Extension of Generalized Haag's Theorem on Spaces with Arbitrary Dimensions

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Abstract

Generalized Haag's theorem is proved in $SO(1,k)$ invariant quantum field theory in any n-dimensional space with $n \geq k+1$. These additional dimensions can be noncommutative.

1 Introduction

In this report we consider one of the most important result of axiomatic approach in quantum field theory (QFT) - the generalized Haag's theorem [1], [2].

Let us recall that in the usual Hamiltonian formalism it is assumed that asymptotic fields at any time are related with interacting fields by unitary transformation. The Haag's theorem shows that in accordance with Lorentz invariance of the theory in this case the interacting fields are also trivial which means that corresponding S-matrix is equal to unity. So the usual interaction representation can not exist in the Lorentz invariant theory. In four dimensional case in accordance with the generalized Haag's theorem four first Wightman functions coincide in two related by the unitary transformation theories.

It is known that n-point Wightman functions $W(x_1, \ldots, x_n)$ are $\langle \Psi_0, \varphi(x_1) \ldots \varphi(x_n) \Psi_0 \rangle$, where Ψ_0 is a vacuum vector. Actually in accordance with translation invariance Wightman functions are functions of variables $\xi_i = x_i - x_{i+1}$, $i = 1, \ldots, n-1$. At first Haag's theorem is proved in $SO(1,3)$ invariant theory in four dimensional case.

Now the theories in spaces of arbitrary dimensions are widely considered. In last years noncommutative generalization of QFT - NC QFT - attracts great interest of physicists as in some cases NC QFT is a low-energy limit of string theory [3].

NC QFT is defined by the Heisenberg-like commutation relations between coordinates

$$[x^\mu, x^\nu] = i\,\theta^{\mu\nu}, \tag{1}$$

where $\theta^{\mu\nu}$ is a constant antisymmetric matrix.

It is very important that NC QFT can be formulated in commutative space if the usual product between operators (precisely between corresponding test functions) is substituted by

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the $\star$ (Moyal-type) product [4], [5]. Let us recall that the $\star$-product between two functions $\varphi(x)$ and $\psi(y)$ is defined as follows:

$$\varphi(x) \star \psi(y) = \exp\left(\frac{i}{2}\,\theta^{\mu\nu}\,\frac{\partial}{\partial x^{\mu}}\,\frac{\partial}{\partial y^{\nu}}\right)\varphi(x)\psi(y). \tag{2}$$

Thus it is interesting to consider Haag's theorem in the general case of $k+1$ commutative variables (time and k spatial coordinates) and arbitrary number of other coordinates, which can include noncommutative coordinates as well. This extension of the Haag's theorem is a goal of our work.

2 Generalized Haag's theorem in four dimensional case

Note that in axiomatic QFT there is no field operator defined in a point [2]. Only the smoothed operators written symbolically as

$$\varphi_f \equiv \int \varphi\,(x)\,f\,(x)\,d\,x, \tag{3}$$

where $f\,(x)$ are test functions, can be rigorously defined.

In the derivation of Haag's theorem it is necessary to assume that fields smeared on the spatial coordinates are also well defined operators.

Let us recall Haag's theorem in four dimensional case [1], [2].

Generalized Haag's Theorem *Let $\varphi_1(f,t)$ and $\varphi_2(f,t)$ belong to two sets of irreducible operators in Hilbert spaces H_1 and H_2 correspondingly. We assume that both theories are Poincare invariant, that is*

$$U_j(a,\Lambda)\,\varphi_j(x)U_j^{-1}(a,\Lambda) = \varphi_j(\Lambda x + a), \tag{4}$$

$$U_j(a,\Lambda)\Psi_{0j} = \Psi_{0j}, \qquad j = 1,2. \tag{5}$$

Assume that there exists the unitary transformation V, which relates fields in question as well as vacuum states in two theories at any t:

$$\varphi_2(f,t) = V\varphi_1(f,t)V^{-1}, \tag{6}$$

$$c\Psi_{02} = V\Psi_{01}, \tag{7}$$

where c is a complex number with module one.

It is also supposed that vacuum states in two theories are invariant under this unitary transformation.

If in two theories there are not states with negative energies, then four first Wightman functions coincide in two theories.

Let us give the idea of the proof.

The invariance of the vacuum states implies that Wightman functions coincide at equal time:

$$(\Psi_{01}, \varphi_1(t,x_1), \ldots, \varphi_1(t,x_n)\Psi_{01}) = (\Psi_{02}, \varphi_2(t,x_1), \ldots, \varphi_2(t,x_n)\Psi_{02}) \tag{8}$$

Let us recall that in accordance with spectral properties of Wightman functions, which imply non-existence of tachyon states, they are analytical functions in tubes [1], [2]. Then from $S\,O\,(1,3)$ symmetry it follows (Bargman-Hall-Wightman theorem) that, actually, Wightman functions are analytical functions in the dilated domain - so called extended tubes [1], [2], [6]. It is very important, that extended tubes contain real points - Jost points. The most important property of these points is the following one: the interval between two arbitrary Jost points r_k and r_l is space-like:

$$(r_k - r_l)^2 < 0. \tag{9}$$

Thus any Jost point belongs to the set of Jost points with its vicinity. So Jost points fully determine Wightman functions, i.e. two Wightman functions, which coincide at Jost points, precisely, in the open subset of these points, coincide identically.

Let us notice that at equal time all points x_i belong to the set of Jost points. It can be shown that the equality of Wightman functions at equal time and their analyticity lead to equality of four first Wightman functions in two theories related by unitary transformation at equal time.

Let us stress that noncommutative coordinates belong to the boundary of analyticity of Wightman functions. Thus the same derivation of Haag's theorem can be done in the presence of arbitrary number of these coordinates.

3 Derivation of Haag's theorem in the $SO(1,k)$ invariant theory

As in the derivation of Haag's theorem only transformations of coordinates, which belong to the domain of analyticity, are essential, we omit the dependence of vectors under consideration on noncommutative variables.

First let us notice that as at $n > k$ vectors $\xi_i = (0, \vec{\xi_i})$ are linear dependent, then vectors related by them with Lorentz transformation are linear dependent too and thus they can not form the open subset of Jost points. Thus they can not determine Wightman functions.

Let us show that if $n \leqslant k$, then corresponding Jost points fully determine Wightman functions.

Indeed, if $n \leqslant k$, one can choose n linear-independent vectors $\tilde{\xi_j} = \left(0, \vec{\tilde{\xi_j}}\right), j = 1, \ldots, n$. Actually, as $\vec{\tilde{\xi_1}}$ is an arbitrary vector, we can choose $\vec{\tilde{\xi_2}}$ in such a way that $\tilde{\xi}_1 \perp \tilde{\xi}_2$. In a similar manner we can construct vector $\tilde{\xi}_3$ such that $\tilde{\xi}_3 \perp \tilde{\xi}_1$, $\tilde{\xi}_3 \perp \tilde{\xi}_2$. Continuing this procedure, which is similar to the well-known Gram-Shmidt orthogonalization procedure, we obtain the set of orthogonal vectors $\left\{\hat{\xi_j}\right\} : \hat{\xi_i} \perp \hat{\xi_j}, i \neq j$. Let us recall that Whigtman functions in two theories coincide on the set of vectors $\left\{\hat{\xi_j}\right\}$ and thus they coincide on the vectors, which are obtained from them by Lorentz transformation.

Evidently $\alpha_i \hat{\xi_i} \perp \alpha_j \hat{\xi_j}$ and the vectors, which are obtained from them by Lorentz transformation, will be orthogonal: $\xi_{j\,\alpha} = \Lambda\left(\alpha \hat{\xi_j}\right), (j = 1, \ldots, n; \alpha \in \mathbb{R})$. Set of orthogonal nonzero vectors is linear-independent, thus vectors $\xi_{j\,\alpha}$ form open subset of Jost points.

As in two theories related by an unitary transformation at equal time first $k+1$ Wightman functions coincide on the open set of Jost points, then these functions coincide in all points. So we have proved the coincidence of all Wightman functions till k-point one (in difference variables). Passing from difference variables to usual ones $(x_1, \ldots, x_n)$, we obtain, that in case of $SO(1,k)$-invariance for and any number of noncommutative variables the first $k+1$ Wightman functions coincide in two theories.

4 Conclusion

The generalized Haag's theorem has been proved in $SO(1,k)$ invariant theory, where k is arbitrary. Let us stress that this result is valid in the space, which includes also arbitrary number of noncommutative coordinates.

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