \quad BR(Z \to \gamma\Upsilon(10579)) = 6.3 \cdot 10^{-5}\,,$$
$$BR(Z \to \gamma\Upsilon(10860)) = 1.12 \cdot 10^{-4}\,, \quad BR(Z \to \gamma\Upsilon(11020)) = 5.9 \cdot 10^{-5}\,. \tag{69}$$

As noted in Section 2.1, there are no theoretical grounds for saturating T_q with the resonances only. Furthermore the prediction $BR(Z \to \gamma\Upsilon(1S)) = 1.28 \cdot 10^{-4}$, see Eq. (69), contradicts the experiment value $BR(Z \to \Upsilon(1S)X) < 4.4 \cdot 10^{-5}$ on CL=95 % [11] thus bearing the theoretical reason.

When $x = -1$, the resonances do not contribute to T_q at all. As this takes place,

$$\sum_{\psi} BR(Z \to \gamma\psi) = 5.69 \cdot 10^{-6}\,,$$
$$BR(Z \to \gamma J/\psi(1S)) = 1.62 \cdot 10^{-6}\,, \quad BR(Z \to \gamma\psi(3686)) = 1.22 \cdot 10^{-6}\,,$$
$$BR(Z \to \gamma\psi(3770)) = 1.7 \cdot 10^{-7}\,, \quad BR(Z \to \gamma\psi(4040)) = 9 \cdot 10^{-7}\,,$$
$$BR(Z \to \gamma\psi(4160)) = 9.8 \cdot 10^{-7}\,, \quad BR(Z \to \gamma\psi(4415)) = 8 \cdot 10^{-7} \tag{70}$$

and

$$\sum_{\Upsilon} BR(Z \to \gamma\Upsilon) = 7.34 \cdot 10^{-5}\,,$$
$$BR(Z \to \gamma\Upsilon(1S)) = 3.08 \cdot 10^{-5}\,, \quad BR(Z \to \gamma\Upsilon(10023)) = 1.3 \cdot 10^{-6}\,,$$
$$BR(Z \to \gamma\Upsilon(10355)) = 7.4 \cdot 10^{-6}\,, \quad BR(Z \to \gamma\Upsilon(10579)) = 8.7 \cdot 10^{-6}\,,$$
$$BR(Z \to \gamma\Upsilon(10860)) = 1.65 \cdot 10^{-5}\,, \quad BR(Z \to \gamma\Upsilon(11020)) = 8.7 \cdot 10^{-6}\,. \tag{71}$$

Zeros in $BR(Z \to \gamma J/\psi(1S)$ at $x = 0.466$ and in $BR(Z \to \gamma\Upsilon(1S)$ at $x = 0.133$ are striking, see Eqs. (65) and (67). In this case

$$\sum_{\psi \neq J/\psi} BR(Z \to \gamma\psi) = 1.39 \cdot 10^{-6}\,, \qquad \sum_{\Upsilon \neq \Upsilon(1S)} BR(Z \to \gamma\Upsilon) = 2.01 \cdot 10^{-6}\,, \tag{72}$$

and

$$T_c(Res) = 0.28 T_c\,, \quad T_b(Res) = 0.33 T_b\,. \tag{73}$$

It follows from Eq. (73) that the continues spectra dominate the saturation of the T_c and T_b amplitudes, but zeros, see Eq. (62),

$$T^c_{J/\psi(1S)}|_{x=0.466} = 0 \text{ and } T^b_{\Upsilon(1S)}|_{x=0.133} = 0 \tag{74}$$

require a rather bizarre dynamics, as I believe.

3 Summary

As is evident from the foregoing, the lower bounds of $\sum_\psi BR(Z \to \gamma\psi) = 1.95 \cdot 10^{-7}$ and $\sum_\Upsilon BR(Z \to \gamma\Upsilon) = 7.23 \cdot 10^{-7}$ are reached when the derivatives of the $Z \to c\bar{c} \to \gamma\gamma^*$ and $Z \to b\bar{b} \to \gamma\gamma^*$ amplitudes are saturated with the resonances in the γ^* low energy region. As this takes place, the branching ratios for the production of the ground states, $BR(Z \to \gamma J/\psi(1S)) = 1.64 \cdot 10^{-7}$ and $BR(Z \to \gamma\Upsilon(1S)) = 4.27 \cdot 10^{-7}$, more or less agree with the quark model predictions.

These lower bounds are "the equilibrium points " of Eqs. (64) and (66) at $x = 0$. The minima are rather sharp, especially in the Υ family case. Thus for the 20 percentage reduction of the Υ family contribution to the triangle diagram amplitude, $x = -0.2$, $\Sigma_\Upsilon BR(Z \to \gamma\Upsilon) = 3.6\times10^{-6}$ and $BR(Z \to \gamma\Upsilon(1S) = 2.7\times10^{-6}$ are resulted from Eqs. (66) and (67), that could be probably measured at LHC. As to the ψ family, only the 70 percentage reduction of its contribution to the triangle diagram amplitude, $x = -0.7$, leads to similar results: according to Eqs. (64) and (65) $\Sigma_\psi BR(Z \to \gamma\psi) = 2.9 \times 10^{-6}$ and $BR(Z \to \gamma J/\psi(1S)) = 10^{-6}$, that also could be probably measured at LHC.

The angular distributions expected in the center-of-mass system of the $q\bar{q} \to Z \to \gamma V$ and $e^+e^- \to Z \to \gamma V$ reactions follows from Eqs. (9) and (11).

$$W(\theta) = \frac{3}{8}\cdot\frac{1+\cos^2\theta + (2m_q^2/M^2)\sin^2\theta}{1+m_V^2/M^2} \approx \frac{3}{8}(1+\cos^2\theta), \tag{75}$$

where θ is the angle between the γ quantum momentum and the beam axis. For more details, see the Appendix.

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Appendix

Angle distributions

The angular distributions expected in the $Z \to \gamma V$ decays in the rest frame of the Z boson follow from Eqs.(9) and (11).

If not to be interested in the photon and V meson polarizations from Eqs. (9) and (11) it is received, neglecting members $\sim (m_V/M_Z)^2$,

$$W\Big(\mathbf{e}(Z)\,,\mathbf{n}\Big) = (3/4)\bigg(\Big(\mathbf{e}(Z)^*\cdot\mathbf{e}(Z)\Big) - \Big(\mathbf{n}\cdot\mathbf{e}(Z)^*\Big)\Big(\mathbf{n}\cdot\mathbf{e}(Z)\Big)\bigg) \tag{76}$$

or

$$W(S_z = 1\,,\ \theta) = W(S_Z = -1\,,\ \theta) = (3/8)(1+cos^2\theta)\,, \tag{77}$$

$$W(S_z = 0\,,\ \theta) = (3/4)\sin^2\theta\,, \tag{78}$$

where S_z is the z component of the Z boson spin in its rest frame, θ is the angle between the γ quantum momentum and the z axis in the Z boson rest frame.

If to be interested in polarization of the photon only from Eqs. (9) and (11) it is received, neglecting members $\sim (m_V/M_Z)^2$,

$$W\Big(\mathbf{e}(Z)\,,\mathbf{n}\,,\mathbf{e}(\gamma)\Big) = (3/4)\bigg(\mathbf{n}\cdot\Big[\mathbf{e}(\gamma)\times\mathbf{e}(Z)\Big]\bigg)\bigg(\mathbf{n}\cdot\Big[\mathbf{e}(\gamma)\times\mathbf{e}(Z)\Big]\bigg)^* \tag{79}$$

or

$$W(S_z = 1\,,\ S_\gamma = +1\,,\ \theta) = W(S_z = -1\,,\ S_\gamma = -1\,,\ \theta) = (3/16)(1+\cos\theta)^2\,, \tag{80}$$

$$W(S_z = 1\,,\ S_\gamma = -1\,,\ \theta) = W(S_z = -1\,,\ S_\gamma = +1\,,\ \theta) = (3/16)(1-\cos\theta)^2\,, \tag{81}$$

$$W(S_z = 0\,,\ S_\gamma = +1\,,\ \theta) = W(S_z = 0\,,\ S_\gamma = -1\,,\ \theta) = (3/8)\sin^2\theta\,, \tag{82}$$

where S_γ is the photon helicity.

Note that Z boson with $S_z = 0$ is not produced if the z axis is the axis of the e^+e^- or $q\bar{q}$ beams in their center-of-mass system. This results in Eqs. (75), (83), (84), and (85).

In that event, the angular distributions expected in the center-of-mass system of the $e^+e^- \to Z \to \gamma V$ and $q\bar{q} \to Z \to \gamma V$ reactions, $W^{e^+e^-}_{S_\gamma}(\theta)$ and $W^{q\bar{q}}_{S_\gamma}(\theta)$ respectively, are

$$W^{e^+e^-}_{S_\gamma=\pm1}(\theta) = \frac{3}{16N_e}\left[(1/2-\xi)^2(1\mp\cos\theta)^2 + \xi^2(1\pm\cos\theta)^2\right] , \tag{83}$$

where $N_e = (1/2-\xi)^2+\xi^2$, $\xi = \sin^2\Theta_W = 0.23$ [11], the z axis is put in the electron momentum direction,

$$W^{u\bar{u}}_{S_\gamma=\pm1}(\theta) = \frac{3}{16N_u}\left[(1/2-e_u\,\xi)^2(1\mp\cos\theta)^2 + e_u^2\,\xi^2(1\pm\cos\theta)^2\right] , \tag{84}$$

where $N_u = (1/2-e_u\,\xi)^2 + e_u^2\,\xi^2$, $e_u = 2/3$; the z axis is put in the u quark momentum direction, and

$$W^{d\bar{d}}_{S_\gamma=\pm1}(\theta) = \frac{3}{16N_d}\left[(1/2-e_d\,\xi)^2(1\mp\cos\theta)^2 + e_d^2\,\xi^2(1\pm\cos\theta)^2\right] , \tag{85}$$

where $N_d = (1/2-e_d\,\xi)^2 + e_d^2\,\xi^2$, $e_d = -\,1/3$; the z axis is put in the d quark momentum direction.

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