

# Enhancement of density perturbations near horizon crossing in braneworld cosmology

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## Abstract

We study, by numerical methods, the time evolution of scalar perturbations in radiation era of Randall-Sundrum braneworld cosmology. Our results confirm an existence of the amplification of perturbation amplitudes (near horizon crossing), discovered recently. We argue that the enhancement factor is asymptotically constant, as a function of scale, in the limit when perturbations have so small scales that they cross horizon in the high energy regime.

## 1 Introduction

During last decade, braneworld cosmological scenarios, in which our 4D Universe is realized as a hypersurface embedded in a higher-dimensional spacetime have attracted much attention. In first scenarios of this kind, suggested as early as in 1980's [1, 2], it had been shown that matter fields can be confined to a field-theoretical domain wall (topological defect) in a world with non-compact extra dimensions. The progress in string theory in subsequent years, especially the discovery of D-branes, has revived interest to the idea of braneworlds. In general, the string theory is quite promising, it may provide a unified description of gauge interactions and gravity. In the present context, it is most important that it predicts the existence of  $p$ -branes,  $(p + 1)$ -dimensional sub-manifolds of the 10 (or 11) - dimensional spacetime on which open strings end. Gauge particles and fermions which correspond to string end points can only move along these  $p$ -branes, while gravitons can propagate in the full spacetime ("bulk"). It is tempting to assume that our  $(3 + 1)$ -dimensional spacetime is such a 3-brane. If only gravity can probe the bulk, the extra dimensions can be very large (in comparison with the smallest length scale tested, so far, in particle physics,  $\sim 10^{-16}$  cm). It had been assumed in [3], that the extra dimensions are compact, in analogy with the old Kaluza-Klein (KK) picture [4]. Slightly later, in works by Randall and Sundrum [5, 6], it was pointed out that this condition is not necessary and the extra dimension may be even non-compact.

The Randall-Sundrum (RS) model is of particular interest due to its relative simplicity, in spite of the fact that it includes nontrivial gravitational dynamics. In the RS2 model [6] a single brane is embedded in a anti - de Sitter (AdS) bulk and, although the 5th dimension extends infinitely, the warped structure of the bulk geometry (i.e., the curvature of the bulk spacetime) leads to a recovery of the standard General Relativity (GR) on the brane at scales larger than the bulk curvature scale  $\ell$ . In particular, Newton's law is recovered at large distances and the Friedmann's equation for the evolution of the Universe is obtained at low energy.

At high energies, i.e., in the very early Universe, the Friedmann equation differs substantially from GR by a correction term which is proportional to  $\rho/\sigma$ , where  $\rho$  is the density of brane

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matter, and  $\sigma$  is the brane tension. This term leads to a faster Hubble expansion at high energies. Inflationary expansion of the Universe is also modified in brane cosmology: the evolution of the inflaton field is more strongly damped, and the brane Universe inflates at much faster rate than what is expected from standard cosmology. Another important effect at high energies is the excitation of KK-modes which escape from our brane into the 5D bulk, leading, in particular, to the suppression of the power spectrum of inflationary gravitational wave background.

Cosmological perturbation theory in braneworld cosmology also has some distinct features [7, 8, 9, 10, 11, 12]. The equations of the perturbation theory contain high-energy corrections ( $\sim \rho/\sigma$ ) similar to those in the Friedmann equation and, in addition, the correction terms arising from the fluctuations of the bulk geometry. Perturbations on brane, e.g., the scalar perturbations (which we are interested in) are coupled with the bulk perturbations. Technically, in a case of the scalar perturbations and AdS bulk, the problem is reduced to the solution of a system of equations for the density contrast variable and the so-called master variable (it appears that all quantities describing the bulk perturbations are written in terms of this variable [7, 13, 8]).

In the context of braneworld models, a question about existence and evolution laws of the higher-dimensional black holes is very interesting and important. In a model with the 5th large extra dimension, a physically meaningful black hole solution is the 5D-Schwarzschild [14, 15], if the horizon size is sufficiently small compared with an effective size of the extra dimension. Really, it is natural to assume that primordial braneworld black holes formed in the early Universe with a horizon size  $r_s \ll \ell$  would be described by a 5D Schwarzschild metric because in this case the AdS curvature has very little effect on the geometry. Numerical calculations support the existence of static solutions for such small  $r_s$  [16]. However, the results of these calculations cannot be extrapolated to the case  $r_s \sim \ell$ .

Unfortunately, an exact solution representing a localized and stable black hole is known only in 4D braneworld model [17], whereas the corresponding solution in the 5D braneworld model has not been found. The process of the gravitational collapse on the brane is very complicate, due to, in particular, gravitational interaction between the brane and the bulk (see, e.g., [18]). Even in the simplest case of RS-type brane, and Oppenheimer-Snyder (OS) - like collapse, braneworld gravity introduces important new features in the black hole formation process (the high energy- and KK-corrections to the field equations of GR, i.e., the same corrections which affect the expansion of the early Universe, are also efficient here). These features lead to a non-static exterior of the black hole [19] in the case of the OS-collapse. Moreover, there are arguments [20, 21] based on AdS/CFT-correspondence, that the non-static behavior exists also in a general collapse. If, really, the black hole solutions in braneworld scenarios, for a black hole larger than AdS radius, are quite different from those in 4D GR (i.e., if, as authors of [20, 21] argue, these solutions are necessarily non-static and predict short lifetime of large black holes due to the strongly enhanced evaporation), there is an unique chance to probe the extra dimension by astronomical observations of massive black holes.

Predictions for an evolution of the small ( $r_s \ll \ell$ ) black holes are less dramatic (and less speculative). The differences from the 4D case are reduced to a larger probability of accretion, in the high energy regime (due to the fact that in this regime the radiation density is proportional to  $t^{-1}$  rather than  $t^{-2}$ ) and to a relative increase of the primordial black hole (PBH) lifetime, for a given initial mass. In particular, initial mass of PBHs evaporating today can be  $10^9 - 10^{10}$  g rather than  $\approx 10^{15}$  g as predicted by GR.

For the PBHs having small masses, there are astrophysical constraints on their abundance, based, e.g., on studies of extragalactic photon and neutrino backgrounds. These constraints give, as usual, the information about primordial density perturbations (we assume that PBHs form from these perturbations). For an extraction of this information one must know the evolution of these perturbations in radiation era. In the recent work by Cardoso et al [22] it had been shown that the density perturbations with short wavelengths are amplified during

horizon re-entry. The magnitude of this enhancement depends, clearly, on a scale of the density perturbations. The smaller is the scale, the earlier the perturbation crosses horizon, and, if comoving wave number  $k$  is larger than some critical value  $k_c$ , this crossing happens at high energy regime. The straightforward calculation of the enhancement factor, in the region of scales which are relevant for PBHs with small masses, evaporating near today, is quite difficult, even numerically, due to a very complicate machinery of cosmological perturbation theory in braneworld cosmology.

In the present paper we study the dependence of the enhancement factor on the comoving size of the density perturbations. We carried out detailed numerical calculations of gauge invariant amplitudes of curvature perturbations as functions of the scale factor and the corresponding enhancement factors. We found the approximate solution (of the equations for perturbation amplitudes), describing the time evolution of the amplitudes near horizon crossing. According to this solution, the magnitude of the enhancement factor doesn't depend, in the high energy region, on the comoving scale. Using this conclusion it is possible to calculate the enhanced perturbation amplitudes for arbitrarily small scale.

The plan of the paper is as follows. In the second section the equations of perturbation theory in RS2 braneworld cosmology which are necessary for curvature perturbation calculations are given. In the third section, the approximate solution of these equations in the high energy limit is suggested. In the fourth section the main relations characterizing the PBH evolution in the RS2 braneworld are briefly reviewed. The scheme used in the numerical calculations is presented in Sec. 5. The results of the paper and conclusions are summarized in the last section.

## 2 Scalar perturbations in RS2 model

### 2.1 Braneworld cosmology in RS2 model

More than ten years ago, in works [23, 24, 25, 26], exact cosmological solutions in the brane world had been obtained. It was shown also [27], for the case when the bulk is without matter fields, that five-dimensional geometry of all these cosmological solutions is the well-known [28] Schwarzschild-AdS (Sch-AdS) spacetime (i.e., the spacetime with 5D black hole geometry), having the metric

$${}^{(5)}ds^2 = -h(r)d\tau^2 + \frac{dr^2}{h(r)} + r^2 d\Sigma_K^2. \quad (1)$$

Here,  $d\Sigma_K^2$  is a metric of a unit 3D sphere, plane or hyperboloid (for  $K = +1, 0, -1$ , respectively),

$$h(r) = K + \frac{r^2}{\ell^2} - \frac{M}{r^2}, \quad (2)$$

$K$  is the curvature of the horizon,  $M$  is the mass parameter of the black hole at  $r = 0$ ,  $\ell$  is the AdS curvature radius. The most natural physical interpretation is that a cosmologically evolving brane is moving in this spacetime, while for an observer on the brane this motion will be seen as an expansion of the universe. If the brane trajectory is given by equations  $\tau_b = T(t)$ ,  $r_b = a(t)$ , where  $t$  is the proper time of the brane, the induced metric on the brane becomes

$${}^{(4)}ds^2 = -dt^2 + a^2(t)d\Sigma_K^2, \quad (3)$$

which is the metric of the Friedmann-Lemaitre-Robertson-Walker (FLRW) spacetime.

The parameter  $M$  is unknown, but the value of it can not be too large, for the brane world scenario to be consistent, e.g., with nucleosynthesis data [25]. We suppose that  $M = 0$  and shall consider below only this particular case. Further, we shall consider the spatially flat brane

only, i.e.,  $K = 0$ . Introducing a new spatial coordinate  $z$  by relation  $z = \ell/r$ , the metric (1) with  $M = 0, K = 0$  becomes conformally flat,

$${}^{(5)}ds^2 = \frac{\ell^2}{z^2} (-d\tau^2 + dz^2 + \delta_{ij}dx^i dx^j). \quad (4)$$

On the brane, the connection of  $\tau$  and  $t$  is given by [13]:

$$\tau_b = T(t), \quad \dot{T} = \frac{1}{a} \sqrt{1 + \ell^2 \left(\frac{\dot{a}}{a}\right)^2}, \quad (5)$$

and a  $z$ -coordinate of the brane is  $z_b = \ell/r_b = \ell/a$ .

The main hypothesis of any braneworld model is that the string theory predicts Einstein gravity in the bulk, i.e., the equation

$$G_{AB} = \kappa_5^2 \mathcal{T}_{AB} \quad (6)$$

takes place. In our case, the bulk energy-momentum tensor has the form

$$\mathcal{T}_{AB} = -\frac{\Lambda_5}{\kappa_5^2} G_{AB} + \delta_A^\mu \delta_B^\nu S_{\mu\nu} \delta(y - y_b), \quad (7)$$

where  $\Lambda_5$  is the bulk cosmological constant,  $\kappa_5$  is a 5D gravitational coupling constant,  $S_{\mu\nu}$  is an effective energy-momentum tensor for the brane,  $y_b$  is the brane position (transverse coordinate of the brane). In the Gaussian normal (GN) system,  $y_b = 0$ . The tensor  $S_{\mu\nu}$  consists of a brane tension term and the matter energy-momentum tensor  $T_{\mu\nu}$ ,

$$S_{\mu\nu} = \sigma g_{\mu\nu} + T_{\mu\nu}. \quad (8)$$

Using junction condition [29], one obtains the effective 4D Einstein equation on a brane [30]:

$${}^{(4)}G_{\mu\nu} = -\Lambda_4 g_{\mu\nu} + \kappa^2 T_{\mu\nu} + \kappa_5^4 \Pi_{\mu\nu} - \mathcal{E}_{\mu\nu}. \quad (9)$$

In this equation, the quantities  $\kappa$  and  $\Lambda_4$ , which are 4D gravitational coupling constant and 4D cosmological constant, respectively, are given by relations:

$$\Lambda_4 = \frac{1}{2} \left( \Lambda_5 + \frac{\kappa_5^4}{6} \sigma^2 \right), \quad \kappa^2 = \kappa_5^2 \frac{\sigma}{6}. \quad (10)$$

In AdS bulk,  $\Lambda_5 < 0$ . In addition, we will use the RS fine tuning condition:

$$\Lambda_5 = -\frac{\kappa_5^4}{6} \sigma^2, \quad (11)$$

which is necessary for static solutions to exist in RS2 model. The 4D gravitational constant becomes

$$\kappa^2 = \frac{\sigma \kappa_5^4}{6} = -\frac{\Lambda_5}{\sigma}. \quad (12)$$

At last, the bulk Einstein equations,  $G_{AB} = -\Lambda_5 g_{AB}$ , give the relation between the 5D cosmological constant and the AdS curvature radius,  $\Lambda_5 = -6/\ell^2$ .

Further, the tensor  $\Pi_{\mu\nu}$  in Eq. (9) is given by the expression

$$\Pi_{\mu\nu} = -\frac{1}{4} T_{\mu\alpha} T_\nu^\alpha + \frac{1}{12} T_\alpha^\alpha T_{\mu\nu} + \frac{1}{24} g_{\mu\nu} \left[ 3T_{\alpha\beta} T^{\alpha\beta} - (T_\alpha^\alpha)^2 \right], \quad (13)$$

and  $\mathcal{E}_{\mu\nu}$  is the limiting value on the brane of the electric part of the bulk Weyl's tensor. The latter term is, in the effective Einstein's equations, an external source, with an energy-momentum tensor  $T_{\mu\nu}^\mathcal{E}$ , defined as

$$T_{\mu\nu}^\mathcal{E} = -\frac{1}{\kappa^2} \mathcal{E}_{\mu\nu}, \quad T_\mu^{\mathcal{E}\mu} = 0. \quad (14)$$

In the case, which we consider in the present paper, this tensor is equal to zero because the Weyl tensor,  $C_{ABCD}$ , vanishes for an AdS bulk.

The  $\Pi_{\mu\nu}$ -tensor term in the effective Einstein equations (components of this tensor are quadratic in  $\rho$ ) leads to the following modification of the Friedmann equation ( $\kappa^2 = 8\pi G$ ):

$$H^2 = \frac{8\pi G}{3}\rho \left(1 + \frac{\rho}{2\sigma}\right). \quad (15)$$

Deriving this formula, the fine tuning condition, Eq. (11), and equalities  $K = 0$ ,  $M = 0$  in Eq. (2) are used. The solution of Eq. (15) for a radiation-dominated state ( $p = \rho/3$ ) on the brane is (see, e.g., [31])

$$a(t) = a_{eq} \frac{t^{1/4}(t+t_c)^{1/4}}{t_{eq}^{1/2}}, \quad H(t) = \frac{2t+t_c}{4t(t+t_c)}, \quad \rho(t) = \frac{3}{32\pi G t(t+t_c)}, \quad (16)$$

where  $t_c \equiv \ell/2$ . The conservation equation has the same form as in 4D case:

$$\dot{\rho} = -3H(\rho + p). \quad (17)$$

As one can see, at late times (low energy density), the well-known relations of 4D cosmology are recovered. The time dependence of the horizon mass in RS model is (we put  $c = 1$ )

$$M_h(t) \equiv \frac{4\pi\rho}{3H^3} = \frac{8t^2(t+t_c)^2}{G(2t+t_c)^3}. \quad (18)$$

The transition between the so-called high energy (HE) and low energy (LE) regimes happens at the ‘‘critical’’ epoch, at which  $H\ell = 1$ , and horizon mass at this time (this happens at  $t = \frac{\ell}{2\sqrt{2}} = t_c/\sqrt{2}$ ) is

$$M_h(t_c/\sqrt{2}) \approx 5 \times 10^{25} \text{ g} \left(\frac{\ell}{0.1 \text{ mm}}\right). \quad (19)$$

The critical value of the comoving wave number,  $k_c$ , which corresponds to this critical epoch, can then be written using the known relations between the horizon mass and the comoving wave number (see, e.g., [32]) as

$$k_c \approx k_{eq} \left(\frac{M_h(t_c/\sqrt{2})}{M_{eq}}\right)^{-1/2} \left(\frac{g_{*c}}{g_{*eq}}\right)^{-1/12} \approx 3 \times 10^{10} \text{ Mpc}^{-1} \left(\frac{\ell}{0.1 \text{ mm}}\right)^{-1/2} \left(\frac{g_{*c}}{100}\right)^{-1/12}, \quad (20)$$

where  $k_{eq}$ ,  $M_{eq}$  and  $g_{*eq}$  are wave number, horizon mass and effective number of relativistic degrees of freedom corresponding to the time of matter-radiation equivalence and  $g_{*c}$  is the number of relativistic degrees of freedom corresponding to the critical epoch.

The main parameter of the model,  $\ell$ , can be constrained by Newton’s law tests in table-top experiments. The most recent results [33, 34] give the following limit, which is very important for cosmological implications of the model:

$$\ell \lesssim (0.015 - 0.044) \text{ mm}. \quad (21)$$

The corresponding constraints from astronomical observations are somewhat weaker (see, e.g., [35]).

## 2.2 Scalar perturbations

The case when  $M = 0$  in Eq. (2) corresponds to a pure AdS bulk spacetime. It is known that in this case a study of cosmological perturbations in the bulk and the brane is greatly

simplified. It was shown in [7, 13, 8] that a solution of the perturbed 5D Einstein equations in a vacuum AdS bulk, having only metric perturbations,

$${}^{(5)}\delta G_B^A = 0, \quad (22)$$

can be reduced to a solution of the evolution equation for the “master variable”  $\Omega$  (which depends only on the coordinates in the 2-dimensional orbit space, i.e., on  $\tau, z$ ) whereas all gauge-invariant metric perturbations in the bulk are written in terms of this  $\Omega$ .

In Poincare coordinate system (used above, in Eq. (4)) the wave equation governing the evolution of the master variable in the bulk (the master equation) is

$$-\frac{\partial^2 \Omega}{\partial \tau^2} + \frac{\partial^2 \Omega}{\partial z^2} + \frac{3}{z} \frac{\partial \Omega}{\partial z} + \left( \frac{1}{z^2} - k^2 \right) \Omega = 0. \quad (23)$$

Here, and everywhere below, we work with Fourier transforms (with respect to the  $x^i$ 's) of  $\Omega$  and all the perturbation functions.

The important boundary condition for  $\Omega$  can be obtained from Israel's junction conditions [29]. These conditions take the simplest form in a GN coordinate system, in which the bulk metric is

$${}^{(5)}ds^2 = g_{\mu\nu} dx^\mu dx^\nu + dy^2. \quad (24)$$

The perturbed 5D metric in this system is given, in generalized 5D longitudinal gauge, by the expression

$$g_{AB} = \begin{pmatrix} -n^2(1 + 2\tilde{A}) & 0 & n\tilde{A}_y \\ 0 & a^2 \left[ (1 + 2\tilde{\mathcal{R}}) \delta_{ij} \right] & 0 \\ n\tilde{A}_y & 0 & 1 + 2\tilde{A}_{yy} \end{pmatrix}. \quad (25)$$

All quantities in Eq. (25) and, in particular,  $n, a$ , are functions of GN coordinates  $t, y$ . On the brane one has  $n_b = 1$ ,  $a_b = a(t, y = 0)$ . The functions  $a(y, t)$  and  $n(y, t)$  are known from the solution of the Einstein equations in GN coordinate system [25]. Scalar quantities  $\tilde{A}$ ,  $\tilde{A}_y$ ,  $\tilde{A}_{yy}$ ,  $\tilde{\mathcal{R}}$  in Eq. (25) are gauge invariants. The formulas relating the derivatives in two coordinate systems, are given by

$$\frac{\partial}{\partial y} = \frac{1}{a} \left( -\ell \frac{\dot{a}}{a} \frac{\partial}{\partial \tau} + \sqrt{1 + \left( \frac{\dot{a}}{a} \right)^2} \ell^2 \frac{\partial}{\partial z} \right), \quad (26a)$$

$$\frac{\partial}{\partial t} = \frac{1}{a} \left( \sqrt{1 + \left( \frac{\dot{a}}{a} \right)^2} \ell^2 \frac{\partial}{\partial \tau} - \ell \frac{\dot{a}}{a} \frac{\partial}{\partial z} \right). \quad (26b)$$

Using the expressions for the junction conditions [36], we neglect in them the terms with anisotropic stress perturbation in the perturbed energy-momentum tensor for matter on the brane and, correspondingly, all terms containing the brane bending scalar  $\xi(t, x^i)$  (describing the perturbed position of the brane) in the expression for the perturbed extrinsic curvature tensor. In this approximation, one can introduce the following notations:

$$\Phi = \tilde{A}_b, \quad \Psi = -\tilde{\mathcal{R}}_b, \quad (27)$$

having in mind that these gauge invariant perturbations of the bulk metric coincide, on the brane, with lapse and curvature perturbations in the conventional 4D cosmological perturbation theory.

Junction conditions give the expressions for matter perturbations ( $\delta\rho$ ,  $\delta q$ ,  $\delta p$ ) on the brane through the linear combinations of gauge invariants and their derivatives and, therefore, through the master variable  $\Omega$  and its derivatives. Using these expressions, one can obtain, for  $\Omega$ , a

boundary condition on the brane expressed through the gauge-invariant quantity  $\Delta$  (defined below in Eq. (33)) [12]:

$$\left[ \frac{\partial \Omega}{\partial y} + \frac{1}{\ell} \left( 1 + \frac{\rho}{\sigma} \right) \Omega + \frac{6\rho a^3}{\sigma k^2} \Delta \right]_b = 0. \quad (28)$$

Considering the perturbed effective Einstein equations,

$${}^{(4)}\delta G_{\mu\nu} = \kappa^2 \delta T_{\mu\nu} + \kappa_5^4 \delta \Pi_{\mu\nu} - \delta \mathcal{E}_{\mu\nu}, \quad (29)$$

one can parameterize the perturbations of  $\mathcal{E}_{\mu\nu}$  in the form [10, 37]:

$$\delta \mathcal{E}_0^0 = \kappa^2 \delta \rho_{\mathcal{E}} Y, \quad (30a)$$

$$\delta \mathcal{E}_i^0 = \kappa^2 k Y_i \delta q_{\mathcal{E}}, \quad (30b)$$

$$\delta \mathcal{E}_i^j = -\kappa^2 \left( \frac{1}{3} \delta \rho_{\mathcal{E}} Y \delta_i^j + k^2 \delta \pi_{\mathcal{E}} Y_i^j \right), \quad (30c)$$

$$Y = e^{i\mathbf{k}\cdot\mathbf{x}}, \quad Y_i = -\frac{1}{k} \partial_i Y, \quad Y_{ij} = \frac{1}{k^2} \partial_i \partial_j Y + \frac{1}{3} \delta_{ij} Y, \quad (31)$$

treating the trace free tensor  $\delta \mathcal{E}_{\mu\nu}$  as an additional fluid source term in Eq. (29) with a radiation-like equation of state. This assumption is in full analogy with a case of the tensor  $T_{\mu\nu}$ , where one has

$$\delta T_0^0 = -\delta \rho Y, \quad (32a)$$

$$\delta T_i^0 = -k Y_i \delta q, \quad (32b)$$

$$\delta T_i^j = \delta \rho Y \delta_i^j + k^2 \delta \pi Y_i^j. \quad (32c)$$

The perturbations of the Weyl fluid,  $(\delta \rho_{\mathcal{E}}, \delta q_{\mathcal{E}}, \delta \pi_{\mathcal{E}})$ , transfer effects of the bulk metric perturbations (effects of ‘‘KK degrees of freedom’’) to the brane.

The solution of the perturbed equations (29) is a generalization of the results of standard 4D cosmological perturbation theory. The corresponding formulas are derived in [22] (in approximation  $\delta \pi = 0$ ). These formulas express gauge invariants  $\Phi, \Psi$  in terms of the gauge invariant matter perturbation variables  $\Delta$  (which is a density contrast in the comoving gauge),  $V$  (which is a peculiar velocity in the longitudinal gauge),  $\Delta_{\mathcal{E}}, V_{\mathcal{E}}$  and  $\delta \pi_{\mathcal{E}}$ . These invariants are given by the relations (in the longitudinal gauge):

$$\rho \Delta = \delta \rho - 3H \delta q, \quad a(\rho + p)V = -k \delta q, \quad (33)$$

and analogously for  $\Delta_{\mathcal{E}}, V_{\mathcal{E}}$ . The perturbation variables of the Weyl’s fluid can be written in terms of the master variable [12]:

$$\delta \rho_{\mathcal{E}} = \left( \frac{k^4 \Omega}{3\kappa^2 a^5} \right)_b, \quad (34a)$$

$$\delta q_{\mathcal{E}} = \left( \frac{k^2}{3\kappa^2 a^3} \left[ \dot{\Omega} - \frac{\dot{a}}{a} \Omega \right] \right)_b, \quad (34b)$$

$$\delta \pi_{\mathcal{E}} = \frac{1}{6\kappa^2 a^3} \left( 3\ddot{\Omega} - 3\frac{\dot{a}}{a}\dot{\Omega} + \frac{k^2}{a^2} \Omega - \frac{3}{2} \kappa_5^2 (p + \rho) \Omega' \right)_b, \quad (34c)$$

where the prime denotes  $\partial_y$  and the dot denotes  $\partial_t$ .

Using the results of [22] and Eqs. (34) one can easily obtain the ordinary differential equation for the gauge invariant  $\Delta$ . In the approximation  $c_s^2 = \dot{p}/\dot{\rho} = 1/3$ ,  $w = p/\rho = 1/3$ , one has

$$\ddot{\Delta} + H \dot{\Delta} + \left[ \frac{1}{3} \left( \frac{k}{a} \right)^2 - \frac{4\rho}{\sigma \ell^2} - \frac{18\rho^2}{\sigma^2 \ell^2} \right] \Delta = \frac{4k^4}{9\ell a^5} \Omega_b. \quad (35)$$

This equation contains the term which is proportional to  $\Omega_b$ , in the right-hand side. Therefore, this equation is connected with Eqs. (23) and (28).

Another important gauge invariant is the curvature perturbation on uniform density slices. It is defined by the relation  $\zeta = \psi - H\delta\rho/\dot{\rho}$ , where  $\psi$  is the curvature perturbation. The relation between  $\zeta$  and  $\Delta$  also contains  $\Omega_b$ :

$$\zeta = \left[ \frac{1}{4} + \frac{3\rho a^2(3\rho + 2\sigma)}{4k^2\ell^2\sigma^2} \right] \Delta + \frac{3Ha}{4k^2} \frac{d\Delta}{d\eta} + \frac{k^2}{6\ell a^3} \Omega_b. \quad (36)$$

### 3 The high energy regime

Studying, in the high energy regime of radiation dominated era (when, in particular,  $H \approx \frac{\rho}{\sigma\ell}$ ,  $\partial_y \approx -\partial_t$ ), the dependence of  $\Delta$  and  $\Omega$  on time before horizon crossing, by power series methods, and taking into account only the dominant growing mode, one can obtain (at leading order in  $k\eta$ ) the result [22]:

$$\Delta^{as} \approx \frac{4}{3}(k\eta)^2, \quad \Omega_b^{as} \approx 3\ell a_*^3 k^{-2} (k\eta)^3. \quad (37)$$

Here,  $a_*$  is the scale factor at time of Hubble horizon crossing. In the high energy regime one has

$$\eta = \frac{1}{3aH}, \quad a = a_*(3k\eta)^{1/3}. \quad (38)$$

The connection between  $a_*$  and the comoving wave number is

$$a_* = \frac{k}{H_*} = a_c \cdot (\sqrt{2} - 1)^{1/3} \left( \frac{k_c}{k} \right)^{1/3}, \quad (39)$$

$$a_c \equiv a(t_c/\sqrt{2}) \approx 1.25 \Omega_R^{1/4} \left( \frac{H_0\ell}{c} \right)^{1/2} \approx 10^{-16} \left( \frac{\ell}{0.1 \text{ mm}} \right)^{1/2}. \quad (40)$$

One can rewrite Eqs. (37) in the form:

$$\Delta^{as} \approx \frac{4}{27} \left( \frac{a}{a_*} \right)^6, \quad \frac{1}{\ell} \Omega_b^{as} \approx \frac{1}{9} \left( \frac{a}{a_*} \right)^9 \frac{a_*^3}{k^2}. \quad (41)$$

As one can see from Eqs. (41) and (39), the value of  $\Delta^{as}$  at Hubble horizon crossing is constant and the corresponding value of  $\Omega_b^{as}$  depends only on  $k$ .

We are interested in a behavior of  $\Delta$  and  $\Omega_b$  in a relatively short time interval, from  $a = a_*$  up to  $a \lesssim 3a_*$ . Just near  $a \approx 3a_*$  the  $\Omega, \Delta$ -values reach maximum. At later times, the oscillations begin, and amplitudes of these oscillations are equal, approximately, to the maximum amplitudes reached at the previous period of the smooth behavior.

Our key assumption is the following: the growth of  $\Omega, \Delta$ -amplitudes in the interval  $(a_* - 3a_*)$  can be described by functions which do not depend on the comoving wave number  $k$ . Namely, one assumes that

$$\Delta = \frac{4}{27} \left( \frac{a}{a_*} \right)^6 f_\Delta \left( \frac{a}{a_*} \right), \quad \frac{1}{\ell} \Omega_b = \frac{1}{9} \frac{a_*^3}{k^2} \left( \frac{a}{a_*} \right)^9 f_\Omega \left( \frac{a}{a_*} \right), \quad (42)$$

in the asymptotical limit of the high energy regime,  $k \gg k_c$ . The functions  $f_\Delta, f_\Omega$  decrease with growth of  $a/a_*$  and it is assumed that

$$f_\Delta(1) = f_\Omega(1) = 1, \quad f'_\Delta(1) = f'_\Omega(1) = 0. \quad (43)$$

According to this assumption, the time evolution of  $\Omega$  and  $\Delta$ , starting from the horizon re-entry is the same for all comoving wave numbers  $k$ , and the  $k$ -dependence enters only through



initial conditions at  $a = a_*$ . It may be justified as follows. A general solution of the master equation is given by the expression [38]

$$\Omega = \frac{\ell^3}{z} \int dm S(m) Z_0(mz) e^{-i\omega\tau}, \quad (44)$$

where  $Z_0$  is the linear combination of Hankel functions and  $S(m)$  is the arbitrary function,  $\omega = \sqrt{m^2 + k^2}$ . The variable  $m$  has a physical sense of the KK mass. It is known that in the very high energy case the contribution to  $\Omega$  from the massive KK modes is, in general, significant, i.e., the characteristic values of  $m$  contributing to the integral for  $\Omega$  can be much larger than  $a_*/\ell$ . At the same time, the physical sizes of perturbations are smaller than  $\ell$ ,  $a_*/k \ll \ell$ , i.e.,  $k \gg a_*/\ell$ . So,  $m$ -values are of the same order as  $k$ -values, or even larger, and the  $k$ -dependence of  $\Omega$  can be effectively masked,  $\omega = \sqrt{m^2 + k^2} \approx m$ .

Substituting these expressions for  $\Delta$  and  $\Omega_b$  in all the equations used for the numerical calculations, one can see that all of them become *independent on  $k$* .

In the high energy regime, when  $a \sim t^{1/4}$ , the following useful relation holds:

$$H^2 a^2 = \left(\frac{a}{a_*}\right)^{-6} k^2. \quad (45)$$

Using this relation and ansatzes (42), one obtains, from Eq. (36) for  $\zeta$ :

$$\begin{aligned} \zeta = & \frac{1}{27} \left(\frac{a}{a_*}\right)^6 f_\Delta \left(\frac{a}{a_*}\right) + \frac{1}{3} f_\Delta \left(\frac{a}{a_*}\right) + \\ & + \frac{1}{9} \left[ 6f_\Delta \left(\frac{a}{a_*}\right) + \frac{a}{a_*} f'_\Delta \left(\frac{a}{a_*}\right) \right] + \frac{1}{54} \left(\frac{a}{a_*}\right)^6 f_\Omega \left(\frac{a}{a_*}\right). \end{aligned} \quad (46)$$

At  $a = a_*$ , one has, as it must be,  $\zeta \approx 1$ . At  $a/a_* = 3$ , i.e., near the maximum, one has

$$\zeta_{max} = \zeta \left(\frac{a}{a_*} \approx 3\right) = \frac{1}{27} 3^6 \left( f_\Delta(3) + \frac{1}{2} f_\Omega(3) \right). \quad (47)$$

If  $f_\Delta(3) \approx f_\Omega(3) \sim 0.15$ , one obtains that  $\zeta_{max} \approx 6$ .

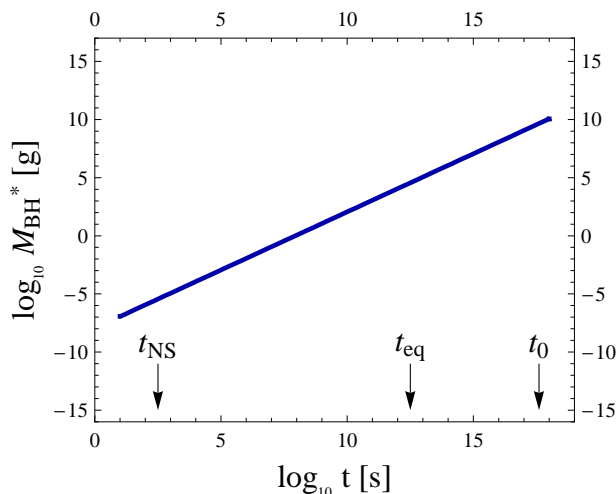


Figure 1: Black hole mass  $M_{BH}^*$  versus the moment of time at which it evaporates, assuming the case of RS cosmology with  $\ell = 0.1$  mm. Labels “ $t_{NS}$ ”, “ $t_{eq}$ ” and “ $t_0$ ” show, correspondingly, the nucleosynthesis, matter-radiation equivalence and present epochs.

The equation (35) for  $\Delta$ , after substituting of Eqs. (42), becomes

$$\begin{aligned} \left(\frac{a}{a_*}\right)^{-6} \left\{ a^2 \frac{d^2 \Delta}{da^2} - 2a \frac{d\Delta}{da} \right\} + \frac{4}{81} \left( 1 - 54 \left(\frac{a}{a_*}\right)^{-6} \right) \left(\frac{a}{a_*}\right)^6 f_\Delta \left(\frac{a}{a_*}\right) = \\ = \frac{4}{81} \left(\frac{a}{a_*}\right)^6 f_\Omega \left(\frac{a}{a_*}\right). \end{aligned} \quad (48)$$

Neglecting in Eq. (48) the terms with derivatives, which, as can be checked, are subdominant, one obtains the approximate result

$$f_\Delta \left(\frac{a}{a_*}\right) \approx f_\Omega \left(\frac{a}{a_*}\right). \quad (49)$$

From the equation (28) for the boundary condition one obtains

$$-\frac{8}{9} f_\Omega \left(\frac{a}{a_*}\right) - \frac{1}{9} \frac{a}{a_*} f'_\Omega \left(\frac{a}{a_*}\right) + \frac{24}{27} f_\Delta \left(\frac{a}{a_*}\right) = 0. \quad (50)$$

For consistency with Eq. (49), the function  $f_\Omega(a/a_*)$  must obey the inequality

$$\frac{a}{a_*} f'_\Omega \left(\frac{a}{a_*}\right) \ll f_\Omega \left(\frac{a}{a_*}\right), \quad (51)$$

at all  $a$  including  $a \approx 3a_*$ .

## 4 Characteristics and evolution of 5D black holes

The formation and evolution of PBHs in RS2 cosmology had been investigated in [39, 40, 41, 42, 43, 44, 31].

Supposing that the braneworld PBHs localized on the brane are represented by the 5D Schwarzschild solution [14, 15] for the metric, one obtains, for the induced 4D metric on the brane, the expression

$$ds^2 = - \left[ 1 - \left(\frac{r_s}{r}\right)^2 \right] dt^2 + \left[ 1 - \left(\frac{r_s}{r}\right)^2 \right]^{-1} dr^2 + r^2 d\Omega^2, \quad (52)$$

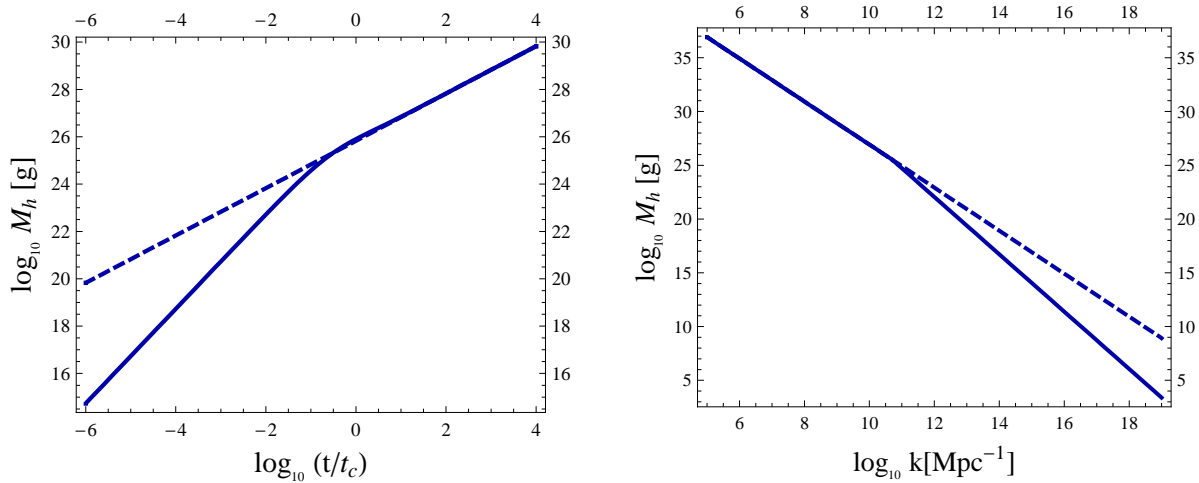


Figure 2: The dependence of horizon mass on cosmic time  $t$  (*left panel*) and comoving wave number  $k$  (*right panel*). Solid curves show the case of RS cosmology assuming that  $\ell = 0.1$  mm while dashed curves are for the case of standard 4D-cosmology.

which does not coincide with the 4D Schwarzschild metric. Correspondingly, the relation between PBH mass  $M_{BH}$  and radius,

$$r_s = \sqrt{\frac{8}{3\pi}} \left(\frac{\ell}{\ell_4}\right)^{1/2} \left(\frac{M_{BH}}{M_4}\right)^{1/2} \ell_4, \quad (53)$$

is different from the analogous relation in the 4D case (throughout this section, we will use, following [39], the convenient notations, in which  $M_4$  is the Planck mass,  $\ell_4 = M_4^{-1}$  is the Planck length,  $t_4 = \ell_4$  is the Planck time).

It follows from Eq. (53) that if PBH's radius at its formation is smaller than AdS radius  $\ell$ , the following inequality for PBH's mass holds:

$$\frac{M_{BH}}{M_4} < \frac{\ell}{\ell_4}. \quad (54)$$

We assume, as usual, that PBHs form with masses equal, approximately, to the horizon mass  $M_h$  at the time of formation,

$$M_{BH} \approx M_h. \quad (55)$$

Using the expression for  $M_h$  (Eq. (18)) and the relation  $t_c = \ell/2$ , one can see that the equality (55) is consistent with the inequality (54) only if  $t < t_c$ , i.e., in the high energy regime. It means that PBHs which form in the high energy regime are 5D black holes.

A rate of a loss of the PBH's mass, due to the 5D-evaporation, is proportional to  $r_s^{-2}$  for the evaporation in the brane as well as in the bulk. So, one has  $dM_{BH}/dt \sim M_{BH}^{-1}$ . Resulting lifetime of the black hole,  $t_{evap}$ , is proportional to  $M_{BH}^2$  rather than  $M_{BH}^3$  as in the 4D case,

$$\frac{t_{evap}}{t_4} \sim \frac{\ell}{\ell_4} \left(\frac{M_{BH}(t_c, t_{evap})}{M_4}\right)^2. \quad (56)$$

In this formula  $t_c$  is the time of the onset of evaporation,  $M_{BH}(t_c, t_{evap})$  is the PBH mass at  $t = t_c$  which evaporates at  $t = t_{evap}$  (one assumes that  $t_c \ll t_{evap}$ ). Here we assume, following [42], that in the relatively short period of time from the PBH formation,  $t_i$ , up to the end of the high energy regime,  $t_c$ , black hole does not evaporate, but increases its mass due to the accretion. The increase of mass due to the accretion is determined by the equation [41, 40]  $dM/dt \sim qM/t$  ( $q$  is the (unknown) parameter of an efficiency of the accretion,  $0 < q < 1$ ). It is assumed, for simplicity, in a derivation of Eq. (56) that at  $t = t_c$  the accretion process ends completely, giving place for the pure evaporation.

One can check that if AdS radius is too small, the condition for 5-dimensionality of PBHs,  $r_s < \ell$ , can not be satisfied. Comparing the mass-lifetime relation (56) with the expression for the radius (53), one can determine, for a given lifetime, the minimal possible value of  $\ell$ , given by the relation

$$\ell_{min} \sim \left(\frac{t_{evap}}{t_4}\right)^{1/3} \ell_4. \quad (57)$$

If, e.g., PBH evaporates today,  $t_{evap} \approx t_0 \sim 10^{17}$  s, one has  $\ell_{min} = 10^{20} \ell_4$ .

The BH mass at the onset of the evaporation ( $t = t_c$ ), if the age of the black hole is equal to the age of the Universe, is

$$M_{BH}(t_c, t_0) \equiv M_{BH}^*(t_0) \approx 5 \times 10^9 \left(\frac{\ell}{0.1 \text{ mm}}\right)^{-1/2} \text{ g}, \quad \ell > 10^{20} \ell_4. \quad (58)$$

and, in general, for PBH evaporating at time  $t$ ,

$$M_{BH}^*(t) \approx 5 \times 10^9 \left(\frac{\ell}{0.1 \text{ mm}}\right)^{-1/2} \left(\frac{t}{t_0}\right)^{1/2} \text{ g}, \quad \ell > \left(\frac{t}{t_4}\right)^{1/3} \ell_4. \quad (59)$$

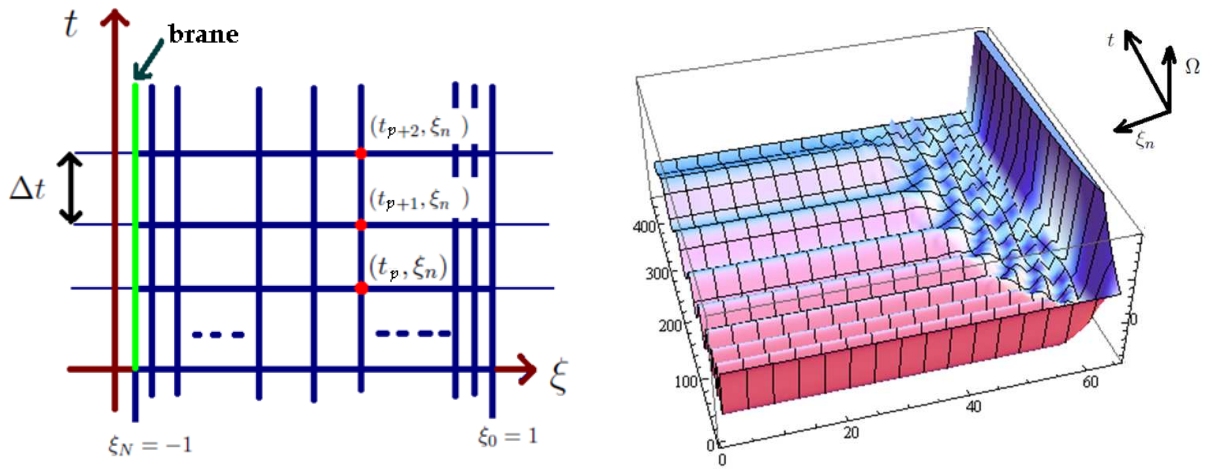


Figure 3: (*left panel*) The computational grid on which we solve the system of equations (35) and (63) with boundary condition (28). The physical brane is at  $\xi = \xi_N = -1$  while the regulatory one is at  $\xi = \xi_0 = 1$ . (*right panel*) An example of calculation results for master variable  $\Omega$  (in this case, we took  $k = 30k_c$  and  $N = 64$ , so brane is at  $\xi = \xi_{64} = -1$ ). The grid over  $n$  is shown to be homogenous, however, note, that the actual  $\xi_n$ -grid is inhomogeneous due to Eq. (67).

The dependence of  $M_{BH}^*$  on  $t$  for  $\ell = 0.1$  mm is shown in Fig. 1.

One should note, in a conclusion of this section, that if the accretion efficiency is not small, the initial masses of the PBHs are smaller than its masses at the onset of the evaporation. And, even without any accretion, initial masses of the 5D PBHs are, for the same values of total lifetime, much smaller than initial masses in the standard cosmology. It means that in the 5D case the known astrophysical constraints on the PBH abundance correspond to primordial perturbations on smaller scales.

The dependence of horizon mass on time and  $k$  in brane cosmology is shown in Fig. 2. It is seen that PBHs with mass less than  $\sim 10^{25}$  g are produced in the high energy regime. The corresponding comoving wave numbers are larger than  $10^{11}$  Mpc $^{-1}$ . If  $\ell \approx 0.1$  mm then PBHs evaporating today correspond to  $k \gtrsim 10^{17}$  Mpc $^{-1} \sim 10^6 k_c$ .

## 5 Numerical scheme

It follows from Sec. 4 that in brane cosmology PBHs having, at formation, relatively small masses ( $\lesssim (10^9 - 10^{10})$  g) and, in particular, evaporating near today, had been produced long before the critical epoch,  $t_{form} \ll t_c$ . The corresponding comoving sizes of perturbed regions are also small,  $k^{-1} \sim (10^{-16} - 10^{-17})$  Mpc. Therefore, it is practically important to determine the enhancement factors for rather large values of  $k$ ,  $k \gtrsim (10^6 - 10^7)k_c$ . The straightforward numerical calculation of these factors for such large  $k$  are quite difficult and unreliable, but, luckily, the numerical calculations for moderately large  $k$ ,  $k \sim (10 - 30)k_c$ , show the flattening and the possible saturation of the dependence of the enhancement factor  $\mathcal{Q}$  on  $k$ . We argue now that, really,  $\mathcal{Q}(k)$  does not depend on  $k$  in the limit of very large  $k$ ,  $k \gg k_c$ .

For the numerical solution of the system of equations (23) and (35) with boundary condition (28), a pseudo-spectral calculation method was employed. Such methods are often used in the tasks of hydrodynamics and a detailed description can be found, e.g., in [45].

To be able to perform a spectral transformation over the set of Chebyshev polynomials we do a following change in the variables:

$$\Omega(\tau, z) \rightarrow \Omega(t, \xi), \quad (60)$$

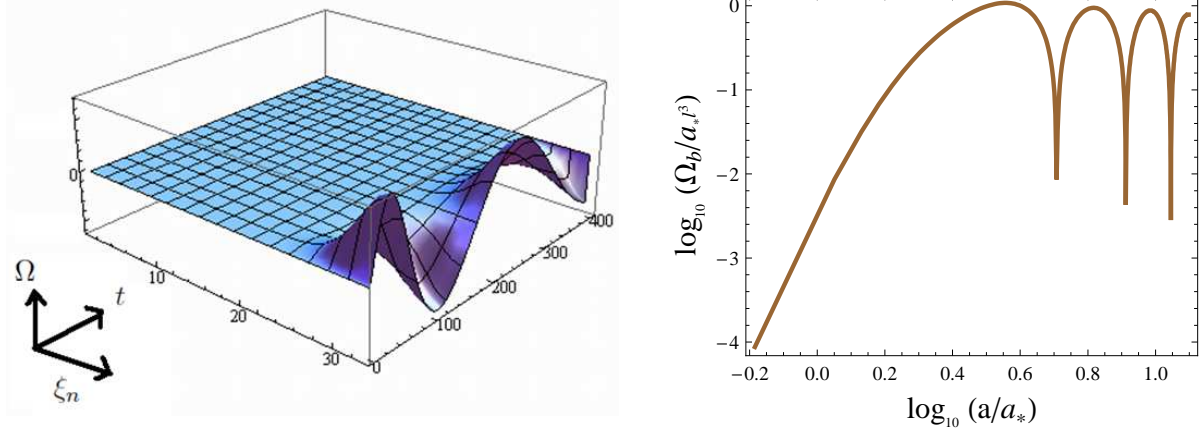


Figure 4: The result of the numerical calculation of master variable  $\Omega$  for  $k = 3k_c$ . *Left panel:* the illustration of  $\Omega(\xi_n, t)$  (here,  $N = 32$ , units over time and  $\Omega$ -axis are arbitrary); *right panel:* the value of  $\Omega$  on the brane,  $\Omega_b$ . The normalization is given by  $\zeta = 1$  for  $a \ll a_*$ .

$$\xi = \frac{2z - (z_{reg} + z_b(t))}{z_{reg} - z_b(t)}, \quad -1 \leq \xi \leq 1, \quad (61)$$

where  $t$  has the meaning of cosmic time on the brane and is related to  $\tau$  by

$$\frac{d\tau}{dt} = \frac{\sqrt{1 + \ell^2 H(t)^2}}{a(t)}, \quad (62)$$

while  $z_{reg}$  is the position of the regulatory boundary (artificial cutoff that is introduced to make a computational domain finite, see, e.g., [46, 47]).

The equation for  $\Omega$  (23) is rewritten in new variables as

$$\frac{\partial^2 \Omega}{\partial t^2} + K_{t\xi} \frac{\partial^2 \Omega}{\partial t \partial \xi} + K_{\xi\xi} \frac{\partial^2 \Omega}{\partial \xi^2} + K_t \frac{\partial \Omega}{\partial t} + K_\xi \frac{\partial \Omega}{\partial \xi} + K\Omega = 0, \quad (63)$$

where

$$K_{t\xi}(t, \xi) = 2 \frac{\partial \xi}{\partial \tau} \left( \frac{dt}{d\tau} \right)^{-1}, \quad (64a)$$

$$K_{\xi\xi}(t, \xi) = \left[ \left( \frac{\partial \xi}{\partial \tau} \right)^2 - \left( \frac{2}{z_{reg} - z_b(t)} \right)^2 \right] \left( \frac{dt}{d\tau} \right)^{-2}, \quad (64b)$$

$$K_t(t, \xi) = 2 \frac{d^2 t}{d\tau^2} \left( \frac{dt}{d\tau} \right)^{-2}, \quad (64c)$$

$$K_\xi(t, \xi) = \left( 2 \frac{\partial^2 \xi}{\partial \tau^2} - \frac{6}{z(t, \xi)(z_{reg} - z_b(t))} \right) \left( \frac{dt}{d\tau} \right)^{-2}, \quad (64d)$$

$$K(t, \xi) = - \left( \frac{1}{z(t, \xi)^2} - k^2 \right) \left( \frac{dt}{d\tau} \right)^{-2}. \quad (64e)$$

Further, a new variable  $\chi$  related to time derivative of  $\Omega$  is introduced [46] to reduce the task to two first-order equations:

$$\frac{\partial \Omega}{\partial t} = \chi - K_{t\xi} \frac{\partial \Omega}{\partial \xi} \equiv F(\chi, \Omega'_\xi; t, \xi), \quad (65)$$

$$\frac{\partial \chi}{\partial t} = -K_{\xi\xi} \frac{\partial^2 \Omega}{\partial \xi^2} + \left( \frac{\partial K_{t\xi}}{\partial t} - K_\xi \right) \frac{\partial \Omega}{\partial \xi} - K_t \left( \chi - K_{t\xi} \frac{\partial \Omega}{\partial \xi} \right) - K\Omega \equiv G(\chi, \Omega, \Omega'_\xi, \Omega''_{\xi\xi}; t, \xi). \quad (66)$$

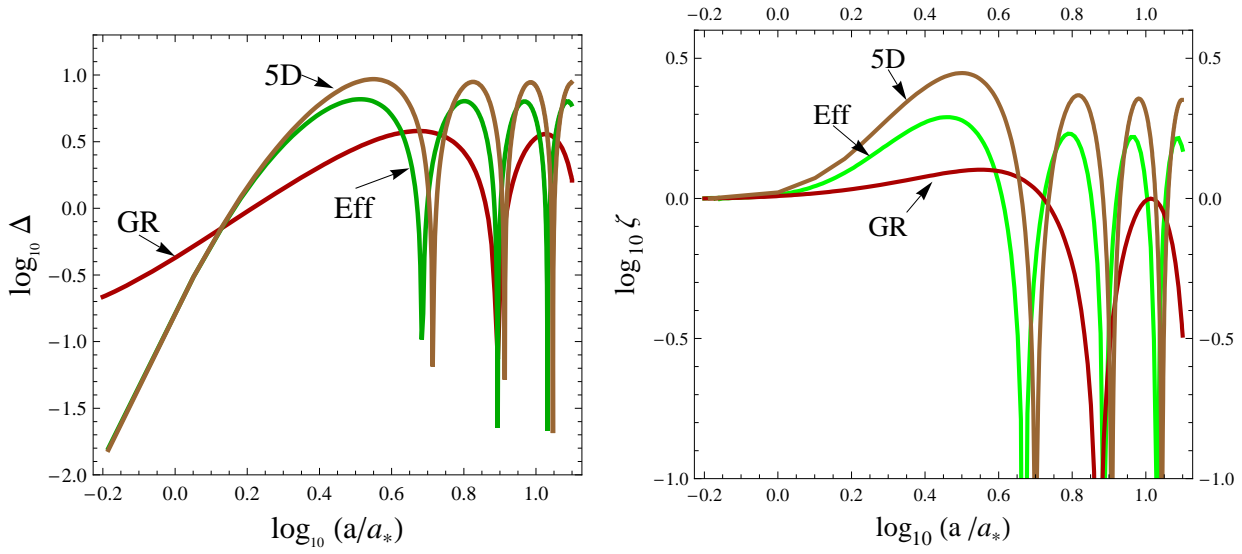


Figure 5: The calculation of the density contrast and curvature perturbation on the brane for  $k = 3k_c$  for the cases of full 5-dimensional calculation (curves labeled “5D”), using effective approach (approximation of  $\Omega_b = 0$ , curves labeled “Eff”) and General Relativity, i.e. standard cosmology (“GR”). Note that the value of  $a_*$  is the same for “5D” and “Eff” cases, but is different for “GR” because of the different evolution of the background quantities. *Left panel:* Comoving density contrast  $\Delta$  as a function of the scale factor, normalized to  $\zeta = 1$  in super-horizon regime. *Right panel:* Curvature perturbation  $\zeta$  calculated using the same three approaches.

To solve this system using difference method, the transformation of all quantities over set of Chebyshev polynomials is done for the  $\xi$  (and  $\chi$ ) - axis. This is done at each time step so that system of partial differential equations (65, 66) reduces to the system of ordinary differential equations.

Thus, at each point  $(t_p, \xi_n)$ , the following quantities are known:

$$\chi, \Omega, \Omega'_\xi, \Omega''_{\xi\xi}, F, G,$$

and also known are Chebyshev transforms

$$\tilde{\chi}_n, \tilde{\Omega}_n, (\tilde{\Omega}'_\xi)_n, (\tilde{\Omega}''_{\xi\xi})_n, \tilde{F}_n, \tilde{G}_n.$$

The grid (see Fig. 3 for illustration) based on Gauss-Lobatto points is used here,

$$\xi_n = \cos\left(\frac{\pi n}{N}\right), \quad n = 0, 1, \dots, N, \quad (67)$$

because it allows to perform fast Fourier transforms (FFTs) between the set of values of any variable (e.g.,  $\Omega$ ) in Gauss-Lobatto points and its Chebyshev components  $\tilde{\Omega}_n$ . Chebyshev transforms of derivatives, such as  $(\tilde{\Omega}'_\xi)_n$  and  $(\tilde{\Omega}''_{\xi\xi})_n$ , are also easily obtained using recurrence relations from the Chebyshev components of the function (see [45] for details).

Equations that are actually solved on each time step are:

$$\frac{d\tilde{\Omega}_n}{dt} = \tilde{F}_n(t); \quad \frac{d\tilde{\chi}_n}{dt} = \tilde{G}_n(t). \quad (68)$$

For points on the brane,  $\Delta(t_p)$  is also evaluated at each step and Eq. (35) is solved using a finite difference method (in our case, a 4-th order Adams-Bashforth-Moulton scheme).

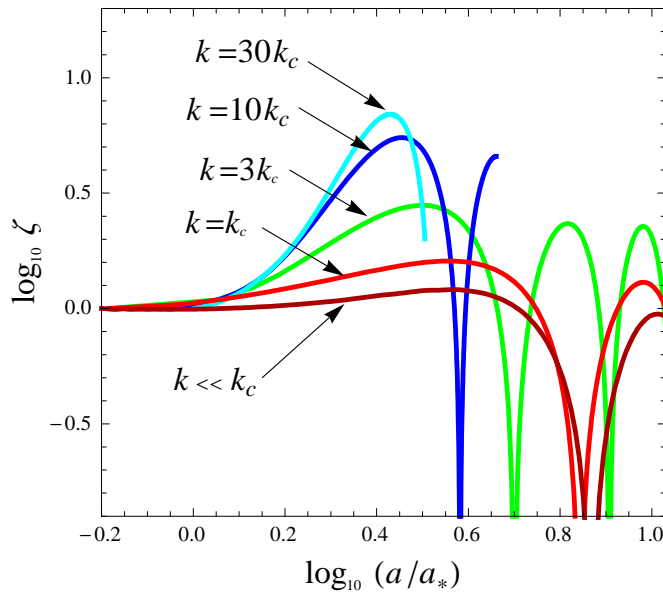


Figure 6: The result of the numerical calculation of  $\zeta$  for different values of  $k$ .

The boundary conditions are imposed on the values of the highest two components of the master variable,  $\tilde{\Omega}_N$  and  $\tilde{\Omega}_{N-1}$ . This is done by demanding the following:

$$\Omega'_\xi(-1) = \sum_{n=0}^N \tilde{\Omega}_n T'_n(-1), \quad (69a)$$

$$\Omega'_\xi(1) = \sum_{n=0}^N \tilde{\Omega}_n T'_n(1), \quad (69b)$$

where  $T_n(\xi)$  is the  $n$ -th order Chebyshev polynomial. The value of  $\Omega'_\xi(1)$  is assumed to be zero (condition on the regulatory brane) while the value on the physical brane,  $\Omega'_\xi(-1)$ , is related to other quantities by the boundary condition (28) which can be expanded as

$$\left( \frac{\partial \Omega}{\partial \xi} \right)_{\xi=-1} = \left( \frac{\frac{H\ell\chi}{\sqrt{1+H^2\ell^2}} - \frac{1}{\ell} \left( 1 + \frac{\rho}{\sigma} \right) \Omega - \frac{6\rho a^3 \Delta}{\sigma k^2}}{\frac{2\sqrt{1+H^2\ell^2}}{a(z_{reg}-z_b(t))} - \frac{H\ell}{a} \frac{\partial \xi}{\partial \tau} + \frac{K_{t\xi} H\ell}{\sqrt{1+H^2\ell^2}}} \right)_{\xi=-1}. \quad (70)$$

Following the approach of [46], we also use the following additional condition:  $\tilde{\chi}_N = \tilde{\chi}_{N-1} = 0$ .

## 6 Results and conclusions

The results of the calculations for the particular wave number of the perturbation are shown in Figs. 4, 5. In Fig. 4, the result for the master variable calculation is shown. In Fig. 5, density and curvature perturbation amplitudes are shown for three cases: full (5-dimensional) numerical calculation, for calculation using approximation of effective theory ( $\Omega_b = 0$ ) and for conventional calculation for 4-dimensional General Relativity case.

It is seen from Fig. 5 that the amplitudes of perturbations get enhanced compared to the standard (General Relativity) case. The sources of this enhancement are additional terms in equations for perturbations (see Sec. 2). It is also seen that the full calculation gives a larger enhancement in perturbation amplitude, compared with the effective theory approach, which only takes into account the quadratic corrections to the equations, neglecting the contribution of bulk degrees of freedom.

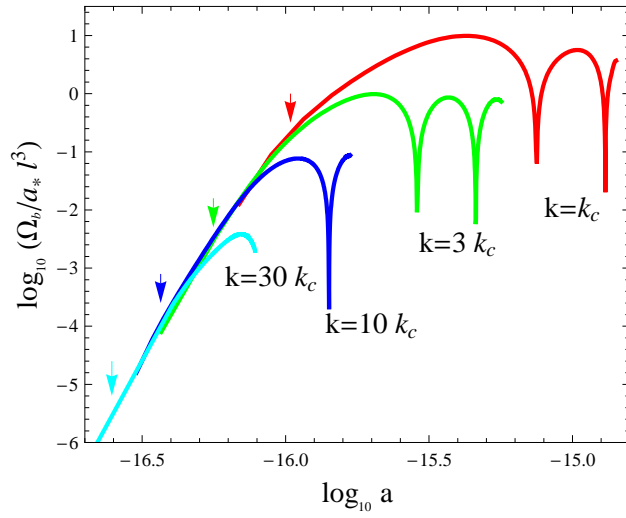


Figure 7: The result of the numerical calculation of  $\Omega_b$  for different values of  $k$ , normalized to  $\zeta_k = 1$  in the super-horizon regime. Arrows show the horizon crossing time ( $k = aH$ ) for each mode.

We can define the following factors, that show the degree of enhancement of the perturbation amplitude [22]:

$$\mathcal{Q}_{eff} = \frac{\Delta_{eff}}{\Delta_{GR}}, \quad \mathcal{Q}_{\mathcal{E}} = \frac{\Delta_{5D}}{\Delta_{eff}}, \quad \mathcal{Q}_{5D} = \frac{\Delta_{5D}}{\Delta_{GR}} = \mathcal{Q}_{eff} \mathcal{Q}_{\mathcal{E}}. \quad (71)$$

In a case of the effective theory, the enhancement reaches an asymptotic value,  $\mathcal{Q}_{eff} \approx 3$ , at  $k \sim 100k_c$ . However, the direct calculation of  $\mathcal{Q}_{5D}$  (or, equivalently,  $\mathcal{Q}_{\mathcal{E}}$ ) for very large wave numbers  $k \gg k_c$  is not easy, due to a quite complicate behavior of  $\Omega$  in the bulk (see Fig. 3 for an illustration: the larger value of  $k$ , the more frequent are oscillations in the bulk). Due to limitations of computing resources, we have been able to make direct calculations in 5D case only for a limited range of  $k \lesssim 100k_c$ .

However, to explore PBH production for masses  $M_{BH}^*(t_0) \sim 10^9$  g (such PBHs, as we have

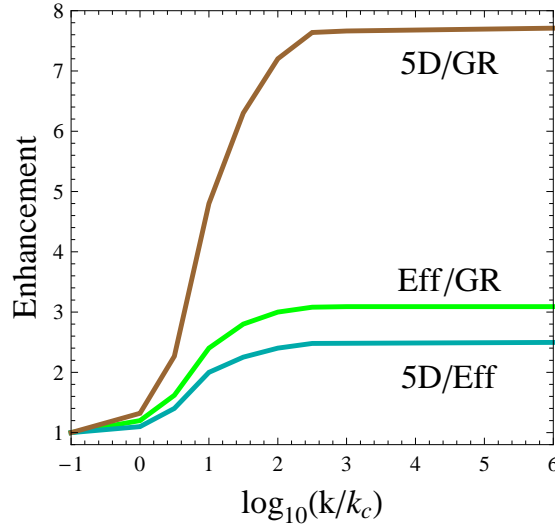


Figure 8: Enhancement factors that show the degree of increasing of the perturbation amplitude after horizon entry. From bottom to top, curves show the enhancement of the amplitude of 5-dimensional calculation compared to the effective one, effective theory compared to General Relativity and 5-dimensional calculation compared to General Relativity.



seen in Sec. 4, evaporate near today, if the value of  $\ell$  is close to its upper bound (21)), we need information about cosmological perturbations for  $k \gtrsim 10^6 k_c$ . To perform calculations for such large wave numbers, we have used an approximate approach according to which  $\Omega_b(a/a_*)$  has the same form for all rather large wave numbers (see discussion in Sec. 3). The time dependence of  $\zeta$  and  $\Omega_b$  for several values of  $k$  is shown in Figs. 6 and 7. For calculations at large values of  $k$  ( $k \gtrsim 100k_c$ ) we have used the required function  $f_\Omega(a/a_*)$  from numerical calculation for  $k = 30k_c$ . Using this approach, we have calculated the enhancement factors for large values of  $k$  ( $k \gtrsim 100k_c$ ), for which the direct use of numerical scheme described in Sec. 5 is not possible. The results of the calculation are shown in Fig. 8.

In summary, we have analyzed the model parameters set and have shown that to treat PBH formation in the early braneworld Universe, for small PBH masses (such as PBHs that should evaporate today), information about behavior of cosmological perturbations at very large wave numbers (up to  $(10^6 - 10^7)k_c$ ) is needed. Because for such wave numbers the full direct calculation of perturbations does not seem to be possible, we have developed an approximate approach to treat such large wave numbers. We have calculated the enhancement factors for perturbation amplitudes using this approach, for a wide range of cosmological scales,  $\sim (10^{10} - 10^{18}) \text{ Mpc}^{-1}$ .

We have presented the details of numerical scheme (using a pseudo-spectral calculation method) that can be implemented to treat scalar cosmological perturbations on the brane and in the bulk numerically. The basic perturbation equations have been adopted here to numerical calculation and the detailed expressions were given.

We have shown, using our approximate approach, that the enhancement factor, as a function of  $k$ , is constant, in the asymptotic limit of high energies.

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