Consequences of the Generalized Haag's Theorem

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Abstract

Consequences of the generalized Haag's theorem are obtained in $SO(1,1)$, $SO(1,3)$ invariance of the theory as well as in the general case of $SO(1,k)$ symmetry with arbitrary k. For two fields connected by unitary transformation at equal time it is proved that:
in case of $SO(1,1)$ symmetry from triviality of one of fields the triviality of the other follows;
in case of $SO(1,3)$ symmetry elastic cross sections and so total ones are equal;
in case of $SO(1,k)$, $k > 3$ not only elastic and total cross sections are equal, but also amplitudes of some inelastic processes coincide.

1 Introduction

Let us recall that Haag's theorem [1], [2] considers two theories, in which quantum fields operators at equal time as well as corresponding vacuum states Ψ_0^i are related by the unitary operator:

$$1)\quad \varphi_f^2(t) = V\,\varphi_f^1(t)\,V^*, \tag{1}$$

$$2)\quad \Psi_0^2 = C\,V\,\Psi_0^1, \quad C \in \mathbb{C}, \quad |C| = 1. \tag{2}$$

In accordance with the generalized Haag's theorem four first Wightman functions coincide in two theories in usual Lorentz invariant case. Let us recall that by definition Wightman functions are:

$$W(x_1, \dots, x_n) = \langle\, \Psi_0, \varphi(x_1)\dots\varphi(x_n)\,\Psi_0\,\rangle.$$

The main consequence of the generalized Haag's theorem is that from triviality of one of the fields in question it follows that other one is trivial too, which implies that corresponding S-matrix is equal to unity.

It is known that in axiomatic quantum field theory (QFT) there is no field operator defined in a point [2]. Only the smoothed operators written symbolically as

$$\varphi_f \equiv \int \varphi(x)\,f(x)\,d\,x, \tag{3}$$

where $f(x)$ are test functions, can be rigorously defined.

In the formulation of Haag's theorem it is assumed that the formal operators $\varphi_i(t, \vec{x})$ can be smeared only on the spatial variables. This assumption is natural also in noncommutative

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case if time commutes with spatial coordinates (space-space noncommutativity). NC QFT is defined by the Heisenberg-like commutation relations between coordinates

$$[x^\mu, x^\nu] = i\,\theta^{\mu\nu}, \tag{4}$$

where $\theta^{\mu\nu}$ is a constant antisymmetric matrix. In the case of space-space noncommutativity $\theta^{0i} = 0$. Let us point out that, moreover, in this case there exists one spatial variable, which commutes with the rest variables [3].

It is very important that NC QFT can be formulated in commutative space if the usual product between operators (precisely between corresponding test functions) is substituted by the $\star$ (Moyal-type) product [4], [5].

At first Haag's theorem is proved in $SO(1,3)$ invariant theory in four dimensional case. The proof of the generalized Haag's theorem in general case of $SO(1,k)$ symmetry is done in our report "Extension of Generalized Haag's Theorem on Spaces with Arbitrary Dimensions", published in these Proceedings. Let us stress that besides of $k+1$ commutative variables (time and k spatial coordinates) the space under consideration can contain arbitrary number of other coordinates, which can include noncommutative coordinates as well.

In space-space NC QFT in four dimensional space Haag's theorem was considered in [6]and[7].

In this report we obtain separately the consequences from the generalized Haag's theorem in $SO(1,1)$, $SO(1,3)$ and $SO(1,k)$, $k>3$ invariant cases. In two first cases our consideration is a generalization of the proof given in [7].

2 $SO(1,1)$ invariant theory

First let us consider the generalized Haag's theorem in the $SO(1,1)$ invariant field theory. In accordance with the result obtained in the above mentioned our report, in two $SO(1,k)$ invariant theories related by an unitary transformation at equal time first $k+1$ Wightman functions coincide. Thus equality of only two-point Wightman functions takes place in $SO(1,1)$ invariant theory.

Let us prove that if one of considered theories is trivial, that is the corresponding S-matrix is equal to unity, then another is trivial too.

Let us point out that in the $SO(1,1)$ invariant theory it is sufficient that the spectral condition, which implies non existence of tachyons, is valid only in respect with commutative coordinates. Also it is sufficient that translation invariance is valid only in respect with the commutative coordinates. The equality of two-point Wightman functions in two theories leads to the following conclusion: if local commutativity condition in respect with commutative coordinates is fulfilled and the current in one of the theories is equal to zero, then another current is zero as well.

Indeed as $W_1(x^1, x^2) = W_2(x^1, x^2)$, then also

$$\left\langle \Psi_0^1, j_{\bar{f}}^1 j_f^1\ \Psi_0^1 \right\rangle \;=\; \left\langle \Psi_0^2, j_{\bar{f}}^2 j_f^2\ \Psi_0^2 \right\rangle, \tag{5}$$

where

$$j_f^i = (\Box + m^2)\,\varphi_f^i.$$

If, for example, $j_f^1 = 0$, then in the space with positive metric

$$j_f^2\,\Psi_0^2 = 0. \tag{6}$$

From the latter formula and local commutativity condition it follows that [2]

$$j_f^2 \equiv 0. \tag{7}$$

Our statement is proved.

3 $SO(1,3)$ invariant theory

Let us proceed now to the $SO(1,3)$ invariant theory, precisely, to the theory with four commutative coordinates and arbitrary number of noncommutative ones. In this case we show that from the equality of the four-point Wightman functions for the fields $\varphi_f^1(t)$ and $\varphi_f^2(t)$, related by the conditions (1) and (2), which takes place in the theory in question, an essential physical consequence follows. Namely, for such fields the elastic scattering amplitudes in the corresponding theories coincide, hence, due to the optical theorem, the total cross-sections coincide as well. In the derivation of this result the local commutativity condition is not used. In particular, if one of these fields, for example, φ_f^1 is a trivial field, also the field φ_f^2 is trivial.

The statement follows directly from the Lehmann-Symanzik-Zimmermann reduction formulas [8].

Here and below in order not to complicate formulas we consider operators $\varphi_1(x)$ and $\varphi_2(x)$ as they are given in a point.

Let $< p_3, p_4|p_1, p_2 >_i,\ i = 1,2$ be elastic scattering amplitudes for the fields $\varphi_1(x)$ and $\varphi_2(x)$ respectively. Owing to the reduction formulas,

$$< p_3, p_4|p_1, p_2 >_i \sim$$

$$\int d\,x_1 \cdots d\,x_4\, e^{i\,(-p_1\,x_1 - p_2\,x_2 + p_3\,x_3 + p_4\,x_4)} \cdot \prod_{j=1}^{4} (\Box_j + m^2)\ < 0|T\,\varphi_i(x_1)\ \cdots\ \varphi_i(x_4)|0 >, \tag{8}$$

where $T\,\varphi_i(x_1)\ \cdots\ \varphi_i(x_4)$ is a chronological product of operators.

From the equality

$$W_2(x_1, \ldots, x_4) = W_1(x_1, \ldots, x_4)$$

it follows that

$$< p_3, p_4|p_1, p_2 >_2 = < p_3, p_4|p_1, p_2 >_1 \tag{9}$$

for any p_i. Having applied this equality for the forward elastic scattering amplitudes, we obtain that, according to the optical theorem, the total cross-sections for the fields $\varphi_1(x)$ and $\varphi_2(x)$ coincide. If now the S-matrix for the field $\varphi_1(x)$ is unity, then it is also unity for the field $\varphi_2(x)$. Let us point out that in four dimension space the equality of the four-point Wightman functions in two theories, related by the unitary transformation, is valid only in the commutative field theory, but not in the noncommutative case.

4 General Case

Now let us proceed to the general case, i.e. to $SO(1,k)$ invariant theory. We prove that in addition to equality of elastic and total cross sections the equality of amplitudes of some inelastic processes takes place. In accordance with the corresponding reduction formula

$$< p_3, p_4, \ldots p_n|p_1, p_2 >_i \sim$$

$$\int d\,x_1 \cdots d\,x_n\, e^{i\,(-p_1\,x_1 - p_2\,x_2 + p_3\,x_3 + p_4\,x_4 + \cdots + p_n\,x_n)} \cdot \prod_{j=1}^{n} (\Box_j + m^2)\ < 0|T\,\varphi_i(x_1)\ \cdots\ \varphi_i(x_n)|0 > . \tag{10}$$

Let us consider $2 \Rightarrow n$ processes. We see that owing to the above mentioned our result amplitudes of these processes coincide in two theories if $n \leqslant k + 1$.

5 Conclusions

We have proved that if one of $SO(1,1)$ invariant theories describes a trivial field, then field, connected with the first one by an unitary transformation at equal time, is trivial as well.

In $SO(1,3)$ invariant theory we have proved equality of elastic and consequently total cross sections in two theories under consideration.

In $SO(1,k)$ invariant theory in addition we have proved the equality of amplitudes of some inelastic processes.

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