Non-perturbative formulation of the cosmological perturbation theory

V. N. Strokov[a*]
[a] *Astrospace Centre of the Lebedev Physical Institute*
ul. Profsoyuznaya, 84/32, Moscow, 117997 Russia

Abstract

A common way to consider the cosmological perturbation theory is decomposing the metrics and, hence, the energy–momentum tensor into background and perturbation, that is, to proceed from the left-hand side of the Einstein equations to the right-hand side. However, the material term of the gravitational field equations offers us natural non-perturbative variables, energy density and pressure, from the very beginning. Therefore, we propose to invert the procedure and find such geometrical variables in the left-hand side of the Einstein equations that would include both background and perturbations.

1 Introduction

Having made its first appearance in the pioneering work by Lifshitz [1], the theory of cosmological perturbations deals with small ripples on homogeneous and isotropical cosmological background. As it often happens in theoretical physics, once zeroth approximation is simple and well studied and there is some kind of smallness in a theory, one can decompose equations with respect to perturbations and make conclusions about a slightly more complicated case. In the theory of cosmological perturbations it was soon realized that the splitting into background and perturbations is not unique. In fact, each observer defines their own background depending on their reference frame, or gauge. However, the spirit of general relativity, which is the base of the cosmological perturbation theory, prescribes to consider invariant quantities. In 1980's this would lead to the development of the aptly named gauge-invariant formalism [2, 3] of the theory. The whole set of perturbations falls into three classes – scalar, vector and tensor modes. The scalar mode is especially interesting since the other two vanish in the background equations.

Now, as long as we decompose the gravitational part of Einstein's equations into zeroth time-dependent background and first-order inhomogeneous perturbations, we tend to do the same with the right-hand term. Afterwards we construct gauge-invariant combinations. Nevertheless, the material term offers us invariant quantities from the very beginning. For example, if the Universe is filled with a scalar field, the latter does the job. The ideal fluid also [4] provides scalars: energy density ε and pressure p. These general invariants contain, of course, every order of the cosmological perturbation theory including zeroth and first ones. Therefore, a question arises: what if we revert the scheme and try to formulate the left-hand side of Einstein's equations in variables, which would comprise both background and perturbations?

Since the homogeneous background we start with is appropriately expressed in terms of sections $t = const$ (t is time variable), the natural framework for fulfilling our goal appears to be the Arnowitt–Deser–Misner formalism [5] (ADM). Then we proceed to briefly describe it.

* **e-mail**: strokov@asc.rssi.ru

2 ADM in a nutshell

In the ADM formalism the space–time interval takes the form:[1]

$$ds^2 = (Ndt)^2 - \gamma_{ij}\left(dx^i + N^i dt\right)\left(dx^j + N^j dt\right). \tag{1}$$

Spatial indices are lowered by the 3-metrics γ_{ij}. Let us introduce the 4-vector $n_\alpha \equiv (N, 0, 0, 0)$. Its contravariant components are $n^\alpha = (1/N, -N^i/N)$. Then the Einstein–Hilbert action can be written in terms of the metric variables and matter Lagrangian $L[m]$:

$$S[N, N^i, \gamma_{ij}, m] = \frac{1}{2}\int \sqrt{\gamma}dtd\mathbf{x}\left(2NL + NR^{(3)} + \frac{1}{N}\left(\kappa_{ij}\kappa^{ij} - \kappa^2\right) + 2E^\alpha{}_{;\alpha}\right), \tag{2}$$

where

$$\kappa_{ij} \equiv \frac{1}{2}\left(\partial_t\gamma_{ij} - N_{i|j} - N_{j|i}\right),$$

$$E^\alpha = n_\alpha n^\beta{}_{;\beta} - n^\beta n_{\alpha;\beta}$$

and the semicolon stands for covariant derivative with respect to the full metrics whereas the vertical line and the curvature $R^{(3)}$ refer to the 3-dimensional quantities.

Variating S with respect to the set of variables $[N, N^i, \gamma_{ij}]$ and matter fields m yields Einstein's equations and material equation-of-motion.

3 Application to cosmology

To begin with, one can easily show that under arbitrary gauge transformations keeping the time t intact and treating it as a free parameter,

$$t \to t,$$

$$\mathbf{x} \to \widetilde{\mathbf{x}}^i\,(t, \mathbf{x}),$$

the variable N acts as a scalar and γ_{ij} appears to be genuine tensor.

These transformations are particularly relevant, because they do not alter the comoving condition. Thus, if we take the comoving reference frame it is not entirely fixed, it rather allows the above-mentioned class of transformations. In the cosmological framework the spatial part γ_{ij} is of the form [6] $\widetilde{a}^2(\delta_{ij} + G_{ij})$, where $\widetilde{a}$ is the cosmological scale factor extended to the inhomogeneous case. In the perturbation theory it turns out to be a genuine scalar, even beyond the comoving-friendly gauge transformations. The point is we now have at least two variables (the second is N, being scalar only within the certain group of transformations) that contain both zeroth and first orders — exactly what was intended.

Still holds a question about how to fix the potential vector generated in a scalar sector and, besides, there are the vector and tensor modes (V^i and $\mathfrak{G}_{ij}$, respectively). But in realistic cosmologies all of those appear in the mere first order. Leaving the accurate definition of the quantities to a more detailed paper, we can now represent the action (2) as a functional of the 'cosmological' variables:

$$S = S[N, \widetilde{a}, V^i, \mathfrak{G}_{ij}, m]. \tag{3}$$

[1]Latin indices range from 1 to 3 while the Greek letters run from 0 to 3. Metrics signature is $(+ - --)$ and the gravitational constant is normalized so that $8\pi G = 1$.

4 Conclusion

Speaking about slightly inhomogeneous cosmology in terms of background and perturbations does not seem to be a natural way of describing it. Since the full geometry we live in is formed by the general distribution of matter, rather than separately by homogeneous background and small ripples over it, physically appropriate language should not discriminate between background and perturbations, which is only possible if it includes invariants embracing every order. First of all, it applies to the scalar mode and in this work we proposed a way to define such invariants.

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