Generalized Super-Landau Models

Evgeny Ivanov*
Bogoliubov Laboratory of Theoretical Physics, JINR
141980, Dubna, Moscow Region, Russia

Abstract

It is a brief account of a new class of $\mathcal{N} = 2$ supersymmetric Landau models which generalize the previously studied superplane Landau model by extending it to an arbitrary magnetic field on any two-dimensional manifold M_2. Using an off-shell $\mathcal{N} = 2$ superfield formalism, it is shown that these models are generically characterized by two independent potentials given on M_2. The relevant Hamiltonians are factorizable and in the restricted case, when both the Gauss curvature and the magnetic field are constant over M_2, admit infinite series of factorization chains implying the integrability of the associated systems. For the particular model with $\mathbb{CP}^1$ as the bosonic manifold, the spectrum and eigenvectors are presented.

1 Introduction: definitions and motivations

The original Landau model [1] describes a charged particle moving on a plane orthogonal to a constant uniform magnetic flux. A spherical generalization of this model was given by Haldane [2]. It describes a charged particle on the 2-sphere $S^2 \sim SU(2)/U(1)$ in the background of Dirac monopole placed in the center. The Landau-type models have plenty of applications. In particular, they provide a theoretical basis of the Quantum Hall Effect (QHE) [3].

By definition, *superextended Landau models* are models of non-relativistic particles moving on supergroup manifolds with S^2 or its planar limit as a "body".

Minimal superextensions of the S^2 Haldane model were constructed in [4, 5]. They include:

- Landau problem on the (2|2)-dimensional supersphere $SU(2|1)/U(1|1)$ [5];
- Landau problem on the (2|4)-dimensional superflag $SU(2|1)/[U(1) \times U(1)]$ [4, 5].

Their large S^2 radius limits yield the *planar super-Landau models*. They were introduced and studied in [6, 7] [1].

The most surprising feature of the super-planar Landau models is the presence of hidden worldline $\mathcal{N} = 2$ supersymmetry. One starts with a model invariant under some target supersymmetry and, as a gift, finally finds the existence of the $\mathcal{N} = 2, d = 1$ supercharges which square on the Hamiltonian of the system. Thus, the super-planar Landau models simultaneously provide a class of the supersymmetric quantum mechanics (SQM) models. SQM models [9] have a lot of applications in diverse domains.

Based on this remarkable property, a natural extension of the super-planar Landau models can be constructed in the following way. One takes the fundamental notion of the worldline $N = 2$ supersymmetry as the primary principle and constructs the most general $\mathcal{N} = 2$ SQM model involving the standard superplane Landau model [6] as a particular case. Such a construction

***e-mail**: achasov@math.nsc.ru

[1] See also [8] where an alternative planar super-Landau model based on the contraction of the supergroup $OSp(1|2)$ was presented.

has been recently accomplished in [10], starting from $\mathcal{N}=2, d=1$ superfield formalism. The basic aim of the present Talk is to give a brief account of this new class of super-Landau models.

Most of the results reported and mentioned in the Talk are obtained together with Andrey Beylin, Tom Curtright, Luca Mezincescu and Paul K. Townsend.

2 Bosonic Landau models

Before turning to the main subject, I will recall the salient features of the bosonic Landau-type models and their superextensions.

2.1 Planar bosonic Landau model

The Landau model is described by the following Lagrangian and Hamiltonian:

$$L_b = |\dot{z}|^2 - i\kappa\left(\dot{z}\bar{z} - \dot{\bar{z}}z\right) = |\dot{z}|^2 + \left(A_z\dot{z} + A_{\bar{z}}\dot{\bar{z}}\right) \,, \tag{2.1}$$

$$A_z = -i\kappa\bar{z}, \quad A_{\bar{z}} = i\kappa z, \quad \partial_{\bar{z}}A_z - \partial_z A_{\bar{z}} = -2i\kappa \,.$$

$$H_b = \frac{1}{2}\left(a^\dagger a + a a^\dagger\right) = a^\dagger a + \kappa \,, \tag{2.2}$$

where

$$a = i(\partial_{\bar{z}} + \kappa z), \quad a^\dagger = i(\partial_z - \kappa\bar{z}), \quad [a, a^\dagger] = 2\kappa \,. \tag{2.3}$$

The invariances of this model are "magnetic translations" and $2D$ rotations generated by:

$$\begin{aligned} &P_z = -i(\partial_z + \kappa\bar{z}), \quad P_{\bar{z}} = -i(\partial_{\bar{z}} - \kappa z), \quad F_b = z\partial_z - \bar{z}\partial_{\bar{z}}, \\ &[P_z, P_{\bar{z}}] = 2\kappa, \qquad [H, P_z] = [H, P_{\bar{z}}] = [H, F_b] = 0. \end{aligned} \tag{2.4}$$

The full set of wave functions corresponding to different Landau Levels (LL) is as follows:

- Lowest Landau level (LLL), $H\Psi_{(0)} = \kappa\Psi_{(0)}$:

$$a\Psi_{(0)}(z,\bar{z}) = 0 \Leftrightarrow (\partial_{\bar{z}} + \kappa z)\Psi_{(0)} = 0 \;\rightarrow\; \Psi_{(0)} = e^{-\kappa|z|^2}\psi_{(0)}(z)$$

- n-th excited LL:

$$\Psi_{(n)}(z,\bar{z}) = [i(\partial_z - \kappa\bar{z})]^n e^{-\kappa|z|^2}\psi_{(n)}(z), \;\; H\Psi_{(n)} = \kappa(2n+1)\Psi_{(n)}$$

Each LL is infinitely degenerate due to $(P_z, P_{\bar{z}})$ invariance. The w.f. form infinite-dimensional unitary irreps of this non-compact group, with the basis consisting of the monomials $z^m, m > 0$. They possess invariant norms:

$$||\Psi_{(n)}||^2 \sim \int dz d\bar{z} e^{-2\kappa|z|^2}\overline{\psi}_{(n)}(\bar{z})\psi_{(n)}(z) < \infty \tag{2.5}$$

for any monomial $\psi_{(n)}(z) \sim z^m$.

2.2 Generalization to S^2

An S^2 analog of the planar Lagrangian L_b is

$$L_b = \frac{1}{(1+r^2|z|^2)^2}|\dot{z}|^2 + is\frac{1}{1+r^2|z|^2}\left(\dot{z}\bar{z} - \dot{\bar{z}}z\right). \tag{2.6}$$

The second term is the $d=1$ WZ term on the coset $SU(2)/U(1)$, r being the "inverse" radius of S^2. The first term is the $d=1$ pullback of the S^2 distance.

The wave functions in this case are finite-dimensional $SU(2)$ irreps, $s, s+1, s+2, \ldots$ being their "spins". The LLL wave function is determined by the covariant analyticity condition on S^2

$$\nabla_{\bar{z}} \Psi_{(0)} = 0, \quad \nabla_{\bar{z}} = (1 + r^2|z|^2)\partial_{\bar{z}} + U(1) \text{ connection}\,. \tag{2.7}$$

Each LL is finitely degenerated since the wave functions are $SU(2)$ irreps. The limit $r \to 0$ yields the planar Landau model.

3 Superextensions

3.1 Worldline supersymmetry vs target-space supersymmetry

Super-Landau models are quantum-mechanical models for a charged particle on a homogeneous supermanifold, such that the "bosonic" truncation is either Landau's original model for a charged particle on a plane or Haldane's spherical version of it. There are two approaches to constructing such extensions.

- Worldline supersymmetry:

$$t \Rightarrow (t, \theta, \bar{\theta}), \quad z, \bar{z} \Rightarrow \mathcal{Z}(t, \theta, \bar{\theta}),\ \bar{\mathcal{Z}}(t, \theta, \bar{\theta})\,,$$

$$z, \bar{z} \Rightarrow (z, \bar{z}, \psi, \bar{\psi}, \ldots) - \text{worldline supermultiplet}\,.$$

  This option yields a version of Supersymmetric Quantum Mechanics.

- Target-space supersymmetry:

$$\text{group manifold}: \ (z, \bar{z}) \Rightarrow \text{supergroup manifold}: \ (z, \bar{z}, \zeta, \bar{\zeta})$$

$$(P_z, P_{\bar{z}}, F_b, \kappa) \Rightarrow (P_z, P_{\bar{z}}, \Pi_\zeta, \Pi_{\bar{\zeta}}, F_b, F_f, \kappa, \ldots),$$

$$\Pi_\zeta = \partial_\zeta + \kappa\bar{\zeta},\ \Pi_{\bar{\zeta}} = \partial_{\bar{\zeta}} + \kappa\zeta,\ F_f = \zeta\partial_\zeta - \bar{\zeta}\partial_{\bar{\zeta}},\ \{\Pi_\zeta, \Pi_{\bar{\zeta}}\} = 2\kappa\,.$$

  The geometrical meaning of this procedure in the simplest case is that 2-dimensional plane $(z, \bar{z})$ is extended to a $(2|2)$ dimensional superplane $(z, \bar{z}, \zeta, \bar{\zeta})$, where $\zeta, \bar{\zeta}$ are new complex fermionic coordinates.

3.2 Superplane Landau model

Planar super-Landau models are the large radius limits (contractions) of the supersphere and superflag Landau models. One makes explicit the S^2 radius R, properly rescales Hamiltonians and sends $R \to \infty$. The supersphere $SU(2|1)/U(1|1)$ goes into an $(2|2)$ dimensional superplane.

The superplane Landau model is determined by the following Lagrangian and Hamiltonian:

$$L = L_f + L_b = |\dot{z}|^2 + \dot{\zeta}\dot{\bar{\zeta}} - i\kappa\left(\dot{z}\bar{z} - \dot{\bar{z}}z + \dot{\zeta}\bar{\zeta} + \dot{\bar{\zeta}}\zeta\right), \tag{3.1}$$

$$H = a^\dagger a - \alpha^\dagger \alpha = \partial_{\bar{\zeta}}\partial_\zeta - \partial_z\partial_{\bar{z}} + \kappa\left(\bar{z}\partial_{\bar{z}} + \bar{\zeta}\partial_{\bar{\zeta}} - z\partial_z - \zeta\partial_\zeta\right) + \kappa^2\left(z\bar{z} + \zeta\bar{\zeta}\right), \tag{3.2}$$

where the operators $a, a^\dagger$ were defined in (2.3) and

$$\alpha = \partial_{\bar{\zeta}} - \kappa\zeta\,, \quad \alpha^\dagger = \partial_\zeta - \kappa\bar{\zeta}\,. \tag{3.3}$$

The invariances are generated by $P_z, P_{\bar{z}}, \Pi_\zeta, \Pi_{\bar{\zeta}}$ and by the new spinorial generators

$$Q = z\partial_\zeta - \bar{\zeta}\partial_{\bar{z}}\,, \quad Q^\dagger = \bar{z}\partial_{\bar{\zeta}} + \zeta\partial_z\,, \quad C = z\partial_z + \zeta\partial_\zeta - \bar{z}\partial_{\bar{z}} - \bar{\zeta}\partial_{\bar{\zeta}}\,. \tag{3.4}$$

They generate the supergroup $ISU(1|1)$, contraction of $SU(2|1)$:

$$\{Q, Q^\dagger\} = C\,,\ [Q, P_z] = i\Pi_\zeta\,,\ \{Q^\dagger, \Pi_\zeta\} = iP_z\,. \tag{3.5}$$

3.3 Norms and hidden worldline supersymmetry

The natural $ISU(1|1)$-invariant inner product is defined as

$$< \phi|\psi >= \int d\mu \, \overline{\phi\left(z,\bar{z};\zeta,\bar{\zeta}\right)} \, \psi\left(z,\bar{z};\zeta,\bar{\zeta}\right), \quad d\mu = dzd\bar{z}d\zeta d\bar{\zeta} \,. \tag{3.6}$$

This definition leads to negative norms for some component wave functions. To make all norms not negative we need to introduce the "metric" operator:

$$G = \frac{1}{\kappa}\left[\partial_\zeta\partial_{\bar{\zeta}} + \kappa^2\bar{\zeta}\zeta + \kappa\left(\zeta\partial_\zeta - \bar{\zeta}\partial_{\bar{\zeta}}\right)\right], \quad << \phi|\psi >> \sim \int d\mu \, \overline{(G\phi)} \, \psi \,. \tag{3.7}$$

The full Hamiltonian H commutes with G, so $H = H^\dagger = H^\ddagger$, where $\ddagger$ denotes hermitian conjugation with respect to the new inner product. However, the hermitian conjugation properties of the operators which *do not commute* with G, change.

Let $\mathcal{O}$ be generator of some symmetry, such that $[H, \mathcal{O}] = 0$. Then

$$\mathcal{O}^\ddagger \equiv G\mathcal{O}^\dagger G = \mathcal{O}^\dagger + G\mathcal{O}^\dagger_G \,, \quad \mathcal{O}_G \equiv [G, \mathcal{O}] \,, \tag{3.8}$$

and O_G is another operator such that $[H, \mathcal{O}_G] = 0$. The symmetry generators that do not commute with G thus generate, in general, additional "hidden" symmetries.

In our case G commutes with all $ISU(1|1)$ generators, except for $Q, Q^\dagger$. Thus the conjugation rules of these generators change:

$$Q^\ddagger = Q^\dagger - \frac{i}{\kappa}S \,, \quad S = a^\dagger\alpha = i\left(\partial_z\partial_{\bar{\zeta}} + \kappa^2\bar{z}\zeta - \kappa\bar{z}\partial_{\bar{\zeta}} - \kappa\zeta\partial_z\right), \; S^\ddagger = a\alpha^\ddagger \,, \tag{3.9}$$

where $\alpha^\ddagger = -\alpha^\dagger$. The operators $S, S^\ddagger, H$ can be checked to form $\mathcal{N} = 2, d = 1$ superalgebra:

$$\{S, S^\ddagger\} = 2\kappa H \,, \quad \{S, S\} = \{S^\ddagger, S^\ddagger\} = 0 \,, \quad [H, S] = [H, S^\ddagger] = 0 \,. \tag{3.10}$$

The LLL ground state is annihilated by $S, S^\ddagger$

$$S\psi^{(0)} = S^\ddagger\psi^{(0)} = 0,$$

and so it is a singlet of $\mathcal{N} = 2$ supersymmetry. Hence $\mathcal{N} = 2$ supersymmetry is unbroken and all higher LL form irreps of it.

3.4 Superfield formulation

Superfield formulation of the superplane model was given in [11]. It makes manifest the hidden $\mathcal{N} = 2$ supersymmetry of this model.

The starting setting of this formulation is $\mathcal{N}$=2, d=1 superspace in the left-chiral basis $(\tau \equiv t + i\theta\bar{\theta}, \theta, \bar{\theta})$. The basic objects are $\mathcal{N}$=2, d=1 chiral bosonic and fermionic superfields $\Phi = z(\tau) + \theta\chi(\tau)$, $\Psi = \bar{\zeta}(\tau) + \theta h(\tau)$, with $\chi(\tau)$ and $h(\tau)$ being auxiliary fields:

$$\bar{D}\Phi = \bar{D}\Psi = 0 \,.$$

The superfield action eventually yielding the superplane model action is:

$$S = \int dt d^2\theta \left\{\Phi\bar{\Phi} + \Psi\bar{\Psi} + \rho\left[\Phi D\Psi - \bar{\Phi}\bar{D}\bar{\Psi}\right]\right\}, \quad \rho = 1/(2\sqrt{\kappa}) \,. \tag{3.11}$$

The auxiliary fields h and χ are eliminated by their algebraic equations of motion as $\chi = 2i\rho\,\dot{\zeta}$, $\quad h = -2i\rho\,\dot{\bar{z}}$. Then the action written in terms of physical fields reads

$$S \Rightarrow \int dt\Big[i\kappa\left(z\dot{\bar{z}} - \bar{z}\dot{z} + \zeta\dot{\bar{\zeta}} - \dot{\zeta}\bar{\zeta}\right) + \left(\dot{z}\dot{\bar{z}} + \dot{\zeta}\dot{\bar{\zeta}}\right)\Big]. \tag{3.12}$$

The natural idea was to construct a generalized $\mathcal{N} = 2$ supersymmetric Landau model by going over to the most general $\mathcal{N} = 2$ superfield action, $S \Rightarrow S_{gen}$. It was recently accomplished in [10].

4 From "free" theory to interaction

4.1 Most general $\mathcal{N}=2$ superfield action

A generalization of the superplane model superfield action (3.11) is as follows

$$S_{gen} = \int dt d^2\theta \left\{ K(\Phi,\bar\Phi) + V(\Phi,\bar\Phi)\Psi\bar\Psi + \rho\left(\Phi D\Psi - \bar\Phi\bar D\bar\Psi\right)\right\} = \int dt\mathcal{L}\,. \tag{4.1}$$

It involves two independent superfield potentials, $K(\Phi,\bar\Phi)$, $V(\Phi,\bar\Phi)$, and goes into the superplane model action in the flat limit, when $K \Rightarrow \Phi\bar\Phi$, $V \Rightarrow 1$.

After eliminating auxiliary fields in $\Phi = z + \ldots$, $\Psi = \psi + \ldots$, and setting $4\rho^2 = 1$, the component Lagrangian reads

$$\mathcal{L}_{comp} = V^{-1}\,\dot z\dot{\bar z} + i\left(\dot z K_z - \dot{\bar z}K_{\bar z}\right) + \ \psi - \text{terms}\,. \tag{4.2}$$

By introducing the notation $Z^A = (z,\psi)$, it can be rewritten as

$$\mathcal{L} = \dot Z^A \dot{\bar Z}^{\bar B} g_{\bar B A} + \left(\dot Z^A \mathcal{A}_A + \dot{\bar Z}^{\bar B}\mathcal{A}_{\bar B}\right)\,, \tag{4.3}$$

where, e.g.,

$$g_{\bar z z} = V^{-1}\left(1 - \psi\bar\psi\,\frac{V_z V_{\bar z}}{K_{z\bar z}V}\right),\quad g_{\bar z\psi} = -\frac{V_{\bar z}}{K_{z\bar z}V}\bar\psi\,,\quad \text{etc,}$$

$$\mathcal{A}_z = i(K_z + \psi\bar\psi\,V_z)\,,\quad \mathcal{A}_\psi = iV\bar\psi\,,\quad \text{etc.}$$

The potentials V and K define an M_2 metric and background super gauge field, respectively.

4.2 Quantization

The classical Hamiltonian is given by the following expression

$$H_{class} = \mathcal{P}_A\, g^{A\bar B}\,\mathcal{P}_{\bar B}\,,\quad \mathcal{P}_A = P_A - \mathcal{A}_A\,,\quad P_A = \frac{\partial L}{\partial \dot Z^A}. \tag{4.4}$$

The Noether $\mathcal{N}=2$ supercharges read

$$Q = \frac{1}{i}\mathcal{P}_z\mathcal{P}_\psi,\quad \bar Q = \frac{1}{i}\mathcal{P}_{\bar\psi}\mathcal{P}_{\bar z},$$

$$\{Q,Q\}_{PB} = \{\bar Q,\bar Q\}_{PB} = 0\,,\quad \{Q,\bar Q\}_{PB} = -2iH_{class}\,. \tag{4.5}$$

The quantization follows the standard routine:

$$P_A \to -i\partial_A,\qquad P_{\bar B} \to -i\partial_{\bar B}.$$

The quantum Hamiltonian is expressed as:

$$H_q = \mathcal{P}_z V \mathcal{P}_{\bar z} + \mathcal{P}_z\mathcal{P}_{\bar\psi}V_{\bar z}\bar\psi - V_z\psi\mathcal{P}_{\bar z}\mathcal{P}_\psi + \mathcal{P}_{\bar\psi}(K_{z\bar z} + \psi\bar\psi V_{z\bar z})\mathcal{P}_\psi\,, \tag{4.6}$$

$$\{Q, Q^\dagger\} = 2H_q\,. \tag{4.7}$$

With the above definitions of Q and H, the Hermitian properties are specified with respect to the inner product with a unity measure,

$$< f, g >= \int dz d\bar z d\psi d\bar\psi\; \overline{f(z,\bar z,\psi,\bar\psi)}\, g(z,\bar z,\psi,\bar\psi)\,,\quad < f, Qg >=< Q^\dagger f, g >\,.$$

The general wave function

$$\Psi\left(z,\bar{z},\psi,\bar{\psi}\right) = f_0(z,\bar{z}) + \psi f_1(z,\bar{z}) + \bar{\psi} f_2(z,\bar{z}) + \bar{\psi}\psi f_3(z,\bar{z}),$$

proves to contain four invariant subspaces of H_q:

$$H_q\,\Psi = \lambda\Psi \;\Rightarrow\; H_q\,\psi f_1 = \lambda_1\,\psi f_1\,,\quad H_q\,\bar{\psi} f_2 = \lambda_2\,\bar{\psi} f_2\,,$$

$$H_q\,(f_0 + \bar{\psi}\psi f_3) = \lambda_3\,(f_0 + \bar{\psi}\psi f_3)\,,\; H_q\,(V^{-1} f_3 + \bar{\psi}\psi V f_0) = \lambda_4\,(V^{-1} f_3 + \bar{\psi}\psi V f_0),$$

yielding two sets of the eigenvalues problems

$$-\left(\partial_{\bar{z}} - K_{\bar{z}}\right) V \left(\partial_z + K_z\right) f_1 = \lambda_1\, f_1, \quad -\left(\partial_z + K_z\right) V \left(\partial_{\bar{z}} - K_{\bar{z}}\right) f_2 = \lambda_2\, f_2\,, \tag{4.8}$$

and

$$-\left(\partial_z + K_z\right)\left(\partial_{\bar{z}} - K_{\bar{z}}\right) V f_0^L = \lambda_3' f_0^L\,, \quad -\left(\partial_{\bar{z}} - K_{\bar{z}}\right)\left(\partial_z + K_z\right) V f_0^H = \lambda_4' f_0^H\,, \tag{4.9}$$

with $f_0 = f_0^L + f_0^H$, $f_3 = V\left(f_0^L - f_0^H\right)$. The functions (f_1, f_0^L) and (f_2, f_0^H) form *two* irreducible $\mathcal{N}=2$ multiplets and possess the same spectrum.

The inner product of two wave functions, like in the superplane model, contains states with negative norms

$$< f, g >= \int dz\, d\bar{z}\, d\psi\, d\bar{\psi}\,\left(\bar{\Psi}_f \Psi_g\right) = \int dz\, d\bar{z}\,\left(\bar{f}_1 g_1 - \bar{f}_2 g_2 + 2V \bar{f}_0^L g_0^L - 2V \bar{f}_0^H g_0^H\right).$$

This drawback is cured by introducing the appropriate metric operator, $< f, g > \;\Rightarrow\; << f, g >>=< Gf, g >$,

$$G = \frac{\left[\mathcal{P}_{\bar{\psi}}, \mathcal{P}_{\psi}\right]}{2V} + 2\left(\psi\frac{\partial}{\partial\psi} - \bar{\psi}\frac{\partial}{\partial\bar{\psi}}\right), \quad [G, Q] = [G, Q^\dagger] = [G, H_q] = 0\,. \tag{4.10}$$

With respect to the new product $<< \,,\, >>$, all norms are strictly positive.

4.3 Integrability

One can pose the question in which cases the above eigenvalue problems can be fully solved and the entire energy spectrum of the model can be found.

A salient feature of the quantum theory is that the worldline $\mathcal{N}=2$ supersymmetry implies the factorization property for the component Hamiltonians (modulo constant shifts). These Hamiltonians live on a curved 2-dimensional manifold $M_2 \sim (z, \bar{z})$ and involve couplings to background magnetic field. The factorizable Hamiltonians of this sort were studied by Ferapontov and Veselov [12]. They found that a sufficient condition for such systems to be integrable is the existence of an infinite sequence of factorization chains, which amounts to determining infinite sequences of eigenvalues and eigenvectors of the corresponding Hamiltonians. They proved that this is the case, iff **i)** The Gauss curvature $\mathbb{K}$ of M_2 is a constant:

$$\mathbb{K} = 2g^{z\bar{z}}\partial_z\partial_{\bar{z}}\ln g^{z\bar{z}} = const, \tag{4.11}$$

and **ii)** The corresponding magnetic field is also a constant over M_2:

$$g^{z\bar{z}}\left[\bar{\nabla}_{\bar{z}}, \nabla_z\right] = c = const\,. \tag{4.12}$$

In our case these conditions, with $\mathbb{K} \neq 0$, amount to the relation:

$$K = \frac{1}{2}\left(1 + \frac{c}{\mathbb{K}}\right)\ln V\,. \tag{4.13}$$

The constant Gauss curvature $\mathbb{K}$ is known to be associated with only three types of the manifolds M_2: **a)** 2-plane with $\mathbb{K} = 0, g^{z\bar{z}} = const$; **b)** 2-sphere with $\mathbb{K} > 0, g^{z\bar{z}} \sim (1 + z\bar{z})^2$ and **c)** hyperboloid with $\mathbb{K} < 0, g^{z\bar{z}} \sim (1 - z\bar{z})^2$. The Landau model corresponding to the case **a)** is just superplane model. As an example of non-trivial curved solvable Landau model with the worldline $\mathcal{N}=2$ supersymmetry we studied in detail the option **b)**.

5 $\mathbb{CP}^1$ model

5.1 Lagrangian

The $\mathbb{CP}^1$ model corresponds to the particular choice of the generic $\mathcal{N}=2$ superfield action (4.1), with

$$K(\Phi,\bar\Phi) = -N\ln\left(1+\Phi\bar\Phi\right), \quad V(\Phi,\bar\Phi) = \left(1+\Phi\bar\Phi\right)^2. \tag{5.1}$$

Here N is quantized by the standard cohomology arguments, $N \in (\mathbb{N}, \mathbb{N}+\frac{1}{2})$. The action is invariant under $SU(2)$ transformations

$$\delta\Phi = \varepsilon + i\beta\,\Phi + \bar\varepsilon\,\Phi^2, \quad \delta\Psi = -\left(i\beta + 2\bar\varepsilon\,\Phi\right)\Psi. \tag{5.2}$$

The component Lagrangian, after eliminating the auxiliary fields, takes the form

$$\begin{aligned}\mathcal{L}_{su(2)} =\;& \frac{\dot z\dot{\bar z}}{(1+z\bar z)^2} + N^{-1}(1+z\bar z)^2\left[1+2N^{-1}\psi\bar\psi\,(1+z\bar z)^2\right]\nabla\psi\nabla\bar\psi \\ & - i\left[\frac{N-2\psi\bar\psi\,(1+z\bar z)^2}{1+z\bar z}\left(\dot z\bar z - \dot{\bar z}z\right) - (1+z\bar z)^2\left(\dot\psi\bar\psi - \psi\dot{\bar\psi}\right)\right],\end{aligned} \tag{5.3}$$

where

$$\nabla\psi = \dot\psi + 2\frac{\dot z\bar z}{1+z\bar z}\,\psi, \quad \nabla\bar\psi = \dot{\bar\psi} + 2\frac{\dot{\bar z}z}{1+z\bar z}\,\bar\psi.$$

5.2 Eigenvalue problems

In the considered case it will be convenient to use the manifestly $SU(2)$ covariant inner product

$$<< f,g >>= \int \frac{dz\,d\bar z}{(1+z\bar z)^2}\left[\bar f_1 g_1 + \bar f_2 g_2 + 2(1+z\bar z)^2\left(\bar f_0^L g_0^L + \bar f_0^H g_0^H\right)\right].$$

The relevant eigenvalue equations are:

$$-V\nabla_{\bar z}^{(N+1)}\nabla_z^{(N+1)}f_1 = \lambda_1 f_1, \quad -V\nabla_z^{(N-1)}\nabla_{\bar z}^{(N-1)}f_2 = \lambda_2 f_2, \tag{5.4}$$

$$-\nabla_{\bar z}^{(N-1)}V\nabla_z^{(N-1)}f_0^H = \lambda_3 f_0^H, \quad -\nabla_z^{(N+1)}V\nabla_{\bar z}^{(N+1)}f_0^L = \lambda_4 f_0^L, \tag{5.5}$$

with $V=(1+z\bar z)^2$, $\nabla_z^{(N)} = \partial_z - N\frac{\bar z}{1+\bar z z}$, $\nabla_{\bar z}^{(N)} = \partial_{\bar z} + N\frac{z}{1+\bar z z}$. Ground states are defined by the equations

$$\nabla_z^{(N+1)}f_1 = \nabla_{\bar z}^{(N-1)}f_2 = \nabla_z^{(N-1)}f_0^H = \nabla_{\bar z}^{(N+1)}f_0^L = 0. \tag{5.6}$$

Both the ground states and excited LL states should be normalizable with respect to the above norm. This is a very stringent requirement. It implies the wave functions to carry irreducible $SU(2)$ multiplets with spins related to the number N.

5.3 Ground states

For $N=0$ there are two normalizable singlet ground states

$$f_0^{H,0}(z,\bar z) = \frac{f_0^{H,0}}{1+\bar z z}, \quad f_0^{L,0}(z,\bar z) = \frac{f_0^L}{1+\bar z z}, \tag{5.7}$$

where $f_0^{H,0}$ and $f_0^{L,0}$ are constants. Thus in this case the ground states are $SU(2)$ singlets.

For $N=\frac{1}{2}$, one has normalizable doublet ground states:

$$f_0^{L,0}(z,\bar z) = \frac{A+Bz}{(1+\bar z z)^{\frac{3}{2}}}, \tag{5.8}$$

the constants A and B thus forming spin 1/2 multiplet of $SU(2)$.

For $N \geqslant 1$, one has the following set of the ground states:

$$f_2^0(z,\bar{z}) = \frac{f_2^0(z)}{(1+\bar{z}z)^{N-1}}\,, \qquad N_{max} = 2(N-1)\,, \tag{5.9}$$

$$f_0^{L,0}(z,\bar{z}) = \frac{f_0^{L,0}(z)}{(1+\bar{z}z)^{N+1}}\,, \qquad N_{max} = 2N\,, \tag{5.10}$$

$f_2^0(z)$ and $f_0^{L,0}(z)$ being z-polynomials of the maximum degree N_{max}. Thus the ground states carry $SU(2)$ spins $N-1$ and N.

All ground states are singlets under the $\mathcal{N}=2$ supersymmetry, i.e. for $N \geq 0$, $\mathcal{N}=2$ supersymmetry is *unbroken*.

5.4 Excited states

For the first $\mathcal{N}=2$ multiplet (f_1, f_0^L) one has the following full set of the eigenvalues and normalizable wave functions:

$$E_\ell = \ell(\ell+2N+1), \quad \ell = 0,1,2,\ldots \tag{5.11}$$

$$f_1^1 = \tilde{f}_1^1,\; f_1^\ell = \nabla_z^{(N+3)} \cdots \nabla_z^{(N+2\ell-1)} \tilde{f}_1^\ell\,,\; \ell > 1, \quad f_0^{L,\ell} = \nabla_z^{(N+1)} \hat{f}_1^\ell\,,\; \ell \geq 1\,, \tag{5.12}$$

$$\tilde{f}_1^\ell = \frac{\tilde{f}_1^\ell(z)}{(1+\bar{z}z)^{N+1}}, \qquad \tilde{\hat{f}}_1^\ell = \frac{\hat{\tilde{f}}_1^\ell(z)}{(1+\bar{z}z)^{N+1}}\,. \tag{5.13}$$

The polynomials $\tilde{f}_1^\ell(z)$ and $\hat{\tilde{f}}_1^\ell(z)$ both carry spins $(N+\ell)$. This two-fold degeneracy is related to $\mathcal{N}=2$ supersymmetry which mixes these two states.

Situation with the second $\mathcal{N}=2$ multiplet (f_2, f_0^H) is more intricate and it requires a separate analysis for $N \geq 1$ and $0 \leq N < 1$. For these two cases we have, respectively, the following sequences of the eigenvalues

$$E_\ell^H = \ell(\ell+2N-1), \quad \ell = 0,1\ldots\,, \qquad E_\ell = (\ell+1)(\ell-2N+2), \quad \ell = 0,1\ldots\,. \tag{5.14}$$

For $N=0$ in the second case the system reveals a four-fold degeneracy (like in the superplane Landau model).

For $N=\frac{1}{2}$, there is no ground state for the second $\mathcal{N}=2$ multiplet and $\mathcal{N}=2$ supersymmetry looks as spontaneously broken in this sector. However, no actual breaking occurs because there is a singlet ground state in the first multiplet (f_1, f_0^L) at $N=\frac{1}{2}$.

6 Summary and outlook

The basic results reported in this Talk can be summarized as follows:

- The worldline $\mathcal{N}=2$ supersymmetry defines a general family of quantum super-Landau models in terms of two independent potentials generating a Kähler metric and coupling to magnetic field. The Hamiltonians are factorized, which allows for a general definition of ground states.

- Due to non-canonical second-order kinetic terms for fermions, the states at each excited LL are grouped into two irreducible $\mathcal{N}=2$ multiplets. This is in contrast to the models with the first-order fermionic kinetic terms, where such states span a single $\mathcal{N}=2$ multiplet (see, e.g., [13]).

- The appearance of the negative norms, like in other super-Landau models, can be evaded by *redefining* the inner product.

- In the $\mathbb{CP}^1$ model, the eigenvalues and eigenfunctions are split into two sequences corresponding to two super monopole systems, one with the charge $2N$ and the other with the charge $2(N-1)$. $\mathcal{N}=2$ supersymmetry is unbroken for any strength of the monopole, in contrast to the "minimal" $\mathcal{N}=2$ $\mathbb{CP}^1$ model in which it is spontaneously broken at $N=1/2$ [13].

I finish by mentioning possible physical applications of the new class of super-Landau models presented here and some problems for further study.

- One can wonder whether this kind of models could have some implications in supersymmetric versions of the QHE. The possible physical significance of the superplane $\mathcal{N}=2$ supersymmetric Landau models in this context was discussed by Hasebe [8]. The models addressed in my talk are curved generalizations of the super-planar models, so it is natural to expect that they can be related to some versions of supersymmetric QHE on curved manifolds.

- It is interesting to find out possible relations of the $\mathbb{CP}^1$ and other particular models to integrable structures in $\mathcal{N}=4$ SYM and string theory. In this connection, it would be worthwhile to see whether some of the generalized $\mathcal{N}=2$ models admit hidden internal supersymmetries like the $ISU(1|1)$ symmetry of the superplane model and whether they can be obtained by a dimensional reduction from some higher-dimensional theories.

- The obvious (and solvable) problem is to extend our consideration to higher $\mathcal{N}$ worldline supersymmetries, i.e. to explore the possibility of existence of the corresponding super-Landau models. The relevant worldline supermultiplets can involve more physical bosonic fields, so these models can be related to higher-dimensional QHE, e.g. to QHE in four dimensions [14]. The first example of the super-Landau model with $\mathcal{N}=4$ worldline supersymmetry and four-dimensional bosonic manifold will be soon presented [15].

Acknowledgements

I thank the Organizers of QUARKS-2010 and, personally, Victor Matveev and Valery Rubakov for inviting me to present this talk and creating a very friendly and stimulating scientific atmosphere at the conference. I am indebted to my co-authors in refs. [4]-[7], [10] for the fruitful collaboration. A partial support from the RFBR grants 09-02-01209 and 09-01-93107 is acknowledged.

References

[1] L. Landau, *Diamagnetismus der Metalle* Z. Phys. **64** (1930) 629.

[2] F.D.M. Haldane, *Fractional Quantization of the Hall Effect: A Hicrarchy of Incompressible Quantum Fluid States*, Phys. Rev. Lett. **51** (1983) 605.

[3] *Quantum Hall Effect: Field Theoretical Approach and Related Topics*, World Sci., Singapore, 2000.

[4] E. Ivanov, L. Mezincescu and P. K. Townsend, *A super-flag Landau model*, `arXiv:hep-th/0404108`.

[5] A. Beylin, T. Curtright, E. Ivanov, L. Mezincescu and P. K. Townsend, *Unitary Spherical Super-Landau Models*, JHEP **0810** (2008) 069, `arXiv:0806.4716[hep-th]`.

[6] E. Ivanov, L. Mezincescu and P. K. Townsend, *Planar super-Landau models*, JHEP **0601** (2006) 143, arXiv:hep-th/0510019.

[7] T. Curtright, E. Ivanov, L. Mezincescu and P. K. Townsend, *Planar super-Landau models revisited*, JHEP **0704** (2007) 020, arXiv:hep-th/0612300.

[8] K. Hasebe, *Quantum Hall Liquid on a Noncommutative Superplane*, Phys. Rev. **D 72** (2005) 105017, arXiv:hep-th/0503162.

[9] E. Witten, *Dynamical Breaking of Supersymmetry*, Nucl. Phys. **B 188** (1981) 513.

[10] A. Beylin, T. Curtright, E. Ivanov, and L. Mezincescu, *Generalized N=2 Super-Landau Models*, JHEP **1004** (2010) 091, arXiv:1003.0218[hep-th].

[11] E. Ivanov, *Supersymmetrizing Landau models*, Theor. Math. Phys. **154** (2008) 349, arXiv:0705.2249[hep-th].

[12] E. V. Ferapontov and A. P. Veselov, *Integrable Schrödinger operators with magnetic fields: factorisation method on curved surfaces*, J. Math. Phys. **42** (2001) 590, arXiv:math-ph/0007034.

[13] E. D'Hoker and L. Vinet, *Supersymmetry Of The Pauli Equation In The Presence Of A Magnetic Monopole*, Phys. Lett. **B 137** (1984) 72; V. P. Akulov and A. I. Pashnev, *Supersymmetric Quantum Mechanics And Spontaneous Breaking Of Supersymmetry At The Quantum Level*, Theor. Math. Phys. **65** (1985) 1027 [Teor. Mat. Fiz. **65** (1985) 84]; G. A. Mezincescu and L. Mezincescu, *Factorization method and the supersymmetric monopole harmonics*, J. Math. Phys. **44** (2003) 3595, arXiv:hep-th/0109002.

[14] S.-C. Zhang, J.-P. Hu, *A Four-dimensional generalization of the quantum Hall effect*, Science **294** (2001) 823, arXiv:cond-mat/0110572; H. Elvang, J. Polchinski, *The Quantum Hall Effect on R^4*, arXiv:hep-th/0209104.

[15] V. Bychkov, E. Ivanov, $\mathcal{N} = 4$ *supersymmetric Landau models*, work in preparation.