Analytic properties of high energy production amplitudes in $N = 4$ SUSY

L. N. Lipatov[a,b]

[a] II. Institut für Theoretische Physik, Hamburg University, Germany

[b] St. Petersburg Nuclear Physics Institute, Gatchina, Russia

Abstract

We investigate analytic properties of the six point planar amplitude in $N = 4$ SUSY at the multi-Regge kinematics for final state particles. For inelastic processes the Steinmann relations play an important role because they give a possibility to fix the phase structure of the Regge pole and Mandelstam cut contributions. The analyticity and factorization constraints allow us to reproduce the two-loop correction to the 6-point BDS amplitude in $N = 4$ SUSY obtained yearlier in the leading logarithmic approximation with the use of the s-channel unitarity. The cut contribution has the Möbius invariant form in the transverse momentum subspace. The exponentiation hypothesis for the amplitude in the multi-Regge kinematics is also investigated in LLA.

1 Introduction

The elastic scattering amplitude in QCD at high energies $\sqrt{s}$ and fixed momentum transfers $q = \sqrt{-t}$ for the transition $AB \to A'B'$ with the definite particle helicities λ_i in the leading logarithmic approximation (LLA) has the Regge form [1]

$$A_{2\to 2} = 2\, g\delta_{\lambda_A\lambda_{A'}} T^c_{AA'} \frac{s^{1+\omega(t)}}{t}\, g\, T^c_{BB'}\, \delta_{\lambda_B\lambda_{B'}}\,,\;\; t = -\vec{q}^{\,2}. \tag{1}$$

The gluon Regge trajectory $j(t) = 1 + \omega(t)$ in LLA is given below

$$\omega(-\vec{q}^{\,2}) = -\frac{\alpha_s N_c}{(2\pi)^2}\,(2\pi\mu)^{2\epsilon} \int d^{2-2\epsilon}k\, \frac{\vec{q}^{\,2}}{\vec{k}^2(\vec{q}-k)^2} \approx -\,a\left(\ln\frac{\vec{q}^{\,2}}{\mu^2} - \frac{1}{\epsilon}\right)\,, \tag{2}$$

where we introduced the dimensional regularization ($D = 4 - 2\,\epsilon$) and the renormalization point μ for the t' Hooft coupling constant

$$a = \frac{\alpha_s\, N_c}{2\pi}\,\left(4\pi e^{-\gamma}\right)^{\epsilon}\,. \tag{3}$$

The gluon trajectory is also known in the next-to-leading approximation at QCD [2] and in SUSY gauge models [3].

For finding the total cross-section in LLA it is enough to calculate the production amplitudes in the multi-Regge kinematics for the final state gluons. They have the simple factorized form [1]

$$A_{2\to 2+n} = -2\,s\,g\,\delta_{\lambda_A\lambda_{A'}}\,T^{c_1}_{AA'}\frac{s_1^{\omega(-\vec{q}_1^{\,2})}}{\vec{q}_1^{\,2}} gC_\mu(q_2,q_1)e^*_\mu(k_1)T^{d_1}_{c_2c_1}\frac{s_2^{\omega(-\vec{q}_2^{\,2})}}{\vec{q}_2^{\,2}}\ldots\frac{s_{n+1}^{\omega(-\vec{q}_{n+1}^{\,2})}}{\vec{q}_{n+1}^{\,2}}\,g\,\delta_{\lambda_B\lambda_{B'}}\,T^{c_{n+1}}_{BB'}\,, \tag{4}$$

where

$$s = (p_A+p_B)^2 \gg s_r = (k_r+k_{r-1})^2 \gg \vec{q}_r^{\,2}\,,\;\; k_r = q_{r+1} - q_r\,. \tag{5}$$

The matrices T^a_{bc} are the generators of the $SU(N_c)$ gauge group in the adjoint representation and $C_\mu(q_r, q_{r-1})$ are the effective Reggeon-Reggeon-gluon vertices. In the case when the polarization vector $e_\mu(k_1)$ describes a produced gluon with a definite helicity one can obtain [4]

$$C \equiv C_\mu(q_2, q_1)\, e^*_\mu(k_1) = \sqrt{2}\, \frac{q_2^* q_1}{k_1^*}\,, \tag{6}$$

where the complex notation $q = q_x + iq_y$ for the two-dimensional transverse vectors was used.

The elastic scattering amplitude with vacuum quantum numbers in the t-channel can be calculated with the use of s-channel unitarity [1]. In this approach the Pomeron appears as a composite state of two Reggeized gluons. It is convenient to present transverse gluon coordinates in a complex form together with their canonically conjugated momenta

$$\rho_k = x_k + iy_k\,,\ \rho^*_k = x_k - iy_k\,,\ p_k = i\frac{\partial}{\partial \rho_k}\,,\ p^*_k = i\frac{\partial}{\partial \rho^*_k}\,. \tag{7}$$

In the coordinate representation the Balitsky-Fadin-Kuraev-Lipatov (BFKL) equation for the Pomeron wave function can be written as follows [1]

$$E\,\Psi(\vec{\rho}_1, \vec{\rho}_2) = H_{12}\,\Psi(\vec{\rho}_1, \vec{\rho}_2)\,,\ \Delta = -\frac{\alpha_s N_c}{2\pi}\ \min\, E\,, \tag{8}$$

where Δ is the Pomeron intercept entering in the expression $\sigma_t \sim s^\Delta$ for the high energy asymptotics of the total cross section. The BFKL Hamiltonian has a simple operator representation [5]

$$H_{12} = \ln\,|p_1 p_2|^2 + \frac{1}{p_1 p_2^*}(\ln\,|\rho_{12}|^2)\,p_1 p_2^* + \frac{1}{p_1^* p_2}(\ln\,|\rho_{12}|^2)\,p_1^* p_2 - 4\psi(1) \tag{9}$$

with $\rho_{12} = \rho_1 - \rho_2$ and $\psi(x) = \gamma'(x)/\Gamma(x)$. The kinetic energy is proportional to the sum of two gluon Regge trajectories $\omega(-|p|_i^2)$ $(i = 1, 2)$. The potential energy $\sim \ln\,|\rho_{12}|^2$ is related to the product of two gluon production vertices C_μ. The Hamiltonian is invariant under the Möbius transformation [6]

$$\rho_k \to \frac{a\rho_k + b}{c\rho_k + d}\,, \tag{10}$$

where a, b, c and d are complex parameters. The eigenvalues of two Casimir operators are expressed in terms of the corresponding conformal weights

$$m = \frac{1}{2} + i\nu + \frac{n}{2}\,,\ \widetilde{m} = \frac{1}{2} + i\nu - \frac{n}{2} \tag{11}$$

and for the principal series of unitary representations of $SL(2, C)$ the parameter ν is real and n is integer.

It turns out, that the BFKL pomeron has the positive intercept $\Delta = g^2 N_c \ln 2\,/\pi^2$ in LLA, which is not compatible with the s-channel unitarity. To restore the unitarity one should take into account the diagrams with an arbitrary number of Reggeized gluons in the t-channel. The composite states of these gluons are described by the Bartels-Kwiecinski-Praszalowicz (BKP) equation [7]. In the $N_c \to \infty$ limit the corresponding Hamiltonian has the property of holomorphic separability [8]

$$H = \frac{1}{2}\sum_k H_{k,k+1} = \frac{1}{2}(h + h^*)\,,\ [h, h^*] = 0\,. \tag{12}$$

The holomorphic Hamiltonian is a sum of the BFKL hamiltonians $h_{k,k+1}$

$$h = \sum_k h_{k,k+1}\,,\ h_{12} = \ln(p_1 p_2) + \frac{1}{p_1}\,(\ln \rho_{12})\,p_1 + \frac{1}{p_2}\,(\ln \rho_{12})\,p_2 - 2\psi(1)\,. \tag{13}$$

Consequently, the wave function Ψ has properties of holomorphic factorization [8] and duality symmetry under the transformation [9]

$$p_i \to \rho_{i,i+1} \to p_{i+1} \,. \tag{14}$$

Moreover, in the holomorphic and anti-holomorphic sectors, there are integrals of motion commuting among themselves and with h [5, 10].

The integrability of BFKL dynamics was firstly demonstrated in [10]. It is related to the fact that h in LLA coincides with a local Hamiltonian of the Heisenberg spin model [11].

In the next-to-leading logarithmic approximation the integral kernel for the BFKL equation was constructed in Refs. [3, 12]. Due to its Möbius invariance, solutions of the BFKL and BKP equations can be classified by the anomalous dimension $\gamma = \frac{1}{2} + i\nu$ of twist-2 operators and the conformal spin $|n|$.

The eigenvalue of the BFKL kernel in the next-to-leading approximation was calculated initially in QCD (see ref. [12]). It contains the contributions proportional to the Kronecker symbols $\delta_{n,0}$ and $\delta_{n,2}$. But in $N = 4$ SUSY these nonanalytic terms are cancelled and a simple expression having the property of the hermitian separability was obtained [3, 13]. Futhermore, the final result in two loops is a sum of special functions having the property of maximal transcendentality [13]. In a different context, one-loop anomalous dimension matrix for twist-2 operators in this model was calculated and its eigenvalues turned out to be proportional to $\psi(1) - \psi(j-1)$, which is related to the integrability of the evolution equation for the quasi-partonic operators in $N = 4$ SUSY [14]. The integrability in this model has also been established for other operators and in higher loops [15, 16].

The maximal transcendentality principle suggested in Ref. [13] allowed to extract the universal anomalous dimension up to three loops in $N = 4$ SUSY [17, 18] from the QCD results [19]. This principle was also helpful for finding a closed integral equation for the cusp anomalous dimension in this model [20, 21] satisfying the AdS/CFT correspondence [22, 23, 24]. In the framework of the asymptotic Bethe ansatz with wrapping corrections the maximal transcendentality principle gave a possibility to calculate the anomalous dimension up to five loops [25] in an agreement with the BFKL predictions. Moreover, the intercept of the BFKL Pomeron at a large 't Hooft coupling constant in $N = 4$ SUSY was found in Refs. [18, 26]. Next-to-leading corrections to the BFKL equation can be obtained with the use of the effective action for the reggeized gluon interactions [27, 28].

A simple ansatz for gluon production amplitudes with the maximal helicity violation in a planar limit for $N = 4$ SUSY was suggested by Bern, Dixon and Smirnov [29]. This ansatz for the elastic case at large coupling was confirmed by Alday and Maldacena [31]. However, later for the multi-particle production amplitude these authors obtain the result different from the BDS predictions [**?**]. It was shown in ref. [32], that already in the 6 point case the BDS ansatz is in a disagreement with the Steinmann relations [33] which are equivalent to the requirement, that the production amplitude does not have simultaneous singularities in overlapping channels. The BDS result was not confirmed also by direct two loop calculations [34]. The reason for the breakdown of the BDS ansatz is related to the fact, that the BDS amplitude for the transition $2 \to 4$ in the multi-Regge kinematics does not contain the Mandelstam cut contribution [35]. This new term appears in the j_2-plane of the t_2 channel at the physical kinematical regions, where the invariants in the direct channels have the following signs $s, s_2 > 0$; $s_1, s_3 < 0$ or $s, s_1, s_2, s_3 < 0$; $s_{012}, s_{123} > 0$ [32]. In LLA the cut contribution for the 6-point amplitude was calculated in LLA with the use of the BFKL equation [36]. The corresponding amplitude in the region $s, s_2 > 0$; $s_1, s_3 < 0$ can be written in the factorized form

$$M_{2\to 4} = M_{2\to 4}^{BDS}\,(1 + i\Delta_{2\to 4}) \tag{15}$$

where A^{BDS} is the BDS amplitude [29] and

$$\Delta_{2\to 4} = \frac{a}{2}\sum_{n=-\infty}^{\infty}(-1)^n \int_{-\infty}^{\infty}\frac{d\nu}{\nu^2+\frac{n^2}{4}}\left(\frac{q_3^* k_a^*}{k_b^* q_1^*}\right)^{i\nu-\frac{n}{2}}\left(\frac{q_3 k_a}{k_b q_1}\right)^{i\nu+\frac{n}{2}}\left(s_2^{\omega(\nu,n)}-1\right). \qquad (16)$$

Here k_a, k_b are transverse components of produced gluon momenta, q_1, q_2, q_3 are the momenta of reggeons in the corresponding crossing channels and

$$\omega(\nu,n) = 4a\,\Re\left(2\psi(1)-\psi(1+i\nu+\frac{n}{2})-\psi(1+i\nu-\frac{n}{2})\right). \qquad (17)$$

The correction Δ is Möbius invariant in the momentum space and can be written in terms of the four-dimensional anharmonic ratios [36] in an accordance with the results of refs. [37].

It was shown also, that in a general case of the Mandelstam cut corresponding to a composite state of n reggeized gluons the Hamiltonian coincides with the local Hamiltonian for an open integrable Heisenberg spin chain [38].

In this paper we reproduce some results of ref. [36] using general arguments based only on analyticity and factorization of the 6-point amplitude without any unitarity constraints incorporated in the BFKL approach. Also the exponentiation ansatz with an additional phase factor for the BDS amplitude is investigated in LLA.

2 Dispersion relation in multi-Regge kinematics

The BDS amplitude [29] for the transition $2\to 3$ in the multi-Regge kinematics can be written in the following form compatible with the Steinmann relation (see [32])

$$\frac{M_{2\to 3}^{BDS}}{\Gamma(t_1)\Gamma(t_2)} = (-s_1)^{\omega_{12}}(-s\kappa_{12})^{\omega_2}c_1^{12}+(-s_2)^{\omega_{21}}(-s\kappa_{12})^{\omega_1}c_2^{12}\,,\ \kappa_{12}=|k_a|^2\,, \qquad (18)$$

where $\Gamma(t_i)$ are the reggeized gluon residues, k_a is the transverse momentum of the produced particle and we put the normalization point μ^2 in the Regge factors equal to unity. The gluon Regge trajectories are

$$\omega_r = \omega(|q_r|^2) = -\frac{\gamma_K}{4}\ln\frac{|q_r|^2}{\lambda^2}\,,\ \gamma_K\approx 4a\,,\ a=\frac{g^2\,N_c}{8\pi^2}\,,\ \omega_{12}=\omega_1-\omega_2\,, \qquad (19)$$

where γ_K is the cusp anomalous dimension and $\lambda^2=\mu^2\exp(1/\epsilon)$ for $D=4-2\epsilon$ with $\epsilon\to -0$. The real coefficients c_1^{12}, c_2^{12} are given below [32]

$$c_1^{12} = |\Gamma_{12}|\,\frac{\sin\pi(\omega_1-\omega_a)}{\sin\pi\omega_{12}}\,,\ c_2^{12}=|\Gamma_{12}|\,\frac{\sin\pi(\omega_2-\omega_a)}{\sin\pi\omega_{21}}\,, \qquad (20)$$

where the Reggeon-Reggeon-gluon vertex Γ_{12} in the physical region $s, s_1, s_2 > 0$ is

$$\Gamma_{12}(\ln\kappa_{12}-i\pi) = |\Gamma_{12}|\,\exp(i\pi\,\omega_a)\,,\ \omega_a=\frac{\gamma_K}{8}\,\ln\frac{|k_a|^2\lambda^2}{|q_1|^2|q_2|^2}\,, \qquad (21)$$

$$\ln|\Gamma_{12}| = \frac{\gamma_K}{4}\left(-\frac{1}{4}\ln^2\frac{|k_a|^2}{\lambda^2}-\frac{1}{4}\ln^2\frac{|q_1|^2}{|q_2|^2}+\frac{1}{2}\ln\frac{|q_1|^2|q_2|^2}{\lambda^4}\ln\frac{|k_a^2|}{\mu^2}+\frac{5}{4}\zeta_2\right). \qquad (22)$$

It is well known, that one particle production amplitude with the reggeon exchanges having definite signatures $\tau_1, \tau_2 = \pm 1$ in the crossing channels t_1 and t_2 has the factorized form in all physical regions [39]

$$\frac{M_{2\to 3}^{\tau_1\tau_2}}{\Gamma(t_1)\Gamma(t_2)} = |s_1|^{\omega_1}\xi_1\,V^{\tau_1\tau_2}\,|s_2|^{\omega_2}\xi_2\,,\ V^{\tau_1\tau_2}=\frac{\xi_{12}}{\xi_1}\,c_1^{12}+\frac{\xi_{21}}{\xi_2}\,c_2^{12}\,, \qquad (23)$$

where

$$\xi_1 = e^{-i\pi\omega_1} - \tau_1\,,\ \xi_2 = e^{-i\pi\omega_2} - \tau_2\,,\ \xi_{12} = e^{-i\pi\omega_{12}} + \tau_1\tau_2\,,\ \xi_{21} = e^{-i\pi\omega_{21}} + \tau_1\tau_2\,. \tag{24}$$

Moreover, for two particles production in the multi-Regge kinematics the amplitude with definite signatures τ_i in three crossing channels can be also presented in the factorized form [39]

$$\frac{M_{2\to 4}^{\tau_1\tau_2\tau_3}}{\Gamma(t_1)\Gamma(t_3)} = |s_1|^{\omega_1}\xi_1\, V^{\tau_1\tau_2}\,|s_2|^{\omega_2}\xi_2\, V^{\tau_2\tau_3}\,|s_3|^{\omega_3}\xi_3\,, \tag{25}$$

where $V^{\tau_2\tau_3}$ is obtained from $V^{\tau_1\tau_2}$ (23) with the corresponding substitutions

$$V^{\tau_2\tau_3} = \frac{\xi_{23}}{\xi_2}\,c_1^{23} + \frac{\xi_{32}}{\xi_3}\,c_2^{23}\,. \tag{26}$$

For the second produced gluon with the transverse momentum k_b the coefficients c^{23} and phase ω_b are

$$c_1^{23} = |\Gamma_{23}|\,\frac{\sin\pi(\omega_2-\omega_b)}{\sin\pi\omega_{23}}\,,\ c_2^{23} = |\Gamma_{23}|\,\frac{\sin\pi(\omega_3-\omega_b)}{\sin\pi\omega_{32}}\,, \tag{27}$$

$$\omega_b = \frac{\gamma_K}{8}\ln\frac{|k_b|^2\lambda^2}{|q_2|^2|q_3|^2}\,,\ |k_b|^2 = \left|\frac{s_2 s_3}{s_{123}}\right|\,. \tag{28}$$

In an accordance with the Steinmann relations the Regge hypothesis leads to the following expression for the Regge pole contribution $M_{2\to 4}^{pole}$ [39, 32]

$$\frac{M_{2\to 4}^{pole}}{\Gamma(t_1)\Gamma(t_3)} = (-s_1)^{\omega_{12}}\,(-s_{012}\kappa_{12})^{\omega_{23}}\,(-s\kappa_{12}\kappa_{23})^{\omega_3}\,c_1^{12}\,c_1^{23}$$
$$+(-s_3)^{\omega_{32}}(-s_{123}\kappa_{23})^{\omega_{21}}\,(-s\kappa_{12}\kappa_{23})^{\omega_1}\,c_2^{12}\,c_2^{23} + (-s\kappa_{12}\kappa_{23})^{\omega_2}\,(-s_1)^{\omega_{12}}\,(-s_3)^{\omega_{32}}\,c_1^{12}\,c_2^{23}$$
$$+(-s_2)^{\omega_{21}}(-s_{012}\kappa_{12})^{\omega_{13}}\,(-s\kappa_{12}\kappa_{23})^{\omega_3}\,\frac{\sin\pi\omega_1}{\sin\pi\omega_2}\,\frac{\sin\pi\omega_{23}}{\sin\pi\omega_{13}}\,c_2^{12}\,c_1^{23}$$
$$+(-s_2)^{\omega_{23}}(-s_{123}\kappa_{23})^{\omega_{31}}\,(-s\kappa_{12}\kappa_{23})^{\omega_1}\,\frac{\sin\pi\omega_3}{\sin\pi\omega_2}\,\frac{\sin\pi\omega_{21}}{\sin\pi\omega_{31}}\,c_2^{12}\,c_1^{23}\,. \tag{29}$$

It is valid in all physical regions different by signs of momenta p_A, p_B, k_1 and k_2. Using the identity

$$\frac{\sin\pi\omega_1}{\sin\pi\omega_2}\,\frac{\sin\pi\omega_{23}}{\sin\pi\omega_{13}}\,\frac{\xi_{13}\xi_2}{\xi_{23}\xi_1} + \frac{\sin\pi\omega_3}{\sin\pi\omega_2}\,\frac{\sin\pi\omega_{21}}{\sin\pi\omega_{31}}\,\frac{\xi_{31}\xi_2}{\xi_{21}\xi_3} = 1\,, \tag{30}$$

one can verify the Regge factorization of the signatured amplitudes $M_{2\to 4}^{\tau_1\tau_2\tau_3}$ (25). Note, that there is another useful relation

$$\frac{\sin\pi\omega_1}{\sin\pi\omega_2}\,\frac{\sin\pi\omega_{23}}{\sin\pi\omega_{13}} + \frac{\sin\pi\omega_3}{\sin\pi\omega_2}\,\frac{\sin\pi\omega_{21}}{\sin\pi\omega_{31}} = 1\,. \tag{31}$$

The two-gluon production amplitude in the multi-Regge kinematics can be written as a sum of the Regge pole and Mandelstam cut contributions [32]

$$M_{2\to 4} = M_{2\to 4}^{pole} + M_{2\to 4}^{cut}\,, \tag{32}$$

where $M_{2\to 4}^{cut}$ is non-zero only in two kinematical regions restricted by the inequalities $s, s_2 > 0$; $s_1, s_3 < 0$ and $s, s_1, s_2, s_3 < 0$; $s_{012}, s_{123} > 0$.

The pole term (29) in the region $s, s_2 > 0$; $s_1, s_3 < 0$ is given below

$$\frac{M_{2\to 4}^{pole}}{|s_1|^{\omega_1}|s_2|^{\omega_2}|s_3|^{\omega_3}\,\Gamma(t_1)\Gamma(t_3)} = e^{-i\pi\omega_3}\,c_1^{12}\,c_1^{23} + e^{-i\pi\omega_1}\,c_2^{12}\,c_2^{23} + e^{-i\pi\omega_2}\,c_1^{12}\,c_2^{23}$$

$$+e^{-i\pi\omega_2}\left(e^{i\pi\omega_{13}}\frac{\sin\pi\omega_1}{\sin\pi\omega_2}\frac{\sin\pi\omega_{23}}{\sin\pi\omega_{13}}+e^{-i\pi\omega_{13}}\frac{\sin\pi\omega_3}{\sin\pi\omega_2}\frac{\sin\pi\omega_{21}}{\sin\pi\omega_{31}}\right)c_2^{12}\,c_1^{23}\,. \quad (33)$$

With the use of the relation

$$e^{i\pi\omega_{13}}\frac{\sin\pi\omega_1}{\sin\pi\omega_2}\frac{\sin\pi\omega_{23}}{\sin\pi\omega_{13}}+e^{-i\pi\omega_{13}}\frac{\sin\pi\omega_3}{\sin\pi\omega_2}\frac{\sin\pi\omega_{21}}{\sin\pi\omega_{31}}$$
$$=\cos\pi\omega_{13}+i\,\frac{\sin\pi\omega_1\,\sin\pi\omega_{23}+\sin\pi\omega_3\,\sin\pi\omega_{21}}{\sin\pi\omega_2} \quad (34)$$

this result can be simplified

$$\frac{M_{2\to4}^{pole}}{|s_1|^{\omega_1}|s_2|^{\omega_2}|s_3|^{\omega_3}\,|\Gamma_{12}||\Gamma_{23}|\,\Gamma(t_1)\Gamma(t_3)}=-e^{-i\pi\omega_2}\frac{\sin\pi\omega_{2a}}{\sin\pi\omega_{12}}\left(e^{-i\pi\omega_{1b}}+2i\frac{\sin\pi\omega_1}{\sin\pi\omega_2}\sin\pi\omega_{2b}\right)$$
$$+e^{-i\pi\omega_b}\,\frac{\sin\pi\omega_{1a}}{\sin\pi\omega_{12}}=\frac{2\,\sin\pi\omega_a\sin\pi\omega_b}{i\sin\pi\omega_2}+e^{-i\pi\omega_2}\,e^{i\pi(\omega_a+\omega_b)}\,. \quad (35)$$

We can present M^{pole} in the region $s,s_2>0;\ s_1,s_3<0$ as a sum of three contributions

$$\frac{M_{2\to4}^{pole}}{|s_1|^{\omega_1}|s_2|^{\omega_2}|s_3|^{\omega_3}|\Gamma_{12}||\Gamma_{23}|\,\Gamma(t_1)\Gamma(t_3)}=\frac{2e^{-i\pi\omega_2}\cos\pi\omega_2\,\sin\pi\omega_a\sin\pi\omega_b}{i\sin\pi\omega_2}$$
$$+ie^{-i\pi\omega_2}\sin\pi(\omega_a+\omega_b)+e^{-i\pi\omega_2}\cos\pi\omega_{ab}\,,\ \ \omega_{ab}=\frac{\gamma_K}{4}\,\ln\frac{|k_a||q_3|}{|k_b||q_1|}\,. \quad (36)$$

Here two first terms have the phase structure of the cut contribution $M_{2\to4}^{cut}$ considered below in (40) and can be included in it, which gives a possibility to redefine $M_{2\to4}^{pole}$ in the form

$$\frac{M_{2\to4}^{pole}}{|s_1|^{\omega_1}|s_2|^{\omega_2}|s_3|^{\omega_3}|\Gamma_{12}||\Gamma_{23}|\,\Gamma(t_1)\Gamma(t_3)}=e^{-i\pi\omega_2}\cos\pi\omega_{ab}\,. \quad (37)$$

Indeed, in an accordance with the above discussed representation for planar amplitudes in the multi-Regge kinematics the cut contribution can be presented as follows (it corresponds to the last two terms in the pole contribution (29)) (cf. [38])

$$M_{2\to4}^{cut}\sim(1-\Phi^{\omega_{13}})\,(-s_{012}\kappa_{12})^{\omega_{13}}\,(-s\kappa_{12}\kappa_{23})^{\omega_3}(-s_2)^{\omega_{21}}\int_{-i\infty}^{i\infty}\frac{d\omega_{2'}}{2\pi i}\,\phi(\omega_{2'})\,(-s_2)^{\omega_{2'}}\,. \quad (38)$$

Here we introduced the quantity Φ which coincides with the anharmonic ratio related to the conformal invariance of the production amplitudes in the momentum space

$$\Phi=\frac{ss_2}{s_{012}s_{123}}\,,\ \ 1-\Phi\approx\frac{|k_a+k_b|^2}{s_2} \quad (39)$$

and the partial wave $\phi(\omega_2)$ is real for real ω_2 and depends on various invariants in crossing channels. The above expression for $M_{2\to4}^{cut}$ is non-zero only in two regions, where $\Phi=\exp(\mp2\pi i)$ (really this fact fixes the relative coefficient of two terms at the first factor in (38). From this representation we conclude, that the phase structure of the cut contribution at $s,s_2>0,\ s_1,s_3<0$ (corresponding to $\Phi=\exp(-2\pi i)$) is

$$\frac{M_{2\to4}^{cut}}{|s_1|^{\omega_1}|s_2|^{\omega_2}|s_3|^{\omega_3}|\Gamma_{12}||\Gamma_{23}|\,\Gamma(t_1)\Gamma(t_3)}=i\,e^{-i\pi\omega_2}\int_{-i\infty}^{i\infty}\frac{d\omega_{2'}}{2\pi i}\,f(\omega_{2'})\,e^{-i\pi\omega_{2'}}\,|s_2|^{\omega_{2'}}\,. \quad (40)$$

The redefined partial wave $f(\omega_{2'})$ can contain the pole $\sim1/\omega_{2'}$, which allows one to absorb the terms $\sim i\exp(-i\pi\omega_2)$ from $M_{2\to4}^{pole}$ to $M_{2\to4}^{cut}$, as it was done in transition from (36) to (37).

In a similar way the pole and cut contributions in the region $s,s_1,s_2,s_3>0;\ s_{012},s_{123}>0$ $(\Phi=\exp(2\pi i))$ can be presented in the form

$$\frac{M_{2\to4}^{pole}}{|s_1|^{\omega_1}|s_2|^{\omega_2}|s_3|^{\omega_3}|\Gamma_{12}||\Gamma_{23}|\,\Gamma(t_1)\Gamma(t_3)}=\cos\pi\omega_{ab}\,, \quad (41)$$

$$\frac{M_{2\to4}^{cut}}{|s_1|^{\omega_1}|s_2|^{\omega_2}|s_3|^{\omega_3}|\Gamma_{12}||\Gamma_{23}|\,\Gamma(t_1)\Gamma(t_3)}=-i\int_{-i\infty}^{i\infty}\frac{d\omega_{2'}}{2\pi i}\,f(\omega_{2'})\,|s_2|^{\omega_{2'}}\,. \quad (42)$$

3 Factorization and analytic properties of $M_{2\to 4}$

The BDS amplitude in the multi-Regge kinematics for the physical channel in which $s, s_2 > 0$, $s_1, s_3 < 0$ is given below (see ref. [32])

$$\frac{M_{2\to 4}^{BDS}}{|s_1|^{\omega_1}|s_2|^{\omega_2}|s_3|^{\omega_3}\,|\Gamma_{12}||\Gamma_{23}|\,\Gamma(t_1)\Gamma(t_3)} = C\,e^{-i\pi\omega_2}\,e^{i\pi(\omega_a+\omega_b)} = e^{-i\pi\omega_2}\,e^{i\pi\delta}\,, \tag{43}$$

where

$$\delta = \frac{\gamma_K}{4}\,\ln\frac{|q_1||q_2||k_a||k_b|}{|k_a+k_b|^2|q_2|^2} \tag{44}$$

and we used the following expression for the phase factor C

$$C = \exp\left(\frac{\gamma_K}{4}\,i\pi\ln\frac{|q_1|^2|q_3|^2}{|k_a+k_b|^2\lambda^2}\right). \tag{45}$$

Note, that the phase δ does not contain infrared divergencies and can be written as follows

$$\delta = \frac{\gamma_K}{4}\,\ln\frac{u_2u_3}{(1-u_1)^2}\,, \tag{46}$$

where u_r are anharmonic ratios of invariants in the momentum space

$$u_1 = \Phi = \frac{s\,s_2}{s_{123}s_{012}}\,,\; u_2 = \frac{s_3t_1}{s_{123}t_2}\,,\; u_3 = \frac{s_1t_3}{s_{012}t_2}\,. \tag{47}$$

Correspondingly, in the physical region where $s, s_1, s_2, s_3 < 0$; $s_{012}, s_{123} > 0$ the BDS amplitude can be written as follows

$$\frac{M_{2\to 4}^{BDS}}{|s_1|^{\omega_1}|s_2|^{\omega_2}|s_3|^{\omega_3}\,|\Gamma_{12}||\Gamma_{23}|\,\Gamma(t_1)\Gamma(t_3)} = C\,e^{-i\pi\omega_2}\,e^{i\pi(\omega_a+\omega_b)} = e^{-i\pi\delta}\,, \tag{48}$$

According to the hypothesis formulated in refs. [**?**, 40] the correct expression for $M_{2\to 4}$ can be obtained from $M_{2\to 4}^{BDS}$ by multiplying it by a factor c being a function of these anharmonic relations

$$M_{2\to 4} = c\,M_{2\to 4}^{BDS}\,. \tag{49}$$

The factorization hypothesis together with the above discussed representation of $M_{2\to 4}$ in the form of a sum of the Regge pole and the Mandelstam cut contributions (32) leads to the following relation for c valid in the region $s, s_2 > 0$, $s_1, s_3 < 0$

$$c\,e^{i\pi\,\delta} = \cos\pi\omega_{ab} + i\int_{-i\infty}^{i\infty}\frac{d\omega}{2\pi i}\,f(\omega)\,e^{-i\pi\omega}\,(1-u_1)^{-\omega}\,,\; 1-u_1 \approx \frac{|k_a+k_b|^2}{s_2} \to +0\,. \tag{50}$$

Here $f(\omega)$ is a real function depending on two invariant variables

$$\phi_2 = \frac{u_2}{1-u_1} \approx \frac{|q_1|^2|k_b|^2}{|k_a+k_b|^2|q_2|^2}\,,\; \phi_3 = \frac{u_3}{1-u_1} \approx \frac{|q_3|^2|k_a|^2}{|k_a+k_b|^2|q_2|^2}\,. \tag{51}$$

The phases δ and ω_{ab} also can be expressed in terms of these variables

$$\delta = \frac{\gamma_K}{8}\,\ln(\phi_3\phi_2)\,,\; \omega_{ab} = \frac{\gamma_K}{8}\,\ln\frac{\phi_3}{\phi_2}\,. \tag{52}$$

In a similar way for the production amplitude in the region $s, s_1, s_2, s_3 < 0$; $s_{012}, s_{123} > 0$ one can derive the relation

$$c\,e^{-i\pi\,\delta} = \cos\pi\omega_{ab} - i\int_{-i\infty}^{i\infty}\frac{d\omega}{2\pi i}\,f(\omega)\,(u_1-1)^{-\omega}\,,\; u_1 > 1\,. \tag{53}$$

The above representations for c are valid on the second sheets of the Riemann surface of this function at $u_1 \to 1$. In the quasi-multi-regge kinematics $s_1, s_3 \gg s_2 \sim t_1 \sim t_2 \sim t_3$ the anharmonic ratio u_1 is not close to unity. The first sheet of the Riemann surface for c corresponds to the production amplitude $M_{2\to4}$ in the kinematical region where $s, s_1, s_2, s_3 > 0$. In this region the amplitude is regular at $u_1 = 1$ and has the singularity at $u_1 = 0$.

To illustrate these analytic properties let us consider the BDS amplitude in the region $s, s_1, s_2, s_3 > 0$. It contains the following dependence on u_1 [32]

$$\ln M_{2\to4}^{BDS} = -\frac{\gamma_K}{8}\left(Li_2(1-u_1) + \ln u_1\, \ln(-\sqrt{u_2\, u_3}) + \frac{1}{2}\ln^2 u_1\right) + \ldots\,, \tag{54}$$

where we included also the phase $i\pi(\omega_a + \omega_b)$ and used the identity

$$\frac{|q_1|\,|q_3|\,|k_1|\,|k_2|}{-s_2\,|q_2|^2} = -\sqrt{u_2\, u_3}\,. \tag{55}$$

With the use of the integral representation for the dilogarithm function $Li_2(z)$

$$Li_2(z) = -\int_0^z \frac{dx}{x}\, \ln(1-x) = z\int_1^\infty \frac{dz'}{z'(z'-z)}\, \ln z' \tag{56}$$

we conlude, that the one loop BDS amplitude [29] has singularities at $u_1 = 0$

$$I_6^{(1)} + F_6^{(1)} = -\frac{\gamma_K}{8}\int_{-\infty}^0 \frac{(1-u_1)\, du_1'}{(1-u_1')(u_1-u_1')}\, \left(\ln(1-u_1') - \ln(-\sqrt{u_2\, u_3}) - \ln(-u_1')\right) + \ldots\,. \tag{57}$$

In the multi-Regge regime, where $s, s_2 > 0$; $s_1, s_3 < 0$, the invariant u_1 is close to unity

$$u_1 \approx e^{-2\pi i}\left(1 - \frac{|k_1 + k_2|^2}{s_2}\right) \tag{58}$$

and the amplitude should be continued to the second sheet of the u_1-plane through the lower edge of the cut at $u_1 < 0$, which generates the additional term (cf. [32])

$$\Delta\left(I_6^{(1)} + F_6^{(1)}\right) = -\pi i\, \frac{\gamma_K}{4}\, \left(\ln(1-u_1) - \ln\frac{t_1 t_3}{s_2 \lambda^2} - \ln u_1\right)\,. \tag{59}$$

Note, that this term has also a singularity at $u_1 = 1$ and is pure imaginary in the physical region $u_1 < 1$.

In the next section we consider the two loop contribution. In this case the second order expansion of the BDS exponent on the first sheet also can be presented in a form of the dispersion integral which relates its real and imaginary parts. However, this expression does not agree with the Steinmann relations.

4 Two loop production amplitude $M_{2\to4}$

For the production amplitude in LLA the following expression for $M_{2\to4}$ in the region $s, s_2 > 0$; $s_1, s_3 < 0$ was obtained in two loops with the use of the s-channel unitarity [36]

$$M_{2\to4} = c\, M_{2\to4}^{BDS}\,,\ \ c = 1 + \frac{a^2}{4}\, r_2 + O(a^3)\,, \tag{60}$$

where

$$r_2 \approx Li_2(1-u_1)\, \ln\frac{(1-u_1)}{u_2}\, \ln\frac{(1-u_1)}{u_3}$$

$$+Li_2(1-u_2)\,\ln\frac{(1-u_2)}{u_3}\ln\frac{(1-u_2)}{u_1}+Li_2(1-u_3)\,\ln\frac{(1-u_3)}{u_2}\ln\frac{(1-u_3)}{u_1}\,. \qquad (61)$$

Here we introduced the four-dimensional anharmonic ratios (47) and included additional terms to provide the invariance of $M_{2\to 4}$ under the cyclic permutations. The added contributions are not essential in the multi-Regge kinematics, although for the exact two-loop result they are important. Note, that another physical region $s, s_1, s_2, s_3 < 0$; $s_{012}, s_{123} > 0$, where $u_1 = \exp(2\pi i)$, is also described correctly by the above expression (61) for r_2.

We should take into account also a similar cut contribution to the transition amplitude $3 \to 3$. But in fact it is already contained in eq. (61) due to the relations (cf. [32, 36])

$$\frac{s_{13}s_{02}}{st'_2} = u_2 \to 1 + \frac{|q_1+q_3-q_2|^2}{t'_2}\,,\ \ \frac{1-u_2}{u_1} \to \frac{|q_1+q_3-q_2|^2|q_2|^2}{|q_3|^2|q_1|^2}\,,\ \ \frac{u_1}{u_3} \to \frac{|q_3|^2|q_1|^2}{|k_2|^2|k_1|^2}\,. \qquad (62)$$

Thus, our expression (61) in two loops leads to the correct multi-Regge asymptotics in all channels. Moreover, the conformal invariance in the momentum representation is valid also in higher loops of LLA if we substitute the anharmonic ratios in the two-dimensional transverse subspace by the corresponding four dimensional ratios $u_{2,3}$ and the power of the logarithm $\ln s_2$ at large s_2 by the following expression

$$-2\pi i\frac{\ln^n s_2}{n} \to (-1)^{n-1}\int_0^{1-u_1}\frac{dt}{t}\,\ln^{n-1} t\,\ln(1-t)\,, \qquad (63)$$

which can be written in terms of the polylogarithm function $Li_{n+1}(z)$.

Let us expand the BDS amplitude in the region $s, s_2 > 0$; $s_{012}, s_{123} < 0$ in the perturbation series to investigate a possibility to correct its bad analytic properties with the factor c depending on the anharmonic ratios. It can be presented at this kinematics in the form [32]

$$M^{BDS}_{2\to 4} = C\,\Gamma(t_1)\,(-s_1)^{\omega_1}\,\Gamma(\ln\kappa_{12}-i\pi)\,(-s_2)^{\omega_2}\,\Gamma(\ln\kappa_{23}-i\pi)\,(-s_3)^{\omega_3}\,\Gamma(t_2)\,, \qquad (64)$$

which was simplified above (see (43)). Note, that the phase δ (44) does not contain infrared divergencies and depends on an anharmonic ratio in the two-dimensional momentum space. It can be written also in terms of four-dimensional anharmonic ratios (46).

The first order term of the expansion of the phase in (43) over δ corresponds to the Mandelstam cut contribution in one-loop approximation [32]. The second order term $-\pi^2\delta^2/2$ of the phase factor expansion

$$e^{i\pi\delta} = 1 + i\pi\delta - \pi^2\frac{\delta^2}{2} + ... \qquad (65)$$

contradicts the Steinmann relations and analytic properties for $M_{2\to 4}$ if we would not take into account the additional logarithmic contribution $\sim \ln s_2$ appearing in the factor c. On the other hand, the LLA result for c in the two loop approximation after its analytic continuation to the region $s, s_2 > 0$; $s_1, s_3 < 0$ can be written as follows [36]

$$c \approx 1 - 2\pi i\,\frac{a^2}{4}\,\ln s_2\,\ln\frac{|k_2|^2|q_1|^2}{|k_1+k_2|^2|q_2|^2}\,\ln\frac{|k_1|^2|q_3|^2}{|k_1+k_2|^2|q_2|^2} + ...\,. \qquad (66)$$

It does not contain the phase factor $\exp(-\pi i)$ in the argument of $\ln s_2$ due to the pure imaginary asymptotics of the function $Li_2(1-u_1)$ in eq. (61) at $u_1 \to \exp(-2\pi i)$. To obtain the correct real part for $F_{2\to 4}$ in an accordance with the phase structure of the cut contribution (38) depending on the argument $-s_2$ we should find somewhere the following real term

$$\Delta c = -\frac{a^2\,\pi^2}{2}\,\ln\frac{|k_2|^2|q_1|^2}{|k_1+k_2|^2|q_2|^2}\,\ln\frac{|k_1|^2|q_3|^2}{|k_1+k_2|^2|q_2|^2}\,. \qquad (67)$$

It is remarkable, that this correction is contained already at the BDS factor in eq. (49). Indeed, Δc can be written as follows

$$\Delta c = -\frac{a^2\,\pi^2}{2}\left(\ln^2\frac{|k_1||k_2||q_1||q_2|}{|k_1+k_2|^2|q_2|^2} - \ln^2\frac{|k_1||q_3|}{|k_2||q_1|^2}\right)$$

$$\approx -\frac{f_K^2\,\pi^2}{32}\left(\ln^2\frac{|k_1||k_2||q_1||q_2|}{|k_1+k_2|^2|q_2|^2} - \ln^2\frac{|k_1||q_3|}{|k_2||q_1|^2}\right) = -\pi^2\frac{\delta^2}{2} + \pi^2\frac{\omega_{ab}^2}{2}\,. \tag{68}$$

The first contribution $-\delta^2/2$ is the second order term in the expansion of the phase factor $\exp(i\delta)$ in (43) and the second contribution $\omega_{ab}^2/2$ is opposite in sign to the second order term in the expansion of the factor $\cos\omega_{ab}$ included in the pole contribution $M_{2\to 4}^{pole}$ (37). Note, that the phase factor $\exp(-i\pi\omega_2)$ exists in all three amplitudes $M_{2\to 4}$, $M_{2\to 4}^{pole}$ and $M_{2\to 4}^{cut}$.

Thus, the two-loop result for the two gluon production amplitude in LLA is in a full agreement with analyticity requirements and a factorization hypothesis (49). In fact it follows completely from these properties without any necessity to solve the BFKL equation [36]. Moreover, the BFKL kernel can be calculated from the two loop correction. Note, that the analytic properties of the cut contribution (38) predict the pure imaginary result also for the next-to-leading term in r_2 (not proportional to $\ln s_2$). Recently [41] this prediction was confirmed by an analytic continuation of the exact expression for two loop production amplitude obtained in refs. [42, 43]. It means, that the representation of the six point amplitude in terms of the P-exponents [40] is in an agreement with the Mandelstam cut asymptoticsat least in two loops [36].

In a similar way in the physical region $s, s, s_1, s_2, s_3 < 0$; $s_{012}, s_{123} > 0$ the factor c in two loops can be presented as follows

$$c \approx 1 + 2\pi i\,\frac{a^2}{4}\,(\ln(-s_2) - i\pi)\,\ln\frac{|k_2|^2|q_1|^2}{|k_1+k_2|^2|q_2|^2}\,\ln\frac{|k_1|^2|q_3|^2}{|k_1+k_2|^2|q_2|^2} + \ldots\,. \tag{69}$$

Here the real term $\sim a^2\pi^2$ contradicts the analytic properties for the Mandelstam cut contribution in this region and it is cancelled as above with the two loop expansions of the BDS phase and the pole contribution

$$\Delta c = -\pi^2\frac{\delta^2}{2} + \pi^2\frac{\omega_{ab}}{2}\,. \tag{70}$$

5 Exponentiation hypothesis

As it was argued above, in the region $s, s_2 > 0$; $s_1, s_3 < 0$ for the multi-loop amplitude $M_{2\to 4}$ in LLA and beyond it one can use the relations

$$M_{2\to 4} = c\,M_{2\to 4}^{BDS} = M_{2\to 4}^{pole} + M_{2\to 4}^{cut}\,, \tag{71}$$

where c is an invariant function of three anharmonic ratios in the momentum space. The BDS amplitude is given by eq. (43), $M_{2\to 4}^{pole}$ is known explicitly (see (37)) and the analytic properties of $M_{2\to 4}^{cut}$ are defined by the integral (40).

These relations can be considered as a set of equations for the real functions c and $f(\omega_2)$ although they seem to be incomplete, because for example in two loops we can add to the result the next-to-leading correction of the form (see ref. [41])

$$\Delta M_{2\to 4} = ia^2\,\chi(z_2, z_3)\,,\;\; z_2 = \frac{1-u_1}{u_2}\,,\;\; z_3 = \frac{1-u_1}{u_3}\,. \tag{72}$$

Generalizing the BDS hypothesis one can assume, that the correct amplitude $M_{2\to 4}$ has an exponential form. However, it will be shown below, that the factor c can not be a pure phase in the region $s, s_2 > 0, s_1, s_3 < 0$

$$c \neq e^{i\phi}\,. \tag{73}$$

This conclusion is based on the fact, that in LLA the complex structure of the production amplitude (including the phase of the BDS ansatz) is known.

We start with the dispersion representation for the cut contribution to the production amplitude in LLA (see (40))

$$\frac{M^{cut}_{2\to 4}}{|M^{BDS}_{2\to 4}|} = ia\,\pi\, e^{-i\pi\omega_2}\sum_{n=0}^{\infty}(\ln(-s_2))^n\, c_n\, a^n\,, \tag{74}$$

where the coefficients c_n due to eq. (16) are known in the form of integrals over ν and a sum over n from powers of the eigenvalue (17) of the BFKL kernel for the adjoint representation. For the real part one obtains with a leading accuracy

$$\Re\frac{M^{cut}_{2\to 4}}{e^{-i\pi\omega_2}|M^{BDS}_{2\to 4}|} = a\,\pi^2\sum_{n=0}^{\infty}(\ln(s_2))^{n-1}\, n\, c_n\, a^n\,. \tag{75}$$

We devided the equality with the factor $\exp(-i\pi\omega_2)$ because it is common for all contributions.

On the other hand, using the exponentiation hypothesis with the additional asumption, that the remainder function is a phase

$$c = e^{i\phi}\,,\ \ \phi\approx\Delta_{2\to 4}\,, \tag{76}$$

where $\Delta_{2\to 4}$ is given in eq. (16), one can obtain the coefficients c_n for $n\geq 2$ from the expansion

$$\Re\frac{M^{cut}_{2\to 4}}{e^{-i\pi\omega_2}|M^{BDS}_{2\to 4}|} = -\frac{\pi^2}{2}\left(a\sum_{k=0}^{\infty}(\ln s_2)^k\, c_k\, a^k\right)^2\,, \tag{77}$$

where

$$c_0 = \ln\frac{|q_1|^2|q_3|^2}{|k_1+k_2|^2\lambda^2}+\frac{1}{2}\ln\frac{|k_1|^2\lambda^2}{|q_1|^2|q_2|^2}+\frac{1}{2}\ln\frac{|k_2|^2\lambda^2}{|q_3|^2|q_2|^2} = \frac{1}{2}\ln\frac{|q_1|^2|q_3|^2|k_1|^2|k_2|^2}{|k_1+k_2|^4|q_2|^4}\,. \tag{78}$$

Here the first term appears from the factor C (45) and two last terms are from the phases ω_a (21) and ω_b (28). For the coefficient c_1 we have from the previous section (see (67))

$$c_1 = -\frac{1}{2}\ln\frac{|k_2|^2|q_1|^2}{|k_1+k_2|^2|q_2|^2}\,\ln\frac{|k_1|^2|q_3|^2}{|k_1+k_2|^2|q_2|^2}\,. \tag{79}$$

Thus, from the exponentiation hypothesis (76) we obtain the recurrent relation for c_n at $n\geq 2$

$$n\, c_n = -\frac{1}{2}\sum_{k=0}^{n-1}c_k\, c_{n-1-k}\,. \tag{80}$$

In particular,

$$c_2 = \frac{1}{8}\ln\frac{|q_1|^2|q_3|^2|k_1|^2|k_2|^2}{|k_1+k_2|^2|q_2|^4}\,\ln\frac{|k_2|^2|q_1|^2}{|k_1+k_2|^4|q_2|^2}\,\ln\frac{|k_1|^2|q_3|^2}{|k_1+k_2|^2|q_2|^2}\,. \tag{81}$$

Let us introduce the generating function $y(x)$

$$\frac{M^{cut}_{2\to 4}}{e^{-i\pi\omega_2}|M^{BDS}_{2\to 4}|} = ia\,\pi\, y(a\ln(-s_2))\,,\ \ y(x) = \sum_{n=0}^{\infty}x^n\, c_n\,. \tag{82}$$

This function satisfies the equation

$$\frac{d}{d\,x}\,y(x) = -\frac{1}{2}\,y^2(x)+b\,, \tag{83}$$

where

$$y(0) = c_0\,,\;\; b = c_1 + \frac{c_0^2}{2} = \frac{1}{2}\,\ln^2\frac{|k_2q_1|}{|k_1q_2|}\,. \tag{84}$$

Its solution is

$$y = \sqrt{2b}\,\tanh\left(\sqrt{\frac{b}{2}}x + \delta\right)\;\;,\;\;\coth\delta = \frac{c_0}{\sqrt{2b}} \tag{85}$$

with the perturbative expansion

$$y = c_0 + c_1\,x - \frac{c_0\,c_1}{2}\,x^2 + \ldots\,. \tag{86}$$

This result based on analytic properties of the production amplitude and on the asumption of the exponentiation (76) of $i\Delta_{2\to4}$ in (15) is in an disagreement with the perturbative solution (16) of the BFKL equation corresponding to the factorization property of t-channel partial waves. In particular, the exponent $\exp(-\sqrt{2b}\,x)$ depends only on the module of the anharmonic ratio u_2/u_3 whereas the correct BFKL expression depends also on a phase. It turns out, that already in three loops the leading logarithmic result (16) contains the special functions $Li_3(x)$ and $Li_2(x)$ absent in c_3. Indeed, according to ref. [44] the function $\widetilde{c_3}$ obtained from eq. (16) has the form

$$\widetilde{c_2} = \frac{1}{8}\left(2\ln|w|^2\ln^2|1+w|^2 - \frac{4}{3}\ln^3|1+w|^3 - \frac{1}{2}\ln^2|w|^2\ln|1+w|^2\right)$$

$$+\frac{1}{8}\ln|w|^2\left(Li_2(-w) + Li_2(-w^*)\right) - \frac{1}{4}\left(Li_3(-w) - Li_3(-w^*)\right)\;,\;\; w = \frac{q_3k_1}{k_2q_1}\,. \tag{87}$$

Thus, the factor c in the region $s, s_2;\, s_1, s_3 < 0$ can not be a pure phase in the physical regions with $u_1 = \exp(\pm 2\pi i)$, where the amplitude contains the Mandelstam cuts. The analytic properties of c are presented in eqs. (50 and (53)). They show, that the knowledge of the amplitude in LLA allows one to calculate not only leading corrections to the imaginary part of the factor c, but also - leading corrections to its real part suppressed by the extra factor $\sim a$ (see ref. [44]).

6 Conclusion

In this paper we investigated analytic properties of the planar six point amplitude for $N = 4$ SUSY in the multi-Regge kinematics. This amplitude has the Regge pole and the Mandelstam cut contributions and should satisfy the Steinmann relations. We calculated the two loop correction to the amplitude $M_{2\to4}$ at the region $s, s_2 > 0;\, s_1, s_3 < 0$ in an agreement with the results of the paper [36] using only analyticity constraints and a factorization hypothesis. It was shown, that in the next-to-leading approximation the two loop correction to the factor c in front of the BDS expression should be also pure imaginary. This prediction is confirmed by direct calculations in ref. [41]. We also demonstrated above, that in upper loops the factor c in the Regge kinematics can not be a pure phase (see (73)), because such phase structure would contradict the t-channel Regge factorization incorporated in the BFKL equation.

I thank J. Bartels and A. Prygarin for helpful discussions. This work was supported by the Russian grant RFBR-10-02-01338-a.

References

[1] L. N. Lipatov, Sov. J. Nucl. Phys. **23** (1976) 338;
V. S. Fadin, E. A. Kuraev, L. N. Lipatov, Phys. Lett. B **60** (1975) 50;
E. A. Kuraev, L. N. Lipatov, V. S. Fadin, Sov. Phys. JETP **44** (1976) 443 ; **45** (1977) 199;
I. I. Balitsky, L. N. Lipatov, Sov. J. Nucl. Phys. **28** (1978) 822.

[2] V. S. Fadin, R. Fiore, M. I. Kotsky, Phys. Lett. B **387** (1996) 593.

[3] A. V. Kotikov, L. N. Lipatov, Nucl. Phys. B **582** (2000) 19.

[4] L. N. Lipatov, Nucl. Phys. B **365** (1991) 614.

[5] L. N. Lipatov, Phys. Lett. B **309** (1993) 394.

[6] L. N. Lipatov, Sov. Phys. JETP **63** (1986) 904.

[7] J. Bartels, Nucl. Phys. B **175** (1980) 365;
J. Kwiecinski, M. Praszalowicz, Phys. Lett. B **94** (1980) 413.

[8] L. N. Lipatov, Phys. Lett. B **251** (1990) 284.

[9] L. N. Lipatov, Nucl. Phys. B **548** (1999) 328.

[10] L. N. Lipatov *High energy asymptotics of multi-colour QCD and exactly solvable lattice models*, Padova preprint DFPD/93/TH/70, hep-th/9311037, unpublished.

[11] L. N. Lipatov, JETP Lett. **59** (1994) 596;
L. D. Faddeev, G. P. Korchemsky, Phys. Lett. B **342** (1995) 311.

[12] V. S. Fadin, L. N. Lipatov, Phys. Lett. B **429** (1998) 127;
M. Ciafaloni and G. Camici, Phys. Lett. B **430** (1998) 349.

[13] A. V. Kotikov, L. N. Lipatov, Nucl. Phys. B **661** (2003) 19.

[14] L. N. Lipatov, talk at "Perspectives in Hadronic Physics", Proc. of Conf. ICTP. Triest, Italy, May 1997, talk "Pomeron before and after QCD", Proc. of the Helsinki Workshop on forward physics, 2000.

[15] J. A. Minahan, K. Zarembo, JHEP **0303** (2003) 013.

[16] N. Beisert and M. Staudacher, Nucl. Phys. B **670** (2003) 439.

[17] A. V. Kotikov, L. N. Lipatov, V. N. Velizhanin, Phys. Lett. B **557** (2003) 114.

[18] A. V. Kotikov, L. N. Lipatov, A. I. Onishchenko, V. N. Velizhanin, Phys. Lett. B **595** (2004) 521; [Erratum-ibid. B **632** (2006) 754].

[19] S. Moch, J. A. M. Vermaseren, A. Vogt, Nucl. Phys. B **688** (2004) 101.

[20] B. Eden, M. Staudacher, J. Stat. Mech. **0611** (2006) P014.

[21] N. Beisert, B. Eden, M. Staudacher, J. Stat. Mech. **0701** (2007) P021.

[22] J. M. Maldacena, Adv. Theor. Math. Phys. **2** (1998) 231.

[23] S. S. Gubser, I. R. Klebanov, A. M. Polyakov, Phys. Lett. B **428** (1998) 105.

[24] E. Witten, Adv. Theor. Math. Phys. **2** (1998) 253.

[25] A. V. Kotikov, L. N. Lipatov, A. Rej, M. Staudacher, V. N. Velizhanin, J. Stat. Mech. **0710**, P10003 (2007);
Z. Bajnok, R. Janik, T. Lukowski, arXi:0811.4448 [hep-th];
T. Lukowski, A. Rej, V. N. Velizhanin, arXiv:0912.1624 [hep-th].

[26] R. C. Brower, J. Polchinski, M. J. Strassler, C. I. Tan, JHEP **0712**, 005 (2007).

[27] L. N. Lipatov, Nucl. Phys. B **452**, 369 (1995); Phys. Rept. **286**, 131 (1997).

[28] E. N. Antonov, L. N. Lipatov, E. A. Kuraev, I. O. Cherednikov, Nucl. Phys. B **721**, 111 (2005).

[29] Z. Bern, L. J. Dixon, V. A. Smirnov, Phys. Rev. D **72**, 085001 (2005).

[30] L. F. Alday, J. Maldacena, JHEP **0706:064**, (2007)

[31] L. F. Alday, J. Maldacena, JHEP **0711:068**, (2007)

[32] J. Bartels, L. N. Lipatov, A. Sabio Vera, Phys.Rev.D80:045002,2009,

[33] O. Steinmann, Helv. Physica Acta **33** 257, 349 (1960).

[34] Z. Bern, L. J. Dixon, D. A. Kosower, R. Roiban, M. Spradin, C. Vergu, A. Volovich, arXiv:0803.1465 [hep-th].

[35] S. Mandelstam, Nuovo Cim.**30**, 1148 (1963).

[36] J. Bartels, L. N. Lipatov, A. Sabio Vera, Eur.Phys.J.C65:587-605,2010,

[37] J. M. Drummond, J. Henn, V. A. Smirnov, E. Sokatchev, JHEP **0701** 064 (2007) ; J. M. Drummond, G. P. Korchemsky, E. Sokatchev, Nucl.Phys. **B795**, 385 (2008).

[38] L. N. Lipatov, J.Phys.A42:304020,2009,

[39] I. T. Drummond, P. V. Landshoff, W. J. Zakrzewski, Nucl. Phys. B **11**, 383 (1969); J. H. Weis, Phys. Rev. D **4**, 1777 (1971).

[40] J. M. Drummond, J. Henn, G. P. Korchemsky, E. Sokatchev, Nucl. Phys. B **826** 337 (2010.)

[41] L. N. Lipatov and A. Prygarin, ”Mandelstam cuts and light-like Wilson loops in N=4 SUSY”, in print.

[42] V. Del Duca, C. Duhr and V. A. Smirnov, JHEP **1003**, 099 , **1005** 084 (2010).

[43] A. B. Goncharov, M. Spradlin, C. Vergu and A. V. Volovich, [archiv:10006.5703 [hep-th]].

[44] L. N. Lipatov and A. Prygarin, ”Mandelstam cuts and six point MHV amplitude in N=4 super Yang-Mills model”, in print.