Dark halos built of scalar gravitons: numerical study

Yu. F. Pirogov[a, b]*, I. Yu. Polev[b]†

[a] *Theory Division, Institute for High Energy Physics*
Protvino, Moscow Region, Russia

[b] *Faculty of General and Applied Physics, Moscow Institute of Physics and Technology*
Dolgoprudny, Moscow Region, Russia

Abstract

In a previous article due to one of the present authors (YFP), an extension to General Relativity, violating general covariance to the residual unimodular one, was proposed. As a manifestation of such a violation, there appears the (massive) scalar graviton in addition to the massless tensor one. The former was proposed as a candidate on the dark matter in the Universe. In a subsequent article (Yu. F. Pirogov, MPLA 24, 3239, 2009), an application of the extension was developed. Particularly, a regular solution to the static spherically symmetric equations in empty space was studied by means of analytical methods. This solution was proposed as a prototype model for the galaxy soft-core dark halos, with the coherent scalar-graviton field as dark matter. The present report is a supplement to the aforementioned article. The statements of the latter are verified and visualized by means of numerical analysis and symbolic calculations. The nice validity of analytical results is found.

1 Introduction

In Ref. [1], an extension to General Relativity was proposed. The extension possesses the residual unimodular covariance, and in line with the massless tensor graviton describes the (massive) scalar one. The latter was proposed as a candidate on dark matter in the Universe. The theory was further developed in a series of subsequent articles. In particular, in [2] a regular solution to the static spherically symmetric equations of extended gravity in empty space was qualitatively studied by means of analytical methods. This solution was proposed as a prototype model for the galaxy soft-core dark halos, with the coherent scalar-graviton field as dark matter. For details, we refer the reader to [1, 2]. The present report is a supplement to Ref. [2]. The statements of the latter are verified and visualized by means of numerical analysis and symbolic calculations. The nice consistency of the qualitative analytical study is found. As a by-product, it is found highly plausible that the power series for the regular solution has just a finite radius of convergence.

2 Extended gravity equations

Here, we shortly remind the results of [2] concerning the static spherically symmetric equations of the unimodular extended gravity in empty space. The line element in such a case looks in the polar coordinates $(\tau, r, \theta, \varphi)$ most generally like:

$$ds^2 = ad\tau^2 - bdr^2 - cr^2(d\theta^2 + \sin^2\theta d\varphi^2), \tag{1}$$

* **e-mail**: yury.pirogov@ihep.ru
† **e-mail**: igor.polev@gmail.com

where a, b and c are some metric potentials. In the static limit, all the variables depend only on the radius r, which is still defined ambiguously. To eliminate such an ambiguity impose the coordinate condition $ab = 1$. Defining now the new variables

$$A \equiv a = 1/b, \quad C \equiv r^2 c \tag{2}$$

and supplementing them by the (dimensionless) scalar-graviton field X we get the system of the nonlinear differential equations as follows:

$$\begin{aligned}
\frac{d}{dr}\left(AC\frac{d}{dr}X\right) &= \frac{6}{R_h^2}C\exp\left(-X\right),\\
\frac{d}{dr}\left(C\frac{d}{dr}A\right) &= \frac{6\varepsilon_h^2}{R_h^2}C\exp\left(-X\right),\\
\frac{d}{dr}\left(C\frac{d}{dr}C\right) - \frac{3}{2}\left(\frac{d}{dr}C\right)^2 &= -\frac{\varepsilon_h^2}{2}\left(C\frac{d}{dr}X\right)^2,\\
\frac{d}{dr}\left(C\frac{d}{dr}A\right) - \frac{d}{dr}\left(A\frac{d}{dr}C\right) + 2 &= 0.
\end{aligned} \tag{3}$$

Here, a dimensionless parameter $\varepsilon_h^2 = 2k_h^2/k_g^2$ refers to the Lagrangian of extended gravity, with κ_g being the mass scale for the ordinary gravity (of the order of Planck mass) and κ_h being an additional mass scale characteristic of the unimodular extended gravity. On the contrary, the parameter $R_h > 0$, with the dimension of length, arises as a free integration constant. Physically, it characterizes an internal length scale for vacuum.

The first equation above reflects continuity condition in empty space, with the other equations being a combination of the continuity condition and three gravity equations. Out of four equations, only three are independent and can be chosen at will as a primary system. The remaining equation should, generally, serve as a consistency condition.

At a finite R_h, it is appropriate to choose the scaled distance $\xi = r/R_h$ as an independent variable. Introducing then $t = \xi^2$ and redefining C as $C \equiv tc$ we get equivalently:

$$\begin{aligned}
2t\frac{d}{dt}\left(AC\frac{d}{dt}X\right) + AC\frac{d}{dt}X &= 3C\exp\left(-X\right),\\
2t\frac{d}{dt}\left(C\frac{d}{dt}A\right) + C\frac{d}{dt}A &= 3\varepsilon_h^2 C\exp\left(-X\right),\\
2C\frac{d^2}{dt^2}C + \frac{1}{t}C\frac{d}{dt}C - \left(\frac{d}{dt}C\right)^2 &= -\varepsilon_h^2\left(C\frac{d}{dt}X\right)^2,\\
2t\left(C\frac{d^2}{dt^2}A - A\frac{d^2}{dt^2}C\right) + \left(C\frac{d}{dt}A - A\frac{d}{dt}C\right) + 1 &= 0.
\end{aligned} \tag{4}$$

This system is the main concern of the present investigation.

3 Analytical study

Here, we recapitulate some results of [2] concerning the solution regular at $t = 0$. Looking for such a solution as a power series of t one gets:

$$\begin{aligned}
\tilde{X} &= t - \frac{1}{2}\left(\frac{3}{5} + \varepsilon_h^2\right)t^2 + \left(\frac{1}{35}\left(4 + \frac{41}{3}\varepsilon_h^2\right) + \frac{1}{3}\varepsilon_h^4\right)t^3,\\
\tilde{a} &= 1 + \varepsilon_h^2\left(t - \frac{3}{10}t^2 + \frac{1}{35}\left(4 + \frac{19}{6}\varepsilon_h^2\right)t^3\right),\\
\tilde{c} &= 1 + \varepsilon_h^2\left(-\frac{1}{10}t^2 + \frac{2}{7}\left(\frac{1}{5} + \frac{1}{3}\varepsilon_h^2\right)t^3\right).
\end{aligned} \tag{5}$$

This representation is, generally, valid at the arbitrary $\varepsilon_h \le 1$ and may formally be continued up to the arbitrary powers of t. But being restricted just to a region of t (see later) it does not give apprehension of solution as a whole.

To proceed, we restrict ourselves to the case $\varepsilon_h \ll 1$ which is preferred from astronomical observations. Under this assumption, decompose formally an arbitrary solution as a power series of ε_h^2:

$$X(t) = \sum_{n=0}^{\infty} \varepsilon_h^{2n} X_n(t), \quad a(t) = \sum_{n=0}^{\infty} \varepsilon_h^{2n} a_n(t), \quad c(t) = \sum_{n=0}^{\infty} \varepsilon_h^{2n} c_n(t), \tag{6}$$

with the conditions $a_0 = c_0 = 1$. Substituting this decomposition into (4) we simplify the latter in the leading ε_h-order as follows:

$$\begin{aligned}
\frac{d}{dt}X_0 + \frac{2}{3}t\frac{d^2}{dt^2}X_0 &= \exp\left(-X_0\right), \\
\frac{d}{dt}a_1 + \frac{2}{3}t\frac{d^2}{dt^2}a_1 &= \exp\left(-X_0\right), \\
\frac{3}{t}\frac{d}{dt}c_1 + 2\frac{d^2}{dt^2}c_1 &= -\left(\frac{d}{dt}X_0\right)^2, \\
a_1 - t\frac{d}{dt}a_1 - 2t^2\frac{d^2}{dt^2}a_1 &= c_1 + 5t\frac{d}{dt}c_1 + 2t^2\frac{d^2}{dt^2}c_1.
\end{aligned} \tag{7}$$

Clearly, $a_1 = X_0$ under the proper boundary conditions. Thus, solving the system of the coupled differential equations (4) reduces to solving the ordinary differential equation for X_0 and c_1, with the last equation of system serving as a constraint.

To this end, introducing the new variables

$$Z = X_0 - \sigma, \quad \sigma = \ln\left(3t\right) \tag{8}$$

transform the equation for X_0 to the autonomous (not containing explicitly the independent variable) form

$$\frac{d^2}{d\sigma^2}Z + \frac{1}{2}\frac{d}{d\sigma}Z = \frac{1}{2}\left(\exp(-Z) - 1\right). \tag{9}$$

Putting then $\dot{Z} \equiv dZ/d\sigma$ reduce the second-order equation for Z to system of two first-order equations for Z and $\dot{Z}$ as follows:

$$\begin{aligned}
\frac{d}{d\sigma}Z &= \dot{Z} \\
\frac{d}{d\sigma}\dot{Z} &= -\frac{1}{2}\dot{Z} + \frac{1}{2}\left(\exp\left(-Z\right) - 1\right).
\end{aligned} \tag{10}$$

Such a system is known to be fully characterized by its phase plane $(Z, \dot{Z})$. Of the particular importance are the exceptional points defined by $dZ/\sigma = d\dot{Z}/d\sigma = 0$. In the case at hand, there is just one point of this kind, $Z = \dot{Z} = 0$, and it belongs to the stable focus type. All the trajectories $(Z(\sigma), \dot{Z}(\sigma))$ winds around this point approaching it at $\sigma \to \infty$. Moreover, among the trajectories there is a unique one $(\tilde{Z}(\sigma), \dot{\tilde{Z}}(\sigma))$, with $\dot{\tilde{Z}}$ remaining finite at $\sigma \to -\infty$.

In the original terms, this signifies two important properties of the solutions $X_0(t)$. First, all of them ripple around the exceptional solution

$$\bar{X}_0 = \ln\left(3t\right) \tag{11}$$

approaching the latter at $t \to \infty$. Second, the regular at the origin solution $\tilde{X}_0$ is unique and, supplemented by condition $\tilde{X}_0(0) = 0$, should look at $t \to 0$ as given by (5) with $\varepsilon_h = 0$:

$$\tilde{X}_0 = t - \frac{3}{10}t^2 + \frac{4}{35}t^3. \tag{12}$$

This gives the qualitative picture of the looked-for regular solution $\tilde{X}_0$ as a whole. The same concerns the regular $\tilde{a}_1 = \tilde{X}_0$. As for the regular $\tilde{c}_1$, it have to approach asymptotically the exceptional solution

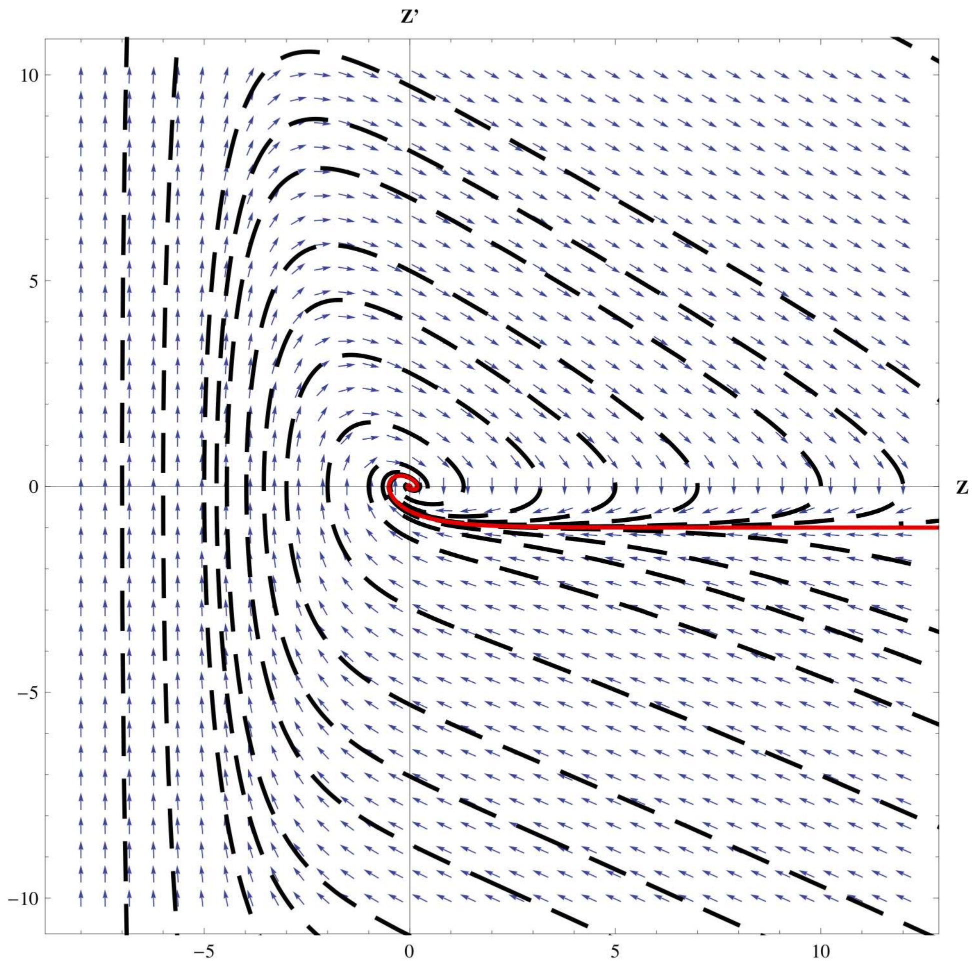
$$\bar{c}_1 = 2 - \ln(3t) \tag{13}$$

and be approximated at $0 \leq t < 1$ by the power series

$$\tilde{c}_1 = -\frac{1}{10}t^2 + \frac{2}{35}t^3. \tag{14}$$

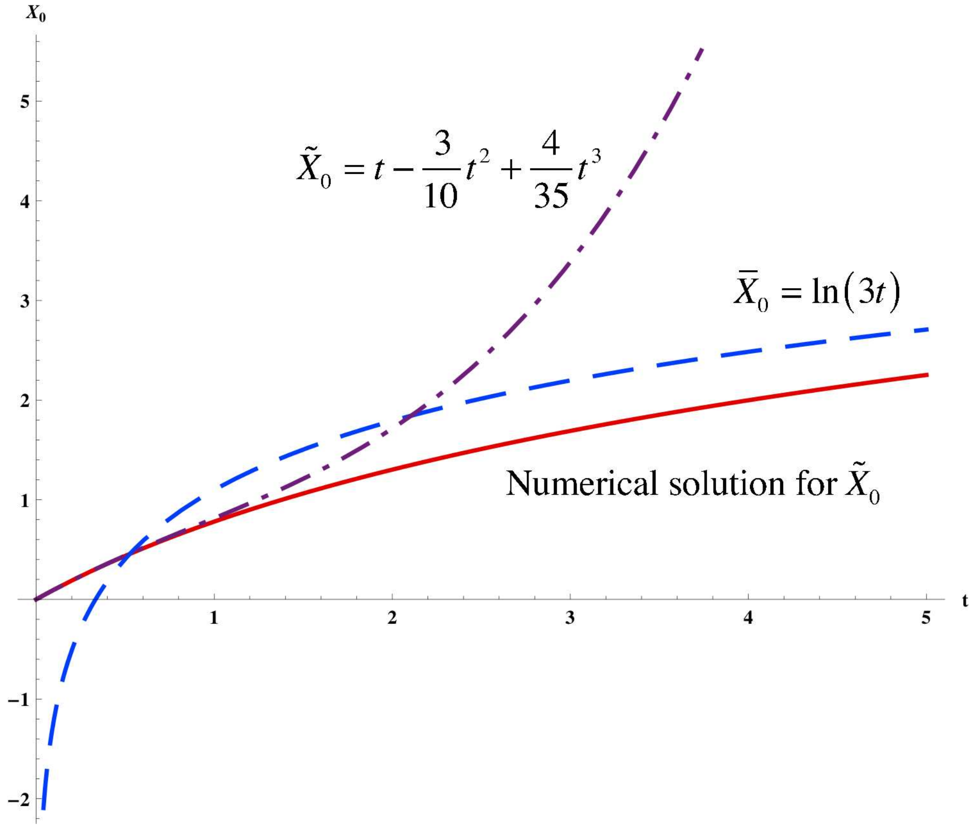
4 Numerical analysis

Here, we numerically verify and visualize the above analytical statements of [2]. The phase plane of system (10) is shown below:

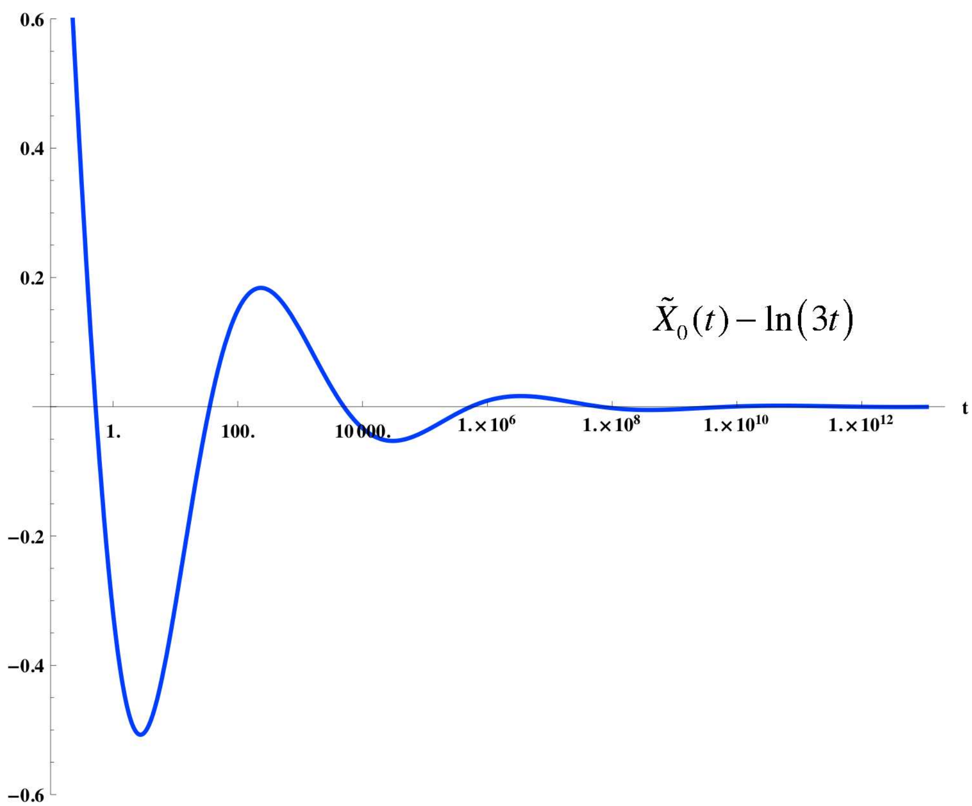


The arrows present the direction field given by $d\dot{Z}/dZ = (d\dot{Z}/d\sigma)/(dZ/d\sigma)$. The trajectories are the lines $\dot{Z}(Z)$ tangential to the direction field in every point. The bulk of trajectories (dashed black lines) possesses $\dot{Z} \to -\infty$ at $\sigma \to -\infty$ resulting in the singular at $t \to 0$ solutions. There is just one trajectory, with the finite $\dot{Z}$ ($\dot{Z} \to -1$ at $\sigma \to -\infty$), indicated by the solid red line. It may be associated with solution $\tilde{X}_0$ regular at $t = 0$. The exceptional trajectory ($\bar{Z} = \dot{\bar{Z}} = 0$) corresponds to exceptional solution $\bar{X}_0$. The picture above explicitly supports the statements made in [2].

To find the exact form of regular solution we integrate the first equation of system (7) numerically, with boundary condition taken from (12). This gives

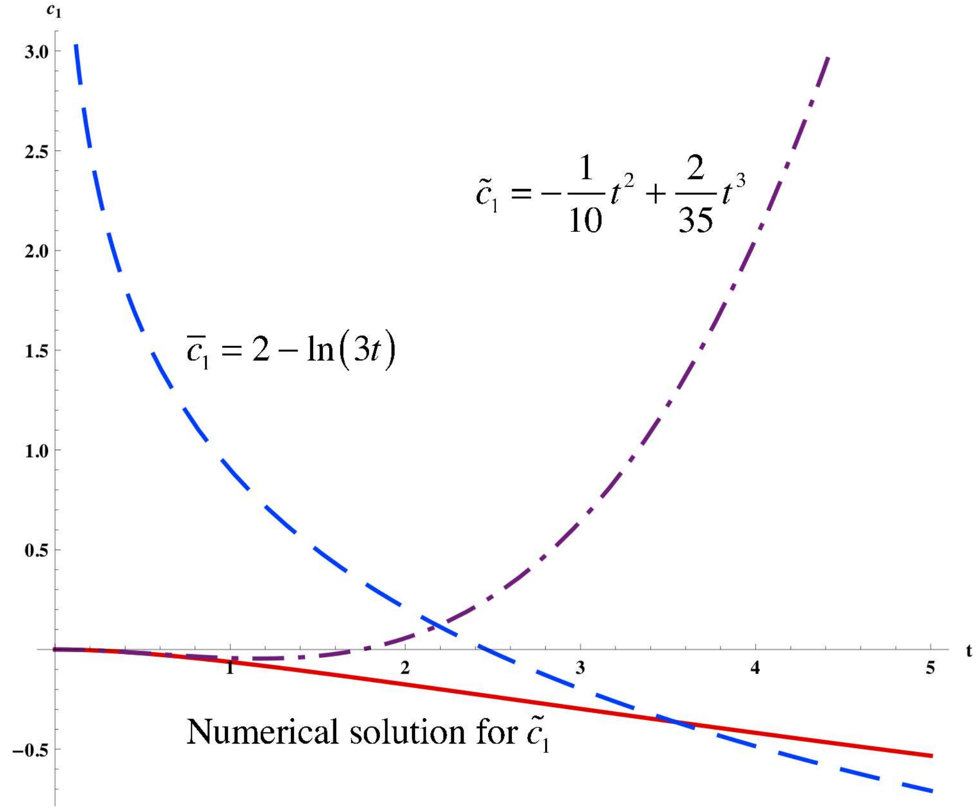


It is well seen that the power series approximates numerical solution very good in interval $0 \le t < 1$. Asymptotically, the regular solution $\tilde{X}_0$ vs. the exceptional one $\bar{X}_0$ looks like:

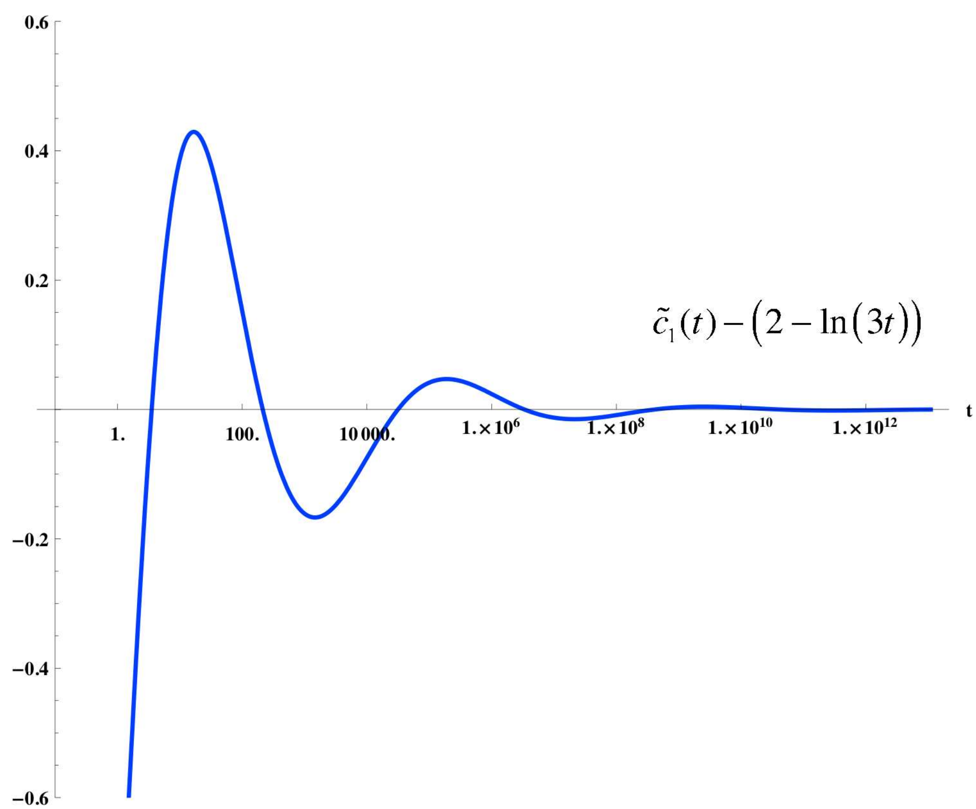


Clearly, $\tilde{X}_0$ ripples around $\bar{X}_0$ approaching the latter at $t \to \infty$. The picture shows that this approach is though extremely slow.

The same concerns $\tilde{c}_1$. Integrating the third equation of system (7) numerically we get solution as follows:

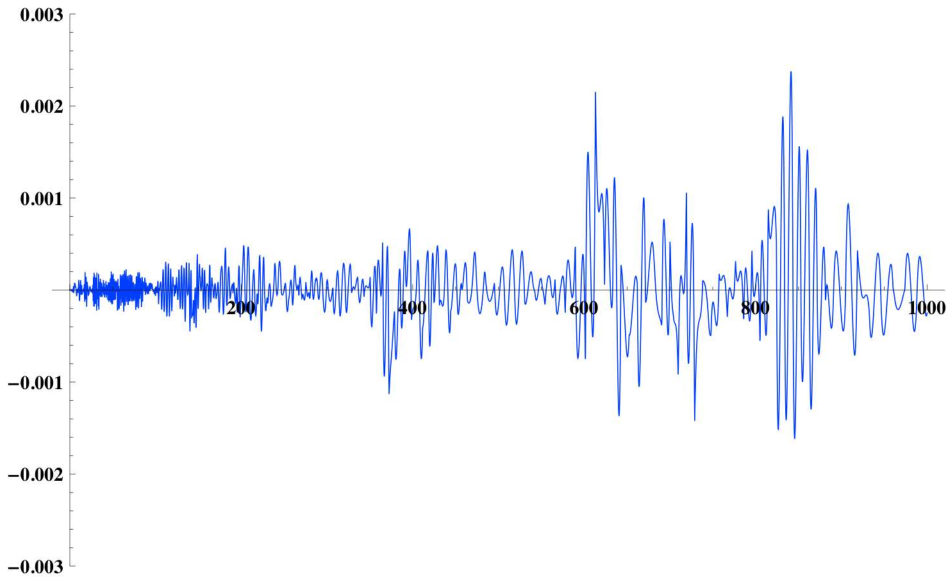


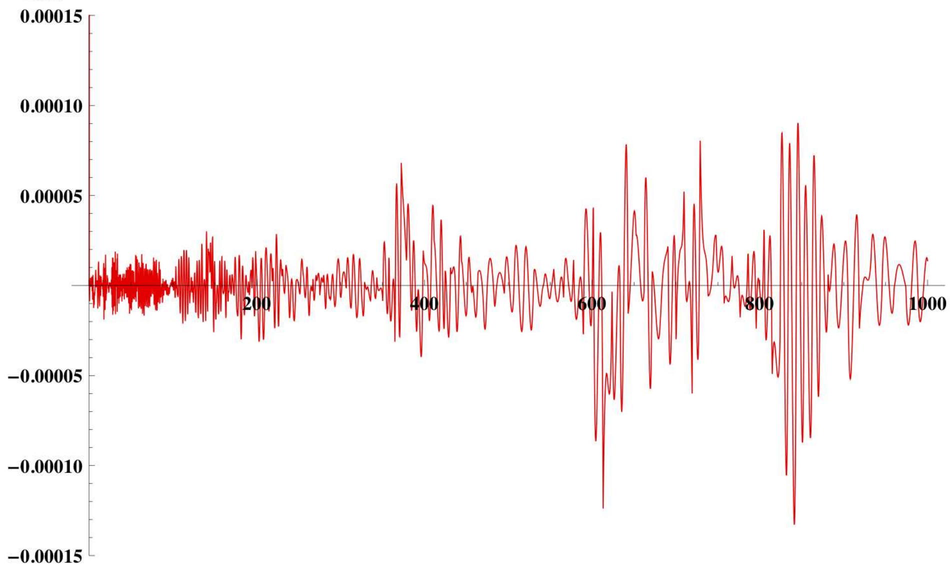
At large t, the solution also ripples around the exceptional one approaching the latter asymptotically:



Thus, numerical study totally confirms all the qualitative statements about regular solution made in [2].

At last, to check the accuracy of numerical calculations we input the found numerical solutions into consistency condition given by the last equation of (7). The absolute (the difference of L.H.S. and R.H.S.) and relative (the ratio of L.H.S. and R.H.S. minus unity) errors of calculations are shown, respectively, by the blue and red lines below:





Evidently, the achieved accuracy of numerical calculations is quite nice.

5 Symbolic calculations

Here, we examine the exact regular solution to exact system (4) in a vicinity of $t = 0$ under an arbitrary ε_h. To this end, decompose the looked-for exact solution $\tilde{X}(t)$ as the power series of t as follows:

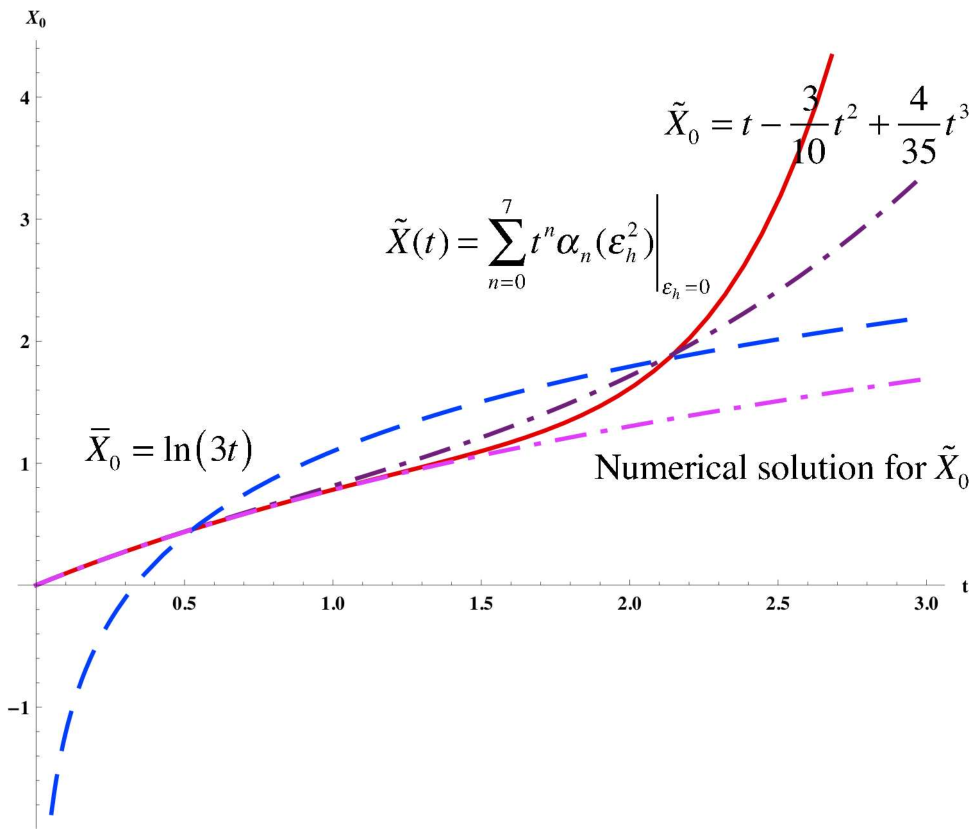
$$\tilde{X} = \sum_{n=0}^{\infty} t^n \alpha_n(\varepsilon_h), \tag{15}$$

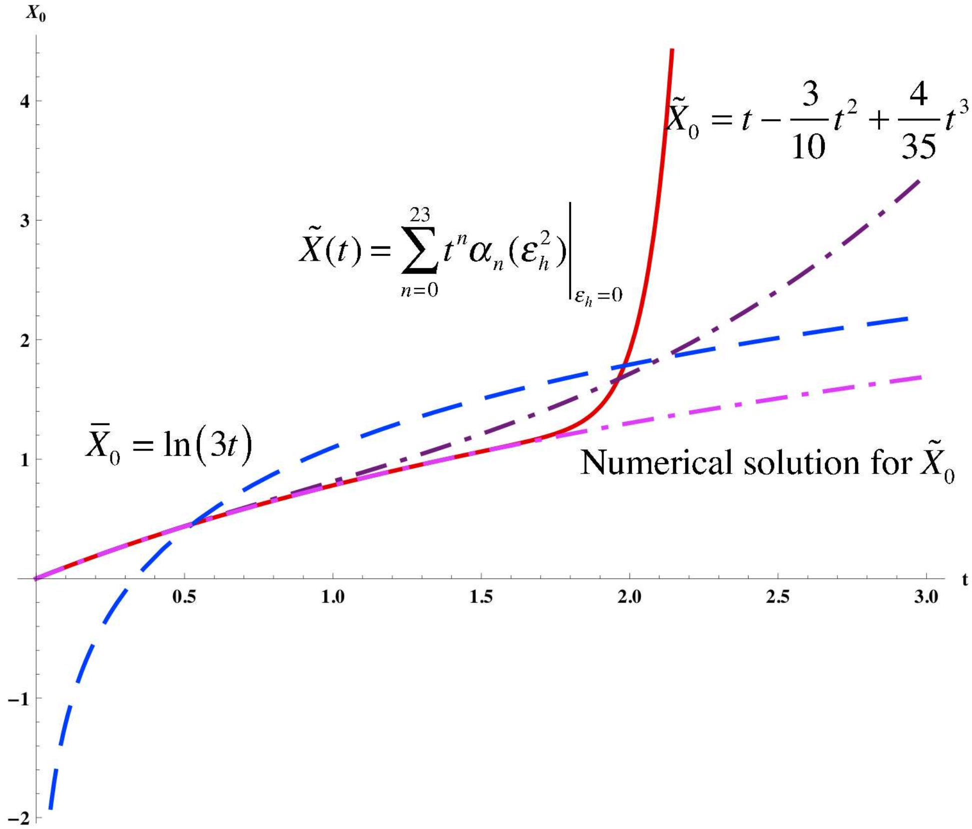
with α_h being some parameters ($\alpha_0 = 0$). We proceed similarly with $a(t)$ and $c(t)$). Implementing a system of symbolic calculations and following the cyclic perturbative procedure proposed in [2] we get in a reasonable time the 24 terms of decomposition, the first five of them being shown below:

$$
\begin{aligned}
\alpha_1 &= 1, \\
\alpha_2 &= -\frac{1}{10}\left(3+5\varepsilon_h^2\right), \\
\alpha_3 &= \frac{1}{105}\left(12+41\varepsilon_h^2+35\varepsilon_h^4\right), \\
\alpha_4 &= -\frac{1}{6300}\left(305+1573\varepsilon_h^2+2735\varepsilon_h^4+1575\varepsilon_h^6\right), \\
\alpha_5 &= \frac{1}{173250}\left(3774+25969\varepsilon_h^2+68120\varepsilon_h^4+79675\varepsilon_h^6+34650\varepsilon_h^8\right) \qquad (16)
\end{aligned}
$$

(and similarly for a_1 and c_1). The first three terms above reproduce those given by (5).

Putting now $\varepsilon_h = 0$ we can compare $\tilde{X}_0$ found previously with the present solution, two approximations to which being shown below:





It is seen that the power-series solution $\tilde{X}|_{\varepsilon_h=0}$ perfectly matches the numerical one $\tilde{X}_0$ up to $t \leq 1.7$. With n increasing, the approximation clearly improves within this region but worsens beyond it. The further analysis reveals the same picture with $\varepsilon_h \neq 0$. Fixing a numerical value for ε_h allows one to calculate much more coefficients α_n in a reasonable time. Thus, calculations with $\varepsilon_h = 0$ up to $n \simeq 3{\cdot}10^7$ make it highly plausible that the area of convergence of power series is limited in this case to $t \leq 1.8$. It seems that similar statement survives with an arbitrary $\varepsilon_h < \mathcal{O}(1)$.

6 Dark halos

Here, we examine validity of analytical results for rotation velocity and the ensuing dark matter profile found in [2]. The circular rotation velocity of a test particle in the spherically symmetric metric (1) is, generally, as follows:

$$v^2(r) = \frac{d}{dr}a(r) \Big/ \frac{d}{dr} \ln\left(r^2 c(r)\right). \tag{17}$$

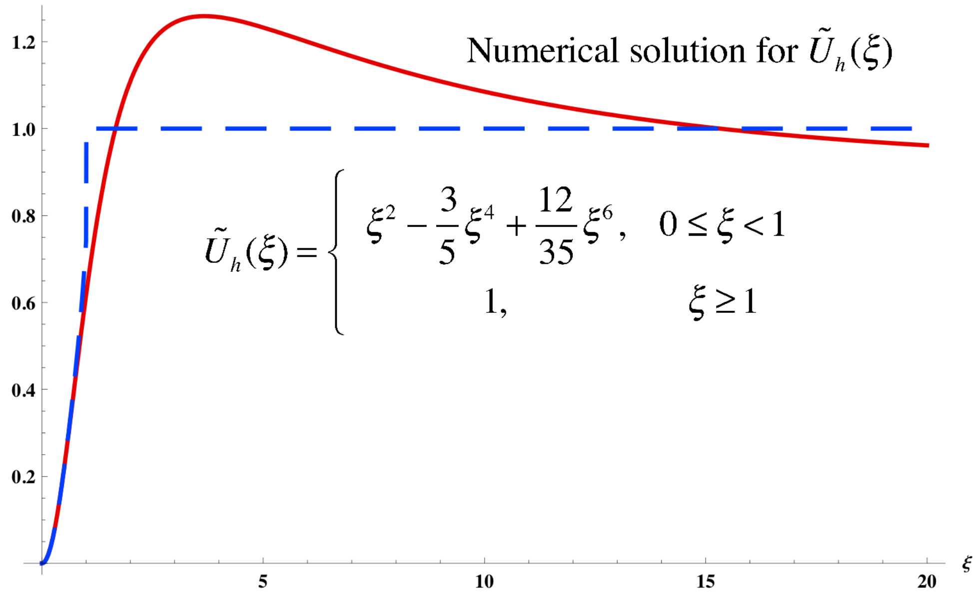
With account for $a_1 = X_0$, the respective velocity squared profile (in terms of $\xi = r/R_h$) in the leading ε_h-order looks like

$$v_h^2(\xi)/\varepsilon_h^2 \equiv U_h(\xi) = (\xi/2)dX_0(\xi)/d\xi. \tag{18}$$

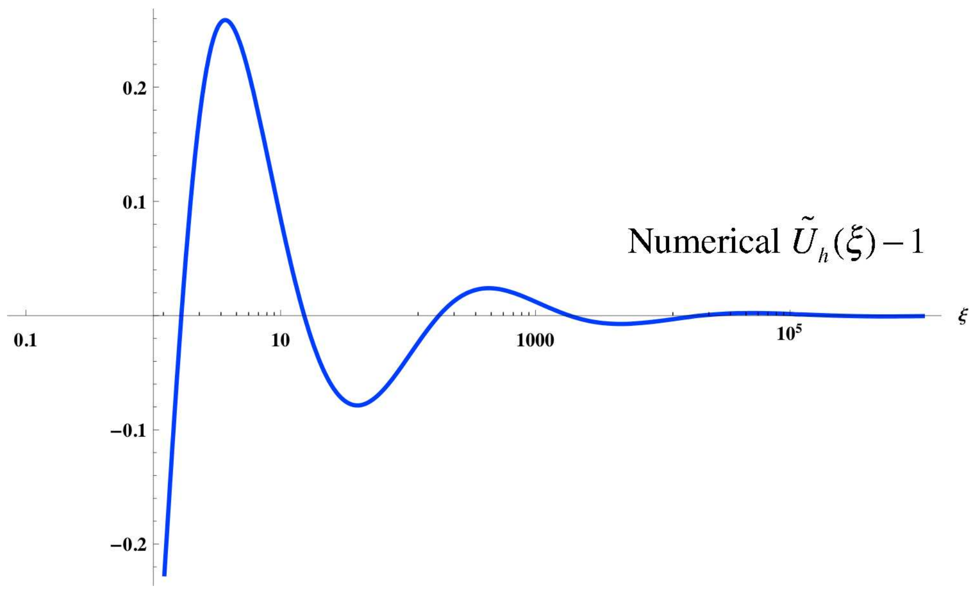
The regular solution $\tilde{X}_0$ results then in:

$$\tilde{U}_h(\xi) = \begin{cases} \xi^2 - \frac{3}{5}\xi^4 + \frac{12}{35}\xi^6, & 0 \leq \xi < 1, \\ 1, & \xi \gg 1, \end{cases} \tag{19}$$

This analytical approximation vs. numerical result is shown below:



with the numerical $\tilde{U}_h$ approaching unity asymptotically like



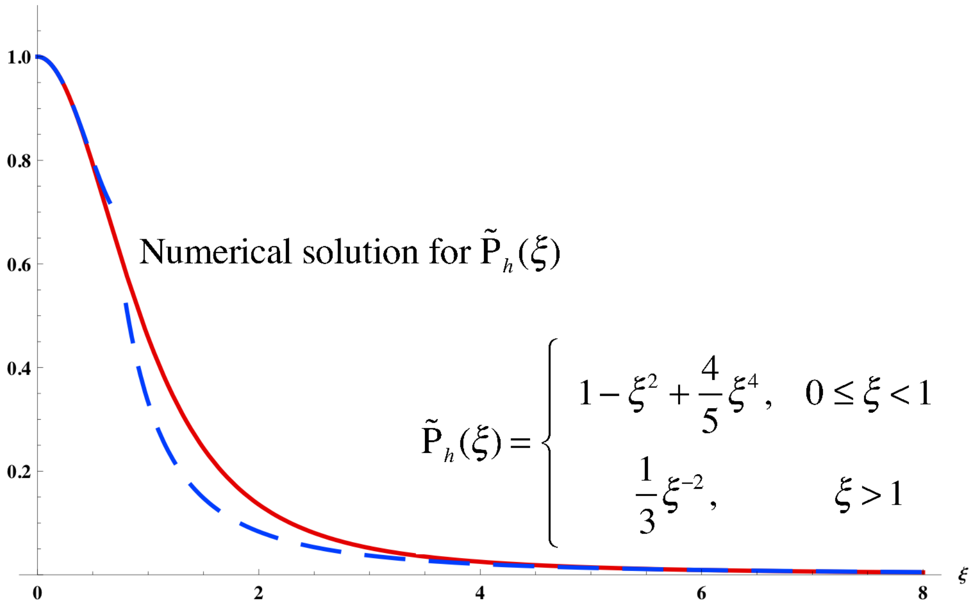
The regular solution implies the dark matter profile as follows:

$$\rho_h(\xi)/\rho_h(0) \equiv \tilde{P}_h(\xi) = \exp\left(-\tilde{X}_0(\xi)\right), \tag{20}$$

where $\rho_h(0) = 6\varepsilon_h^2\kappa_g^2/R_h^2$. Analytically, this results in the soft-core halo profile [2]:

$$\tilde{P}_h(\xi) = \begin{cases} 1 - \xi^2 + \frac{4}{5}\xi^4, & 0 \le \xi < 1, \\ \frac{1}{3}\xi^{-2}, & \xi \gg 1, \end{cases} \tag{21}$$

with the finite central density $\rho_h(0)$. The analytical approximation vs. numerical result is as follows:



Clearly, the accuracy of analytical study is quite reasonable.

7 Conclusion

Summarizing, numerical analysis of regular solution to the static spherically symmetric equations of the unimodular extended gravity in empty space totally confirms and somewhat refines analytical statements made in [2]. As a by-product, symbolic calculations make it highly plausible that the power-series decomposition of solution is valid just interior to a finite convergence radius. We are going to expand the conducted study on the general solutions to the aforementioned equations, with a view to refine the application of theory to the galaxy dark halos started in [2].

References

[1] Yu. F. Pirogov, Phys. At. Nucl. **69**,1338, 2006; gr-qc/0505031.

[2] Yu. F. Pirogov, Mod, Phys. Lett. A **24**, 3239, 2009; arXiv:0909.3311 [gr-qc].