Nonlocal Cosmological Models and Exact Solutions

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Abstract

A general class of gravitational models driven by a nonlocal scalar field with a linear or quadratic potential is considered. The Ostrogradski representation for nonlocal gravitational models and the way of its localization are proposed. We study the action with an arbitrary analytic function $\mathcal{F}(\Box_g)$, which has both simple and double roots. The localization allows to find exact solutions of nonlocal Einstein equations.

1 Introduction

The phenomenological relation between the pressure (Lagrangian density) p and the energy density ϱ: $p = w\varrho$ is typically used to describe various types of cosmic fluids. The function w is called the state parameter.

Contemporary cosmological observational data [1] strongly support that the present Universe exhibits an accelerated expansion providing thereby an evidence for a dominating dark energy component with the state parameter

$$w_{\mathrm{DE}} = -1.0 \pm 0.2. \tag{1}$$

The present cosmological observations do not exclude an evolving parameter w_{DE}. Moreover, the recent analysis of the observation data indicates that the varying in time dark energy with the state parameter w_{DE}, which crosses the cosmological constant barrier, gives a better fit than a cosmological constant [2] (see also [3] and references therein).

One of most important problem in the cosmological models is the instability problem. The cosmological models with $w_{DE} < -1$ violate the null energy condition, which is generally related to the phantom fields appearing. As it has been shown in papers [4, 5], the adding of high order derivative terms leads to the presence of phantoms. From a purely classical perspective, they render the Hamiltonian unbounded below. In the Ostrogradski formulation, the kinetic energy of the system is seen to be non-positive definite. It follows that the theory is disastrously unstable: time evolution will generically drive certain sectors of the system to become arbitrarily excited. The standard quantization of these models leads to instability, which is physically inadmissible. In [6] the theory with $w_{DE} < -1$ has been interpreted as an approximation in the framework of the fundamental theory. Because the fundamental theory must be stable and must admit quantization, this instability can be considered an artefact of the approximation.

One of the first attempts to apply string theory to cosmology [7] was related to the problem of the cosmological singularity [8]. A possible way to avoid cosmological singularities consists of dealing with nonsingular bouncing cosmological solutions. In these scenarios the Universe contracts before the bounce [9]. Such models have strong coupling and higher-order string corrections are inevitable. It is important to construct nonsingular bouncing cosmological solutions in order to make a concrete prediction of bouncing cosmology.

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Recently a wide class of nonlocal cosmological models based on the string field theory[1] [10] (SFT) and the p-adic string theory [11] (that is considered as a toy model for the SFT) emerges and attracts a lot of attention [12]–[33]. Due to the presence of phantom excitations nonlocal models are of interest for the present cosmology. To obtain a stable model with the NEC violation (the state parameter $w_{\rm DE} < -1$) one should construct this model as an effective model, connected with the fundamental theory, which is stable and admits quantization. With the lack of quantum gravity, we can just trust string theory or deal with an effective theory admitting the UV completion.

The purpose of this paper is to study the string field theory inspired gravitational models with a nonlocal scalar field. We consider a general form of nonlocal action for the scalar field with a quadratic or linear potential, keeping the main ingredient, the analytic function $\mathcal{F}(\Box_g)$, which in fact produces the nonlocality, almost unrestricted.

2 Model setup

In this paper we consider nonlocality associated with a scalar field, dynamics of which governed by a Lagrangian, containing infinite order derivative operator. The SFT inspired nonlocal gravitation models [12] are introduced as a sum of the SFT action of tachyon field and gravity part of the action plus the cosmological constant. One cannot deduce this form of the action from SFT, he can just assume the minimal form of gravity interaction of all string modes[2].

We consider a general class of gravity models, which include minimally coupling with a nonlocal scalar field and are described by the following action:

$$S=\int d^4x\sqrt{-g}\left(\frac{R}{16\pi G_N}+\frac{1}{(\alpha')^2g_4}\left(\frac{1}{2}\phi\,\mathcal{F}\left(\alpha'\Box_g\right)\phi-V(\phi)\right)-\Lambda\right), \tag{2}$$

We use the signature $(-,+,+,+)$, $g_{\mu\nu}$ is the metric tensor. G_N is the Newtonian constant: $8\pi G_N=1/M_P^2$, M_P is the Planck mass. α' is the string length squared (the string tension). The scalar field ϕ is dimensionless, a positive number g_4 is a dimensionless four dimensional effective coupling constant related with the ten dimensional open string coupling constant g_o and the compactification scale. The potential $V(\phi)$ is a quadratic polynomial:

$$V(\phi)=C_2\phi^2+C_1\phi+C_0, \tag{3}$$

where C_2, C_1 and C_0 are arbitrary real constants. The nonlocal cosmological models with quadratic potentials have been studied in [16, 17, 23, 24, 26, 28, 29, 30, 31].

The d'Alembert operator $\Box_g$ is applied to scalar functions and can be written as follows

$$\Box_g=\frac{1}{\sqrt{-g}}\partial_\mu\sqrt{-g}g^{\mu\nu}\partial_\nu\,. \tag{4}$$

One can introduce dimensionless coordinates $\bar{x}_\mu=M_sx_\mu$ and rewrite action (2) as follows

$$S=\int d^4\bar{x}\sqrt{-g}\alpha'\left(\frac{\bar{R}}{16\pi G_N}+\frac{1}{\alpha' g_4}\left(\frac{1}{2}\phi\,\mathcal{F}\left(\bar{\Box}_g\right)\phi-V(\phi)\right)-\Lambda'\right), \tag{5}$$

where $\bar{R}$ and $\bar{\Box}_g$ are defined in terms of dimensionless coordinates $\bar{x}$, $\Lambda'\equiv\alpha'\Lambda$.

The function $\mathcal{F}$ is assumed to be an analytic function, which can be represented by the convergent series expansion:

$$\mathcal{F}(\Box_g)=\sum_{n=0}^{\infty}f_n\Box_g^n. \tag{6}$$

[1]One of the main motivations to construct string field theory (SFT) — an off-shell formulation of a string theory — is a possibility to study in this framework non-perturbative phenomena in string theory.

[2]Note the cosmological model [32], in which a nonlocal operator acts both on gravity and on a scalar field.

Hereafter we omit bars for simplicity. From action (5) we obtain the following equations:

$$G_{\mu\nu} = 8\pi G_N \left(T_{\mu\nu} - \Lambda' g_{\mu\nu}\right), \tag{7}$$

$$\mathcal{F}(\Box_g)\phi = \frac{dV}{d\phi}, \tag{8}$$

where $G_{\mu\nu}$ is the Einstein tensor. Modifying values of f_0 and Λ', we transform action (5) with the potential $V(\phi)$ to an action with the linear potential $C_1\phi$. In other words, we can put $C_2 = 0$ and $C_0 = 0$ without loss of generality.

The energy–momentum (stress) tensor $T_{\mu\nu}$ is:

$$T_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}} = \frac{1}{\alpha' g_4}\Big(E_{\mu\nu} + E_{\nu\mu} - g_{\mu\nu}\left(g^{\rho\sigma}E_{\rho\sigma} + W\right)\Big), \tag{9}$$

$$E_{\mu\nu} \equiv \frac{1}{2}\sum_{n=1}^{\infty} f_n \sum_{l=0}^{n-1} \partial_\mu \Box_g^l \phi \partial_\nu \Box_g^{n-1-l}\phi, \qquad W \equiv \frac{1}{2}\sum_{n=2}^{\infty} f_n \sum_{l=1}^{n-1} \Box_g^l \phi \Box_g^{n-l}\phi - \frac{f_0}{2}\phi^2 + C_1\phi.$$

3 The Ostrogradski representation

Our results are generalizations of the Ostrogradski representation. Let us remind the classical results in the Minkowski space [34, 35]. We consider such

$$\mathcal{F}(\Box) = \mathcal{F}_1(\Box) \equiv \prod_{j=1}^{N}\left(1 + \frac{\Box}{\omega_j^2}\right) \tag{10}$$

that all roots of the polynomial $\mathcal{F}_1(\Box)$, which are equal to $-\omega_j^2$, are simple. The d'Alembert operator in the Minkowski space is denoted as $\Box$.

For the local Lagrangian $\mathcal{L}_F$ the Ostrogradski representation is as follows

$$\mathcal{L}_F \equiv \phi\mathcal{F}_1(\Box)\phi \cong L_l = \sum_{j=1}^{N} c_j\phi_j(\Box + \omega_j^2)\phi_j, \tag{11}$$

where the sign $'\cong'$ means equality up to a full derivative. We define

$$\phi_j = \prod_{k=1,k\neq j}^{N}\left(1 + \frac{1}{\omega_k^2}\Box\right)\phi, \qquad \Rightarrow \qquad \left(\Box + \omega_j^2\right)\phi_j = 0. \tag{12}$$

Substituting ϕ_j in L_l, one gets that equality (11) is equivalent to [35]:

$$L_l = \mathcal{L}_F \quad \Leftrightarrow \quad \sum_{k=1}^{N}\frac{c_k\omega_k^4}{\omega_k^2 + \Box} = \frac{1}{\mathcal{F}_1(\Box)} \quad \Rightarrow \quad c_k = \frac{\mathcal{F}_1'(-\omega_k^2)}{\omega_k^4}, \qquad \mathcal{F}_1'(-\omega_k^2) \equiv \frac{d\mathcal{F}_1(J)}{dJ}\Big|_{J=-\omega_k^2}.$$

Let $\mathcal{F}_1(\Box)$ has two real simple roots, it is evident that $\mathcal{F}_1' > 0$ in one and only one root. Therefore, we get a model with one phantom scalar field and one standard scalar field.

4 The initial value problem in the case of nonlocal models

If $\mathcal{F}(\Box)$ is a N degree polynomial, for example $\mathcal{F}(\Box) = \mathcal{F}_1(\Box)$, then the general solution of (8) depends on $2N$ independent parameters, therefore, to uniquely specify a solution it is enough to set values of ϕ and its $(2N-1)$ derivatives at the initial moment t_0.

If $\mathcal{F}(\Box)$ is not a polynomial, but an analytic function with an infinite Taylor series, then it is nontrivial to state the initial value problem [22]. When does equation (8) admit a well-defined initial value problem – even formally, that is ignoring issues of convergence – and how many initial data are required to determine a solution? Such questions are fundamental to any physical application.

At the first sight one can determine the function ϕ and all its derivatives at some moment t_0 by arbitrary way, but it is not correct. Indeed, if we assume that $\phi(t)$ is an analytic function, then we unique define this function in the neighbourhood of some moment of time t_0, setting the values of ϕ and all its derivatives at $t = t_0$. So, we come to conclusion that an arbitrary analytic function is a solution of (8) with the suitable initial data. We get a contradiction.

Note that from the equation (8) it follows not only the condition

$$\mathcal{F}(\Box)\phi(t)|_{t=t_0} = V'(\phi)|_{t=t_0}, \tag{13}$$

but also an infinite number of conditions

$$(\partial_t^n \mathcal{F}(\Box)\phi(t))|_{t=t_0} = \partial_t^n V'(\phi)|_{t=t_0}, \tag{14}$$

which should be satisfied. For cosmological models with a quadratic potential it is possible to localize the Einstein equations and to specify correct initial data, using the technique, proposed in [23]. In the next section we demonstrate the algorithm of localization a nonlocal gravitational model with an arbitrary quadratic or linear potential. By linearizing a nonlinear model about a particular field value, one is able to specify initial data for nonlinear models, which he then evolves into the full nonlinear regime using the diffusion-like equation [24].

5 Localization of Nonlocal Gravitational Models

Our goal is to generalize the Ostrogradski representation on gravitational models with action (5). We assume that an analytic function $\mathcal{F}(\Box_g)$ has simple or double roots.

Let us start with the case $C_1 = 0$ and consider an analytical function $\mathcal{F}(J)$, which has simple roots J_i and double roots $\tilde{J}_k$. A particular solution to $\mathcal{F}(\Box_g)\phi = 0$ is the function

$$\phi_0 = \sum_{i=1}^{N_1} \phi_i + \sum_{k=1}^{N_2} \tilde{\phi}_k, \tag{15}$$

where

$$(\Box_g - J_i)\phi_i = 0, \qquad (\Box_g - \tilde{J}_k)^2 \tilde{\phi}_k = 0. \tag{16}$$

Without loss of generality we assume that for any i_1 and $i_2 \neq i_1$ conditions $J_{i_1} \neq J_{i_2}$ and $\tilde{J}_{i_1} \neq \tilde{J}_{i_2}$ are satisfied. The fourth order differential equation $(\Box_g - \tilde{J}_k)(\Box_g - \tilde{J}_k)\tilde{\phi}_k = 0$ is equivalent to the following system of the second order equations:

$$(\Box_g - \tilde{J}_k)\tilde{\phi}_k = \varphi_k, \qquad (\Box_g - \tilde{J}_k)\varphi_k = 0. \tag{17}$$

The energy–momentum tensor, which corresponds to ϕ_0, has the following form [28]:

$$T_{\mu\nu}(\phi_0) = T_{\mu\nu}\left(\sum_{i=1}^{N_1} \phi_i + \sum_{k=1}^{N_2} \tilde{\phi}_k\right) = \sum_{i=1}^{N_1} T_{\mu\nu}(\phi_i) + \sum_{k=1}^{N_2} T_{\mu\nu}(\tilde{\phi}_k), \tag{18}$$

where all $T_{\mu\nu}$ are given by (9) and

$$E_{\mu\nu}(\phi_i) = \frac{\mathcal{F}'(J_i)}{2}\partial_\mu\phi_i\partial_\nu\phi_i, \qquad W(\phi_i) = \frac{J_i\mathcal{F}'(J_i)}{2}\phi_i^2, \tag{19}$$

$$E_{\mu\nu}(\tilde{\phi}_k)=\frac{\mathcal{F}''(\tilde{J}_k)}{4}\left(\partial_\mu\tilde{\phi}_k\partial_\nu\varphi_k+\partial_\nu\tilde{\phi}_k\partial_\mu\varphi_k\right)+\frac{\mathcal{F}'''(\tilde{J}_k)}{12}\partial_\mu\varphi_k\partial_\nu\varphi_k, \tag{20}$$

$$W(\tilde{\phi}_k)=\frac{\tilde{J}_k\mathcal{F}''(\tilde{J}_k)}{2}\tilde{\phi}_k\varphi_k+\left(\frac{\tilde{J}_k\mathcal{F}'''(\tilde{J}_k)}{12}+\frac{\mathcal{F}''(\tilde{J}_k)}{4}\right)\varphi_k^2, \tag{21}$$

where a prime denotes a derivative with respect to J: $\mathcal{F}'\equiv\frac{d\mathcal{F}}{dJ}$, $\mathcal{F}''\equiv\frac{d^2\mathcal{F}}{dJ^2}$ and $\mathcal{F}'''\equiv\frac{d^3\mathcal{F}}{dJ^3}$.

Let us consider the following local action

$$S_{loc}=\int d^4x\alpha'\sqrt{-g}\left(\frac{R}{16\pi G_N}-\Lambda'\right)+\sum_{i=1}^{N_1}S_i+\sum_{k=1}^{N_2}\tilde{S}_k, \tag{22}$$

where

$$S_i=-\frac{1}{g_4}\int d^4x\sqrt{-g}\frac{\mathcal{F}'(J_i)}{2}\left(g^{\mu\nu}\partial_\mu\phi_i\partial_\nu\phi_i+J_i\phi_i^2\right), \tag{23}$$

$$\begin{aligned}\tilde{S}_k=&-\frac{1}{g_4}\int d^4x\sqrt{-g}\left(g^{\mu\nu}\left(\frac{\mathcal{F}''(\tilde{J}_k)}{4}\left(\partial_\mu\tilde{\phi}_k\partial_\nu\varphi_k+\partial_\nu\tilde{\phi}_k\partial_\mu\varphi_k\right)+\right.\right.\\&\left.\left.+\frac{\mathcal{F}'''(\tilde{J}_k)}{12}\partial_\mu\varphi_k\partial_\nu\varphi_k\right)+\frac{\tilde{J}_k\mathcal{F}''(\tilde{J}_k)}{2}\tilde{\phi}_k\varphi_k+\left(\frac{\tilde{J}_k\mathcal{F}'''(\tilde{J}_k)}{12}+\frac{\mathcal{F}''(\tilde{J}_k)}{4}\right)\varphi_k^2\right).\end{aligned} \tag{24}$$

We can see that solutions of the Einstein equations and equations in ϕ_k, $\tilde{\phi}_k$ and φ_k, obtained from this action, solves the initial system of nonlocal equations (7). Thus, we obtained that special solutions to nonlocal equations can be found as solutions to system of local (differential) equations. The result has been obtained for an arbitrary metric. If $\mathcal{F}(J)$ has an infinity number of roots then one nonlocal model corresponds to infinity number of different local models. In this case the initial nonlocal action (5) generates infinity number of local actions (22).

Remark. We should prove that the way of localization is self-consistent. To construct local action (22) we assume that equations (16) are satisfied. Therefore, the method of localization is correct only if these equations can be obtained from the local action S_{loc}. The straightforward calculations show that

$$\frac{\delta S_{loc}}{\delta\phi_i}=0\quad\Leftrightarrow\quad\Box_g\phi_i=J_i\phi_i;\qquad\frac{\delta S_{loc}}{\delta\tilde{\phi}_k}=0\quad\Leftrightarrow\quad\Box_g\varphi_k=\tilde{J}_k\varphi_k. \tag{25}$$

Using (25) we obtain

$$\frac{\delta S_{loc}}{\delta\varphi_k}=0\qquad\Leftrightarrow\qquad\Box_g\tilde{\phi}_k=\tilde{J}_k\tilde{\phi}_k+\varphi_k. \tag{26}$$

So, the way of localization is self-consistent in the case of $\mathcal{F}(J)$ with simple and double roots [28]. The self-consistence of similar approach for $\mathcal{F}(J)$ with only simple roots has been proven in [23, 26].

In spite of the above-mention equations we obtain from S_{loc} the equations:

$$G_{\mu\nu}=8\pi G_N\left(T_{\mu\nu}(\phi_0)-\Lambda' g_{\mu\nu}\right), \tag{27}$$

where ϕ_0 is given by (15) and $T_{\mu\nu}(\phi_0)$ can be calculated by (18). So, we obtained such systems of differential equations that any solutions of these systems are particular solutions of the initial nonlocal equations (7).

Let us consider functions $\mathcal{F}(J)$ with two and only two simple roots. If $\mathcal{F}(J)$ has two real simple roots, then $\mathcal{F}'(J)>0$ at one root and $\mathcal{F}'(J)<0$ at another root, so we get a quintom model, in other words, local model with one standard scalar field and one phantom scalar field. In the case of two complex conjugated simple roots J_j and J_j^* one gets the following action:

$$S_c=-\frac{1}{2g_4}\int d^4x\sqrt{-g}\left(\mathcal{F}'(J_j)\left(g^{\mu\nu}\partial_\mu\phi_j\partial_\nu\phi_j+J_j\phi_j^2\right)+\mathcal{F}'^*(J_j)\left(g^{\mu\nu}\partial_\mu\phi_j^*\partial_\nu\phi_j^*+J_j^*\phi_i^{*2}\right)\right).$$

We introduce real fields ξ and η such that $\phi_j = \xi + i\eta$, $\phi_j^* = \xi - i\eta$, denote $d_r \equiv \Re e(\mathcal{F}'(J))$, $d_i \equiv \Im m(\mathcal{F}'(J))$, and obtain:

$$S_c = -\frac{1}{2g_4}\int d^4x\sqrt{-g}\Big(d_r g^{\mu\nu}\left(\partial_\mu\xi\partial_\nu\xi - \partial_\mu\eta\partial_\nu\eta\right) + d_i g^{\mu\nu}\left(\partial_\mu\xi\partial_\nu\eta - \partial_\mu\eta\partial_\nu\xi\right) + V_1\Big),$$

where V_1 is a potential term. In the case $d_i = 0$ we get a quintom model, in opposite case the kinetic term in S_c has a nondiagonal form. To diagonalize the kinetic term we make the following transformation:

$$\chi = \upsilon + \tilde{C}\sigma, \qquad \eta = -\tilde{C}\upsilon + \sigma, \qquad \text{where} \qquad \tilde{C} \equiv \frac{d_r + \sqrt{d_r^2 + d_i^2}}{d_i}, \tag{28}$$

and get a quintom model:

$$S_c = -\frac{1}{2g_4}\int d^4x\sqrt{-g}\left(\frac{2\left(d_r^2 + d_i^2\right)}{d_i^2}\left(d_r + \sqrt{d_r^2 + d_i^2}\right)\left(\partial_\mu\upsilon\partial_\nu\upsilon - \partial_\mu\sigma\partial_\nu\sigma\right) + V_1\right).$$

In the case of a real double root $\tilde{J}_k$ we express $\tilde{\phi}_k$ and φ_k in terms of new fields ξ_k and χ_k:

$$\tilde{\phi}_k = \frac{1}{2\mathcal{F}''(\tilde{J}_k)}\left(\left(\mathcal{F}''(\tilde{J}_k) - \frac{2}{3}\mathcal{F}'''(\tilde{J}_k)\right)\xi_k - \left(\mathcal{F}''(\tilde{J}_k) + \frac{2}{3}\mathcal{F}'''(\tilde{J}_k)\right)\chi_k\right), \tag{29}$$

$$\varphi_k = \xi_k + \chi_k, \tag{30}$$

we obtain the corresponding $\tilde{S}_k$ in the following form:

$$\begin{aligned}\tilde{S}_k =& -\frac{1}{2g_4}\int d^4x\sqrt{-g}\left(g^{\mu\nu}\frac{\mathcal{F}''(\tilde{J}_k)}{4}(\partial_\mu\xi_k\partial_\nu\xi_k - \partial_\nu\chi_k\partial_\mu\chi_k) + \right.\\ &+ \frac{\tilde{J}_k}{4}\left((\mathcal{F}''(\tilde{J}_k) - \frac{2}{3}\mathcal{F}'''(\tilde{J}_k))\xi_k - (\mathcal{F}''(\tilde{J}_k) + \frac{2}{3}\mathcal{F}'''(\tilde{J}_k))\chi_k\right)(\xi_k + \chi_k) + \\ &\left. + \left(\frac{\tilde{J}_k\mathcal{F}'''(\tilde{J}_k)}{12} + \frac{\mathcal{F}''(\tilde{J}_k)}{4}\right)(\xi_k + \chi_k)^2\right).\end{aligned}$$

It is easy to see that each $\tilde{S}_k$ includes one phantom scalar field and one standard scalar field. So, in the case of one double root we obtain a quintom model. In the Minkowski space appearance of phantom fields in models, when $\mathcal{F}(J)$ has a double root, has been obtained in [35]. So, we come to conclusion that both two simple roots and one double root of $\mathcal{F}(J)$ generate quintom models (see [3] for reviews of quintom models).

Let us consider the model with action (5) in the case $C_1 \neq 0$. For the string field theory inspired form of $\mathcal{F}(\Box)$ the case $f_0 \neq 0$ has been considered in [24]. In this case we work in a new scalar field $\chi = \phi - C_1/f_0$ and get the energy–momentum tensor (9) with

$$E_{\mu\nu} = \frac{1}{2}\sum_{n=1}^{\infty} f_n \sum_{l=0}^{n-1}\partial_\mu\Box_g^l\chi\partial_\nu\Box_g^{n-1-l}\chi, \quad W = \frac{1}{2}\sum_{n=1}^{\infty} f_n\sum_{l=1}^{n-1}\Box_g^l\chi\Box_g^{n-l}\chi - \frac{f_0}{2}\chi^2 + \frac{C_1^2}{2f_0}.$$

The constant $C_1^2/(2f_0)$ can be consider as a part of the cosmological constant. Thus, in terms of χ we obtain a model without a linear term and can conclude that at $f_0 \neq 0$ the adding of a linear term to the potential shifts the scalar field ϕ on the constant and the changes the value of cosmological constant.

Let us consider the case $f_0 = 0$. In this case $J = 0$ is a root of $\mathcal{F}(J)$. It is easy to show, that the function

$$\tilde{\chi} = \phi_0 + \psi, \tag{31}$$

where ϕ_0 and ψ are solutions of the following equations

$$\mathcal{F}(\Box)\phi_0 = 0, \qquad \Box^m \psi = \frac{C_1}{f_m}, \tag{32}$$

m is the order of the root $J = 0$, satisfies

$$\mathcal{F}(\Box)\tilde{\chi} = C_1. \tag{33}$$

The function ϕ_0 is given by (15), but the sum do not include ϕ_{i_0}, which corresponds to the root $J = 0$, because this function can not be separated from ψ. We consider the cases of $m = 1$ and $m = 2$. In the last case, when $J = 0$ is a double root, we denote the function ψ as $\tilde{\psi}$.

To localize the Einstein equations one should calculate the energy–momentum tensor for $\tilde{\chi}$. The straightforward calculations show [30], that

$$T_{\mu\nu}(\tilde{\chi}) = T_{\mu\nu}(\psi) + T_{\mu\nu}(\phi_0), \tag{34}$$

where

$$W(\psi) = C_1\psi + \frac{f_2 C_1^2}{2 f_1^2}, \qquad E_{\mu\nu}(\psi) = \frac{1}{2} f_1 \partial_\mu\psi\partial_\nu\psi. \tag{35}$$

The function ϕ_0 is given by (15) and satisfies equation (8) with $C_1 = 0$, therefore, we use W_0 instead of W to calculate $T_{\mu\nu}(\phi_0)$ and obtain equality (18).

In the case of the double root $J = 0$ equation

$$\Box^2\tilde{\psi} = \frac{C_1}{f_2}, \iff \left\{ \begin{array}{l} \Box\tilde{\psi} = \tau, \\ \Box\tau = \dfrac{C_1}{f_2}. \end{array} \right. \tag{36}$$

We obtain

$$T_{\mu\nu}(\tilde{\chi}) = T_{\mu\nu}(\tilde{\psi}) + T_{\mu\nu}(\phi_0), \tag{37}$$

$$E_{\mu\nu}(\tilde{\psi}) = \frac{1}{2}\left(f_2(\partial_\mu\tilde{\psi}\partial_\nu\tau + \partial_\nu\tilde{\psi}\partial_\mu\tau) + f_3\partial_\mu\tau\partial_\nu\tau\right), \qquad W(\tilde{\psi}) = \frac{f_2}{2}\tau^2 + C_1\tilde{\psi} + \frac{f_3 C_1}{f_2}\tau. \tag{38}$$

For an arbitrary quadratic potential $V(\phi) = C_2\phi^2 + C_1\phi + C_0$ there exists the following algorithm of localization:

- Change values of f_0 and Λ such that the potential takes the form $V(\phi) = C_1\phi$.
- Find roots of the function $\mathcal{F}(J)$ and calculate orders of them.
- Select an finite number of simple and double roots.
- Construct the corresponding local action. In the case $C_1 = 0$ one should use formula (22). In the case $C_1 \neq 0$ and $f_0 \neq 0$ one should use (22) with the replacement of the scalar field ϕ by χ and the corresponding modification of the cosmological constant. In the case $C_1 \neq 0$ and $f_0 = 0$ the local action is the sum of (22) and either

  $$S_\psi = -\frac{1}{2g_4}\int d^4x\sqrt{-g}\left(f_1 g^{\mu\nu}\partial_\mu\psi\partial_\nu\psi + 2C_1\psi + \frac{f_2 C_1^2}{f_1^2}\right),$$

  in the case of simple root $J = 0$, or

  $$S_{\tilde{\psi}} = -\int d^4x\frac{\sqrt{-g}}{2g_4}\left[g^{\mu\nu}\left(f_2(\partial_\mu\tilde{\psi}\partial_\nu\tau + \partial_\nu\tilde{\psi}\partial_\mu\tau) + f_3\partial_\mu\tau\partial_\nu\tau\right) + f_2\tau^2 + 2C_1\tilde{\psi} + \frac{f_3 C_1}{2f_2}\tau\right]$$

  in the case of double root $J = 0$. Note that in the case $C_1 \neq 0$ and $f_0 = 0$ the local action (22) has no term, which corresponds to the root $J = 0$.
- Vary the obtained local action and get a system of the Einstein equations and equations of motion. The obtained system is a finite order system of differential equations.
- Seek solutions of the obtained local system.

6 Exact Solution in the Friedmann–Robertson–Walker metric

Let us consider the Einstein equations, which correspond to a real simple root J_1 in the Friedmann–Robertson–Walker metric [23]:

$$\left\{ \begin{aligned} 3H^2 &= \frac{4\pi G\mathcal{F}'(J_1)}{\alpha' g_4}\left(\dot{\phi}^2 + J_1\phi^2\right) + 8\pi G\Lambda', \\ \dot{H} &= -\frac{4\pi G\mathcal{F}'(J_1)}{\alpha' g_4}\dot{\phi}^2, \end{aligned} \right. \qquad (39)$$

where a dot denotes a time derivative, the Hubble parameter $H = \dot{a}/a$, a is the scale factor.

Exact real solutions of this system have been obtained in [18, 23]. In our notations these solutions are as follows:

At $J_1 > 0$

$$\phi(t) = \pm\frac{\sqrt{3J_1}\alpha' g_4}{6\pi G\mathcal{F}'(J_1)}(t - t_0), \qquad H(t) = -\frac{J_1\alpha' g_4}{6\pi G\mathcal{F}'(J_1)}(t - t_0), \qquad (40)$$

where t_0 is an arbitrary constant. These solutions exist only at

$$\Lambda' = -\frac{J_1\alpha' g_4}{24G^2\pi^2\mathcal{F}'(J_1)}. \qquad (41)$$

At $J_1 = 0$ summing the first and the second equations of (39), we obtain:

$$\dot{H} = 8\pi G\Lambda' - 3H^2. \qquad (42)$$

The type of solutions depends on a sign of Λ':

- $\Lambda' = 0$

  $$H(t) = -\frac{1}{3(t - t_0)}, \qquad \phi(t) = \tilde{C}_1 \pm \frac{\sqrt{3}\sqrt{\alpha' g_4}}{\sqrt{\pi G\mathcal{F}'(0)}}\ln(t - t_0), \qquad (43)$$

  where t_0 and $\tilde{C}_1$ are arbitrary constants.

- If $\Lambda' > 0$, then we obtain solutions:

  $$H_1(t) = \frac{2\sqrt{6\pi G\Lambda'}}{3}\tanh\left(2\sqrt{6\pi G\Lambda'}(t - t_0)\right), \qquad (44)$$

  $$\phi_1(t) = \pm\sqrt{\frac{-\alpha' g_4}{12\pi G\mathcal{F}'(0)}}\arctan\left(\sinh\left(2\sqrt{6\pi G\Lambda'}(t - t_0)\right)\right) + \tilde{C}_2 \qquad (45)$$

  and

  $$\tilde{H}_1(t) = \frac{2\sqrt{6\pi G\Lambda'}}{3}\coth\left(2\sqrt{6\pi G\Lambda'}(t - t_0)\right), \qquad (46)$$

  $$\tilde{\phi}_1(t) = \pm\sqrt{\frac{\alpha' g_4}{12\pi G\mathcal{F}'(0)}}\ln\left(\tanh\left(\sqrt{6\pi G\Lambda'}\,(t - t_0)\right)\right) + \tilde{C}_2, \qquad (47)$$

  hereafter t_0 and $\tilde{C}_2$ are arbitrary real constants.

- In the case $\Lambda' < 0$ we obtain the following real solution:

  $$H_2(t) = -\frac{2\sqrt{-6\pi G\Lambda'}}{3}\tan\left(2\sqrt{-6\pi G\Lambda'}(t - t_0)\right), \qquad (48)$$

  $$\phi_2(t) = \pm\sqrt{\frac{\alpha' g_4}{12\pi G\mathcal{F}'(0)}}\text{arctanh}\left(\sin\left(2\sqrt{-6\pi G\Lambda'}(t - t_0)\right)\right) + \tilde{C}_2. \qquad (49)$$

One of important questions is the investigation of classical stability of the obtained solutions. The stability of isotropic solutions in the Bianchi I metric in models with minimally coupling one or two scalar (phantom scalar) fields has been studied in [36, 37]. In particular, the stability of the exact solutions, obtained in the Friedmann–Robertson–Walker metric [23], has been analysed in [36]. Anisotropic exact solutions in the Bianchi I metric have been presented in [30].

7 Conclusion

The main result of this paper is the generalization of the Ostrogradski representation on gravitational models with a nonlocal scalar field. The algorithm of localization is proposed for an arbitrary analytic function $\mathcal{F}(\Box_g)$, which has both simple and double roots. We have proved that the same functions solve the initial nonlocal Einstein equations and the obtained local Einstein equations. We have found the corresponding local actions and proved the self-consistence of our approach. In the case of two simple roots as well as in the case of one double root we get a quintom model.

The consideration of simple and double roots allows us to make the conjecture that the existence of local actions, which correspond to a nonlocal action, does not depend on order of $\mathcal{F}(\Box_g)$ roots and the method of localization can be generalized on a nonlocal action with an arbitrary analytic $\mathcal{F}(\Box_g)$. In the case of simple roots exact solutions in the Friedmann–Robertson–Walker metric have been found in [23]. The algorithm of localization does not depend on metric, so it can be used to find solutions in any metrics.

Cosmological perturbations in models with a single nonlocal scalar field have been studied in papers [26, 33]. We construct the equation for the energy density perturbations of the non-local scalar field and explicitly prove that for the free field it is identical to a closed system of local cosmological perturbations equations in a particular model with multiple local free scalar fields.

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