Back reaction of accretion onto black hole

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Abstract

We calculate a back reaction of the accreted matter near the event horizon of the Reissner-Nordström black hole. It is shown that a test fluid approximation for the accreted matter is violated near the extremely charged black hole.

1 Einstein equations

A spherically symmetric gravitational field may be written in the general form with two arbitrary functions [1]):

$$ds^2 = e^\nu dt^2 - e^\lambda dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2). \tag{1}$$

We will find the small corrections to the Reissner-Nordström metric due to a back reaction of the stationary spherically symmetric accreted fluid. For application to the Reissner-Nordström metric of the charged black hole, we define two new metric functions, f_0 and f_1, and also two "mass functions" $m_0(r,t)$ and $m_1(r,t)$:

$$e^\nu = g_{00} = f_0 \equiv 1 - \frac{2m_0(t,r)}{r} + \frac{e^2}{r^2}, \tag{2}$$

$$e^{-\lambda} = g_{11}^{-1} = f_1 \equiv 1 - \frac{2m_1(t,r)}{r} + \frac{e^2}{r^2}, \tag{3}$$

where e — is an electric charge of the black hole. In the case of the pure Reissner-Nordström metric (i. e., in the absence of accreting fluid), both mass functions equal to the black hole mass, $m_0 = m_1 = m = const$, and, respectively, $f_0 = f_1$. A spherically symmetric gravitational field in the general case is defined by the four Einstein equations. Three of the them are the differential equations of a first order and the fourth one is of a second order. These equations for metric (1) have the following form [1]:

$$8\pi T_0^1 = -e^{-\lambda}\frac{\dot\lambda}{r}, \tag{4}$$

$$8\pi T_0^0 = -e^{-\lambda}\left(\frac{1}{r^2} - \frac{\lambda'}{r}\right) + \frac{1}{r^2}, \tag{5}$$

$$8\pi T_1^1 = -e^{-\lambda}\left(\frac{1}{r^2} + \frac{\nu'}{r}\right) + \frac{1}{r^2}, \tag{6}$$

$$8\pi T_2^2 = \frac{e^{-\nu}}{2}\left[\ddot\lambda + \frac{\dot\lambda}{2}\left(\dot\lambda - \dot\nu\right)\right] - \frac{e^{-\lambda}}{2}\left[\nu'' + (\nu' - \lambda')\left(\frac{\nu'}{2} + \frac{1}{x}\right)\right]. \tag{7}$$

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The energy-momentum tensor components in these equations for a spherically symmetric distribution of perfect fluid around the black hole are

$$
\begin{aligned}
T_0^1 &= (\rho+p)u\sqrt{\frac{f_0}{f_1}(f_1+u^2)}, && (8)\\
T_0^0 &= \rho+(\rho+p)\frac{u^2}{f_1}+\frac{e^2}{8\pi r^4}, && (9)\\
T_1^1 &= -\left[(\rho+p)\frac{u^2}{f_1}+p\right]+\frac{e^2}{8\pi r^4}, && (10)\\
T_2^2 &= -p-\frac{e^2}{8\pi r^4}, && (11)
\end{aligned}
$$

where $u = dr/ds$ — is a radial component of the fluid 4-velocity, and respectively, ρ and ρ is an energy density and pressure of fluid in the comoving frame. Below it is supposed an arbitrary equation of state $p = p(\rho)$, relating the fluid pressure and energy density. The Bianci identity holds true for the Einstein equations, and so only three equations from four in (8)–(11) are independent. For these independent equations we choose (4), (5) and (6).

2 Self-consistency of accretion solution

There is a small parameter in the considered problem: the fluid energy density far from the black hole (at the infinity), $\rho_\infty m^2 \ll 1$, or the accretion rate $\dot{m} = 4\pi m^2 A(\rho_\infty + p_\infty)$, which is proportional to the small parameter $\rho_\infty m^2$. The second supposed small parameter is a slowness of the black hole mass changing, $\dot{m}/m \ll c_s/m$ (the stationary limit). This second small parameter come to the first one if a fluid sound velocity c_s is not extremely small. With a small parameter $\rho_\infty m^2 \ll 1$, the test fluid approximation in the background metric is valid in the region $x \ll X_{\max} = (\rho_\infty m^2)^{-1/3}$. The back reaction of fluid may be found by perturbation method due to existence of this small parameter.

Zero approximation corresponds to the electro-vacuum background Reissner-Nordström metric. The first approximation is a stationary spherically symmetric inflow of the test fluid in the background Reissner-Nordström metric. The corresponding solution [2, 3, 6] defines the conserved radial flux of energy $\dot{m}$ and the radial dependance for the 4-velocity component $u = dr/ds = u(r)$, for energy density $\rho = \rho(r)$ and for pressure $p = p(\rho) = p(r)$. Respectively, this solution fixes all components of the energy-momentum tensor T^{ik}. For self-consistency of the accretion problem in the background metric, the radial flux of energy must be small$\dot{m} \ll 1$. In the second approximation we will take into account the linear contributions with respect to $\dot{m} \ll 1$ to the energy momentum-tensor in the Einstein equations. As a result we find the deviation of metric from the background one, i. e. the back reaction with a linear accuracy with respect to $\dot{m}$.

The requested solution in the full space-time depends on specific form of the fluid equation of state. However, it will be shown that in the vicinity of the black hole event horizon the solution is independent on the equation of state in the linear approximation with respect on $\dot{m}$. This universal behavior of the back reaction near the event horizon allow to calculate the radius of modified horizon, i. e. the shift of black hole horizon radius under influence of the accreting matter. Additionally, we find a consistency condition for the test fluid approximation in the accretion problem. Formally, to find a perturbative modification of the black hole horizon, it is needed consider the region, where both $f_0 \ll 1$ and $f_1 \ll 1$.

3 Back reaction in the Schwarzschild metric

As a first step we find the back reaction of accretion near the event horizon of the Schwarzschild black hole. Remind, that in the considered approximation, the value of the energy flux is a small parameter, $\dot{m} \ll 1$. The first Einstein equation (4) defines the conserved radial flux of anergy, i. e. the matter accretion rate,

$$\dot{m} = -4\pi r^2 (\rho + p) u \sqrt{\frac{f_0}{f_1}(f_1 + u^2)} = const. \tag{12}$$

The value of this flux $\dot{m} = 4\pi A m^2 (\rho_\infty + p_\infty)$ is defined (in the first approximation) from the solution of the test fluid accretion in the background metric [2, 3], where ρ_∞ and p_∞ are, respectively, an energy density and pressure of the accreting fluid far from the black hole, at $r \gg m$ and the numerical constant A depends on the fluid equation of state $p = p(\rho)$ (see, e. g. [4, 5, 6]). Under the used linear approximation with respect to $\dot{m} \ll 1$, on the modified event horizon $r = r_+$ the following conditions are satisfied: $f_0(r_+, t) = f_1(r_+, t) = 0$, or, equivalently, $m_0(r_+, t) = m_1(r_+, t)$, and $f_0(r_+, t)/f_1(r_+, t) = 1$. The Einstein equation (4) in the linear approximation with respect to $\dot{m} \ll 1$ takes the simple form

$$\frac{\partial m_1}{\partial t} = \dot{m}. \tag{13}$$

As a result, in the considered approximation it is possible to search the solution of equation (13) in the factorized form $m_1(r, t) = m(t)\, \mu_1(r)$, where the dimensionless function $\mu_1(r)$ is defined so, that $\mu_1(r) = 1$ at $\dot{m} = 0$. After substitution of the factorized ansatz in (13) we obtain

$$\frac{dm_1(t)}{dt}\, \mu_1(r) = \dot{m}. \tag{14}$$

The r.h.s. of this equation is already linear with respect to $\dot{m} \ll 1$. So, for the dimensionless function $\mu_1(r)$ in this equation it is needed to use a zero approximation on $\dot{m}$, i. e. it is needed to put $\mu_1(r) = 1$. In result, the partial differential equation (13) for the function $m_1(r, t)$ reduces to the ordinary differential equation for $m(t)$ in the linear with respect to $\dot{m}$ approximation. The corresponding solution of this reduced equation is

$$m(t) = m(0) + \int_0^t \dot{m}(t')dt'. \tag{15}$$

Here $m(0)$ — is a black hole mass at the initial moment $t = 0$, $m(t)$ — is a current value of black hole mass at time t (a black hole mass is slowly changing due to accretion).

At this step we may find a radial dependance of the function $\mu_1(r)$ with a linear accuracy with respect to $\dot{m}$ with the help of the second Einstein equation (5). In the linear approximation with respect to $\dot{m}$, the functional dependencies of $m_0(r, t)$ and $m_1(r, t)$ on radius r and time t are factorized. For this reason it is useful in the following to use the dimensionless radial variable $x = r/m(t)$ and dimensionless mass functions $\mu_0(r) = m_0(r, t)/m(t)$ and $\mu_1(r) = m_1(r, t)/m(t)$.

The second Einstein equation (5) may be written in the form

$$\frac{d\mu_1}{dx} = 4\pi x^2 \left[\rho + (\rho + p)\frac{u^2}{f_1}\right]. \tag{16}$$

A combined solution of equations (12) and (16) with the using of accretion solution for fluid with equation of state $p = p(\rho)$, defines the requested function μ_1. From (12) and (16) near the event horizon, where $f_0 \ll 1$ and $f_1 \ll 1$, we obtain

$$\frac{d\mu_1}{dx} \simeq \frac{4\pi x^2 (\rho + p) u^2}{f_1} \simeq \frac{2\dot{m}}{x - 2\mu_1(x)}. \tag{17}$$

We transform the nonlinear equation (17) to the linear one, by introducing the new variable $\delta(x) \equiv x - 2\mu_1(x) \ll 1$. After the transformation of derivative

$$\frac{d\mu_1(x)}{dx} = \frac{d\mu_1(x)}{d\delta(x)}\frac{d\delta(x)}{dx} = \frac{d\mu_1(x)}{d\delta}\left[1 - 2\frac{d\mu_1(x)}{dx}\right], \tag{18}$$

we rewrite the differential equation (17) with respect to the variable $\delta(x)$:

$$\frac{d\mu_1}{d\delta} = \frac{\frac{d\mu_1}{dx}}{1 - 2\frac{d\mu_1}{dx}} = \frac{2\dot{m}}{\delta - 4\dot{m}}, \tag{19}$$

where $|\delta| \ll 1$. Solution of the resulting linear equation (19) at $|\delta| \ll 1$ (i. e. near the modified horizon) is:

$$\mu_1(\delta) = \mu_+ + 2\dot{m}\log|1 - \frac{\delta}{4\dot{m}}|, \tag{20}$$

Here $\mu_+ = \mu(x_+)$ is an integration constant, which in general depends on $\dot{m}$. Solution (20) for the inverse function $x = x(\mu_1)$ is written in the explicit form:

$$x(\mu_1) = x_+ + 4\dot{m}\left(1 + \frac{\mu_1 - \mu_+}{2\dot{m}} - \exp\frac{\mu_1 - \mu_+}{2\dot{m}}\right), \tag{21}$$

where $x_+ = 2\mu_+$. It is important to note that this solution describes the perturbed metric in the narrow radial region $|x - 2\mu_1(x)| \ll 1$ around the modified event horizon.

4 Boundary conditions

To find the integration constant $\mu_+ = x_+/2$ in (20)we need to use the boundary conditions. Formally, we may suppose that distribution of fluid around the black hole is a sphere of some finite radius X_0, satisfying the condition $1 \ll X_0 \ll X_{\rm max} = (\rho_\infty m^2)^{-1/3}$. It can be always satisfied, if $\dot{m} \ll 1$. On the boundary of the fluid sphere with the external empty space at $x = X_0$ it is needed to make a smooth connection with an external Schwarzschild solution.

After integration of the r.h.s. of equation (16) in the limits from x_+ to X_0 we obtain:

$$\int_{x_+}^{X_0}\left[4\pi\rho x^2 + \frac{4\pi u^2x^2(\rho+p)}{f}\right]dx \simeq -2\dot{m}\log|4\dot{m}|. \tag{22}$$

The main contribution to this integral $\dot{m} \ll 1$ comes from lower limit at $x = x_+$, where the subintegral function reduces to (19). Actually, the multiplier 4 in the term with logarithm in (22) is in excess of accuracy of the used linear approximation with respect to $\dot{m} \ll 1$, and we will omit in in the following.

We neglect in (22) the contribution from the integral

$$\mu_f = \int_{x_+}^{X_0} 4\pi\rho x^2 dx \simeq (4\pi/3)\rho_\infty m^2 X_0^3 \sim \dot{m}X_0^3, \tag{23}$$

which is a mass of accreting fluid inside the sphere of radius X_0, and also the other contribution from the second integral in (22) near the upper limit, which is of the order $\dot{m}/X_0$, and which is the gravitational "mass defect". The condition $X_0^3 \ll \log|\dot{m}|$, if the parameter $\dot{m}$ is sufficiently small. Under this condition, the dominating contribution to (22) comes from the term with logarithm.

Accordingly, after integration in the l.h.s. of equation (16) we obtain:

$$\int_{x_+}^{X_0}\frac{d\mu_1(x)}{dx}dx \simeq \int_{\mu_+}^{1} d\mu = 1 - \mu_+. \tag{24}$$

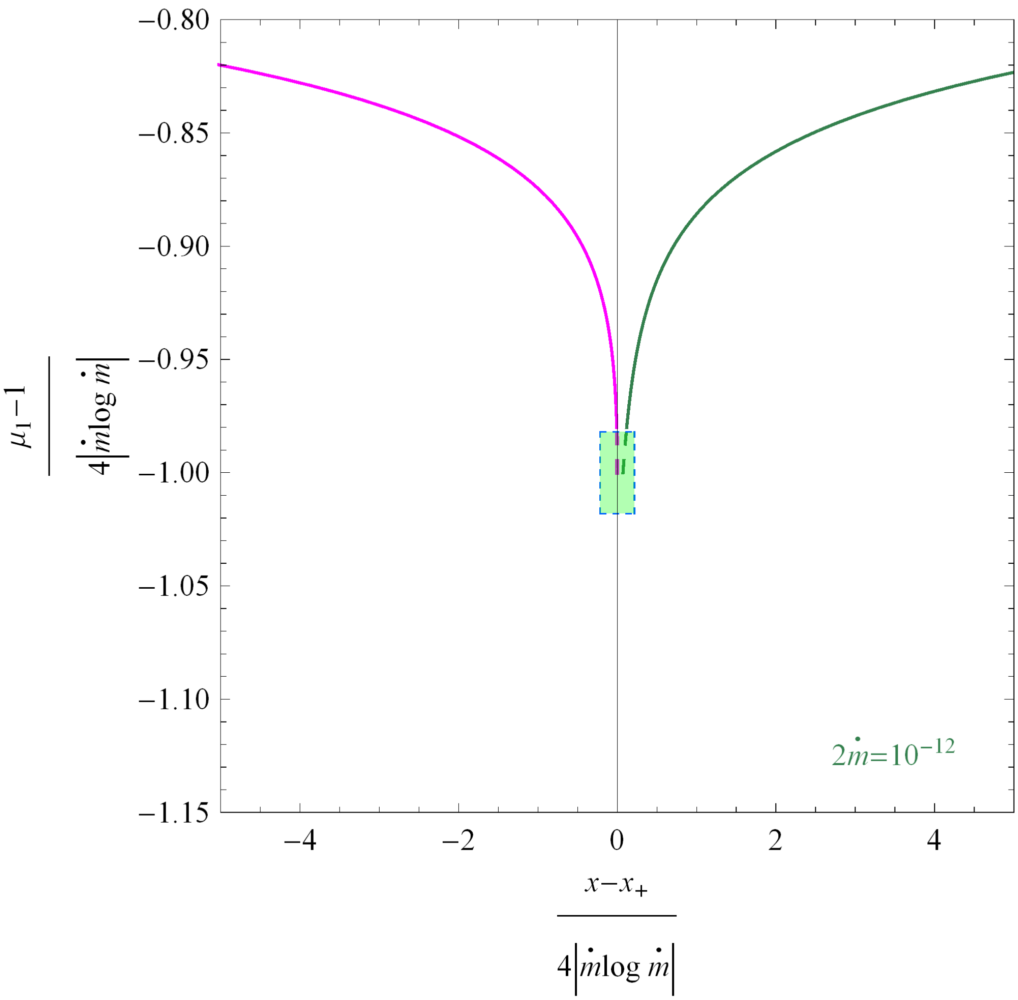

Figure 1: Modification of the Schwarzschild metric by the back reaction from accreted matter. The mass function $\mu_1(x)$ from equation (20) near the event horizon of black hole x_+ in the linear approximation with respect to $\dot{m} \ll 1$. Inside the filled box the used linear approximation is insufficient for the determination of mass function $\mu_1(x)$.

Here we also neglected the contribution from fluid to the integral near the upper limit with respect to μ_+. By combining (24) and (22), we find

$$\mu_+ = \mu_1(x_+) \simeq 1 + 2\dot{m} \log |\dot{m}|. \tag{25}$$

The third Einstein Equation (6) has the form

$$\frac{f_1}{f_0}\mu_0' + \frac{1}{x}\left(\mu_1 - \mu_0\frac{f_1}{f_0}\right) = -4\pi x^2\left[(\rho+p)\frac{u^2}{f_1} + p\right]. \tag{26}$$

Near the event horizon this equation may be written as

$$\mu_0' \simeq -4\pi x^2(\rho+p)\frac{u^2}{f_1} \simeq -\mu_1'. \tag{27}$$

Here it is taken into account that ratio f_0/f_1 near the horizon equals to its background value, $(f_0/f_1)_+ = 1$, at the linear approximation with respect to $\dot{m} \ll 1$. Solution of equation (27) near the event horizon, where $\delta = x - 2\mu_1(x) \ll 1$), with the help of (20) is

$$\mu_0(\delta) \simeq \mu_+ - 2\dot{m}\log|1 - \frac{\delta}{4\dot{m}}|, \tag{28}$$

By comparing (20) and (28), we see that near the horizon it is satisfied the condition

$$\mu_0(\delta) + \mu_1(\delta) = x_+. \tag{29}$$

As a result, in the linear approximation with respect to $\dot{m}$ in the Einstein equations, the corrections to the Schwarzschild metric due to the back reaction of accreted matter are everywhere small, even on the modified horizon, $\dot{m}\log|\dot{m}| \to 0$ at $\dot{m} \to 0$.

5 Back reaction in the Reissner-Nordström metric

Quite similar to the Schwarzschild case, we find in the implicit form the mass function $\mu_1(x)$ near the event horizon of the modified Reissner-Nordström metric (1) — (3) due to the back reaction of the stationary accreted fluid:

$$\mu_1(x) \simeq 1 + \frac{(1+\epsilon)^2}{2\epsilon}\dot{m}\log\left|\frac{(1+\epsilon)^3}{2\epsilon^2}\dot{m} - \delta\right|, \tag{30}$$

where it is used the black extreme parameter $\epsilon = \sqrt{1 - e^2/m^2}$ and is introduced a new variable $\delta(x) \equiv x - [\mu_1(x) + \sqrt{\mu_1(x)^2 - 1 + \epsilon^2}] \ll 1$. The value of the mass function on the modified horizon $\mu_1(x_+)$ corresponds to $\delta = 0$:

$$\mu_1(x_+) \simeq 1 + \frac{(1+\epsilon)^2}{2\epsilon}\dot{m}\log\left|\frac{(1+\epsilon)^3}{2\epsilon^2}\dot{m}\right|, \tag{31}$$

where $x_+^0 = 1 + \epsilon$. The resulting radius of the modified horizon is

$$x_+ = \mu_1(x_+) + \sqrt{\mu_1(x_+)^2 - 1 + \epsilon^2} \simeq (1+\epsilon)\left[1 + \frac{1}{2}\left(\frac{1+\epsilon}{\epsilon}\right)^2 \dot{m}\log\left|\frac{\dot{m}}{\epsilon^2}\right|\right]. \tag{32}$$

From equations (31) and (32) it follows, that a test fluid approximation *is violated* due to the back reaction of the accreted fluid in the limit $\epsilon \to 0$, when a black hole is approaching to the extreme state. Namely, the correction to the radius of the black hole event horizon diverges for an arbitrarily small accretion inflow $\dot{m}$, if $\epsilon \to 0$. This behavior is in agreement with the cosmic censorship conjecture [7] and with the third law of black hole thermodynamics [8]: the extreme state is unattainable in the finite processes or, in other words, it is impossible to transform the black hole into the naked singularity. To resolve the problem with the back reaction of accreting matter on the extreme black hole it is requested solution of the Einstein equations beyond the perturbation level.

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