Wave function of gravitating shell

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Abstract

For the wave function of thin gravitating shell in the Reissner-Nordström geometry it is found an analytical solution in the form of the Meiksner polynoms and calculated discrete spectrum. It's shown that the extreme state in quantum spectrum of gravitating shell is absent, similar to the case of extreme black hole.

In the absent of quantum theory of gravity, a quasi-classical approximation is a convenient method for description of quantum effects on the classical background in general relativity (gravitons, Hawking effect, spectrum of primary perturbations, black hole mass spectrum). Here we describe a thin gravitating shell model, which is very useful for analyzing a black hole mass spectrum [3].

In the thin shell formalism there is a simple method to get a spectrum mass of black holes. This method is based on natural assumption that a mass of gravitating systems (for instance, a mass of black hole) on a space infinity $m_{\rm out}$ is a Hamiltonian of the system. Then, if we know the Hamiltonian of the systems, we may obtain in principle a wave equation and solve it.

The Schwarzschild black hole in General relativity is described by only one parameter, namely the black hole mass. At the same time, on the Carter-Penrose diagram for the eternal Schwarzschild black hole (see Fig. 1), which describes a corresponding global geometry, there are two space infinities, called, correspondingly, R_+ and R_- regions. For this reason, we guess, that in the Schwarzschild geometry for eternal black hole, a spectrum mass may depend on two quantum numbers [4].

Let us consider a dynamical equation for the evolution of thin shell in the Reissner - Nordström geometry. This equation is (see e. g., [5, 6], and also [7, 8, 9, 10]):

$$\sigma_{\rm in}\sqrt{\dot\rho^2+1-\frac{2m_{\rm in}}{\rho}+\frac{Q_{\rm in}^2}{\rho^2}}-\sigma_{\rm out}\sqrt{\dot\rho^2+1-\frac{2m_{\rm out}}{\rho}+\frac{Q_{\rm out}^2}{\rho^2}}=4\pi\rho\,\mu(\rho), \qquad (1)$$

where $\sigma_{\rm in,out}=\pm1$, and $m_{\rm in}$, $m_{\rm out}$, $Q_{\rm in}$, $Q_{\rm out}$ are, correspondingly, a black hole mass and charge inside and outside of the shell, $\rho=\rho(\tau)$ is a shell radius, measured by an observer at rest with respect to this shell and the dot $(\dot{\ })$ defines a derivative with respect to proper time τ of this observer. We consider below only the dust shell, when $\mu(\rho)$ is $\mu(\rho)=A/\rho^2$, where $A>0$ is an integration constant [6]. It is convenient to use notation $M=4\pi A$, which is a total shell mass. The sign condition, $\sigma_{\rm in,out}$, as it is easy seen from equation (1), has a form

$$\sigma_{\rm in} = {\rm sign}\left[m_{\rm out}-m_{\rm in}+\frac{Q_{\rm in}^2-Q_{\rm out}^2+M^2}{2\rho}\right], \qquad (2)$$

$$\sigma_{\rm out} = {\rm sign}\left[m_{\rm out}-m_{\rm in}+\frac{Q_{\rm in}^2-Q_{\rm out}^2-M^2}{2\rho}\right]. \qquad (3)$$

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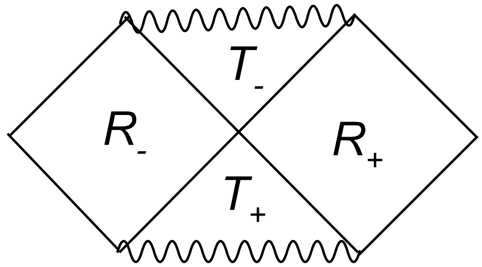

Figure 1: The Carter-Penrose diagram for eternal Schwarzschild black hole.

We consider below a case, when $m_{\rm out} > m_{\rm in}$ and $M^2 \geq Q^2_{\rm out} - Q^2_{\rm in}$. So the sign $\sigma_{\rm in}$ equals to $\sigma_{\rm in} = 1$, in the all space-time, but the sign $\sigma_{\rm out}$ may be equal to 'plus' or to 'minus'. After squaring of this equation (1), we get the expression

$$m_{\rm out} = m_{\rm in} + \frac{Q^2_{\rm out} - Q^2_{\rm in} - M^2}{2\rho} + M\sigma_{\rm in}\sqrt{\dot\rho^2 + 1 - \frac{2m_{\rm in}}{\rho} + \frac{Q^2_{\rm in}}{\rho^2}}. \tag{4}$$

The left part of this expression is a total mass of the system $m_{\rm out}$, which is conserved during the time evolution. Basing on the results of previous works [11, 12, 13, 14], we define this mass as a Hamiltonian of the system H.

To proceed further, we define a new variable $x = M\rho$ in the equation (4) (see also [11, 12, 13]). Then the Hamiltonian is

$$H = \sigma_{\rm in}\sqrt{\dot x^2 + M^2\left(1 - \frac{2m_{\rm in}M}{x} + \frac{Q^2_{\rm in}M^2}{x^2}\right)} + m_{\rm in} + M\frac{Q^2_{\rm out} - Q^2_{\rm in} - M^2}{2x}. \tag{5}$$

The corresponding Lagrangian is

$$\begin{aligned} L &= \sigma_{\rm in}\dot x \ln\left[\dot x + \sqrt{\dot x^2 + M^2\left(1 - \frac{2m_{\rm in}M}{x} + \frac{Q^2_{\rm in}M^2}{x^2}\right)}\right] - \sigma_{\rm in}\sqrt{\dot x^2 + M^2\left(1 - \frac{2m_{\rm in}M}{x} + \frac{Q^2_{\rm in}M^2}{x^2}\right)} \\ &+ M\frac{M^2 - Q^2_{\rm out} + Q^2_{\rm in}}{2x} - \sigma_{\rm in}\dot x \ln M - m_{\rm in}, \end{aligned} \tag{6}$$

and, respectively, a canonical momentum

$$p = \sigma_{\rm in}\ln\left[\frac{\dot x}{M} + \sqrt{\frac{\dot x^2}{M^2} + \left(1 - \frac{2m_{\rm in}M}{x} + \frac{Q^2_{\rm in}M^2}{x^2}\right)}\right]. \tag{7}$$

Now we express the Hamiltonian (5) through a canonical momentum (7). In order to do this, we find from equation (7) the value $\dot x$ and substitute it to the Hamiltonian (5). As a result, the Hamiltonian is

$$H = m_{\rm in} - M\frac{M^2 - Q^2_{\rm out} + Q^2_{\rm in}}{2x} + \frac{\sigma_{\rm in}M}{2}\left[e^{\sigma_{\rm in}p} + \left(1 - \frac{2m_{\rm in}M}{x} + \frac{Q^2_{\rm in}M^2}{x^2}\right)e^{-\sigma_{\rm in}p}\right].$$

In order to write a wave equation $H\phi(x) = m_{\rm out}\phi(x)$, we use a commutation relation $[p, x] = -i$ and identity

$$\exp\left(x_0\frac{\partial}{\partial x}\right)\phi(x) = \phi(x + x_0). \tag{8}$$

As a result, a wave equation is

$$\phi(x - i) + \left(1 - \frac{2m_{\rm in}M}{x} + \frac{Q^2_{\rm in}M^2}{x^2}\right)\phi(x + i) - \frac{M^2 - Q^2_{\rm out} + Q^2_{\rm in}}{x}\phi(x) = 2E\phi(x), \tag{9}$$

where $E = (m_{\rm out} - m_{\rm in})/M$. The distinction of wave equation (9) from the usual Schrodinger one, is that this is a difference equation, not the differential one. This is a result of the quantization, made with respect of the proper time of observer on the shell, but not with respect of time for the observer at infinity.

If we expand the exponent in (8) in series, then the wave equation will transform to the difference equation of infinite order. Hence, the infinite number of boundary conditions are needed to add. These boundary conditions were used in the form

$$\phi^{2l}(0) = 0, \quad l = 0, 1 \ldots. \tag{10}$$

It is requested also that a wave function must be finite at the space infinity.

A general solution of wave equation (9) is expressed through the Meiksner polynoms (see, e. g. [12]), which satisfy the next difference equation

$$\sigma(x)[f(x+1) - 2f(x) + f(x-1)] + \tau(x)[f(x+1) - f(x)] + \lambda f(x) = 0. \tag{11}$$

In this equation $\sigma(x) = x$, $\sigma(x) + \tau(x) = \mu(\gamma + x)$, where μ and γ are constants. In order to solve the wave equation (9), we reduce it to equation (11). At first, we made the coordinate transformation $x \to -ix$, which corresponds to $90^{\rm o}$ rotation in the complex plane. Then, equation (9) transforms to

$$\phi(x+1) + \left(1 - \frac{2m_{\rm in}Mi}{x} - \frac{Q_{\rm in}^2 M^2}{x^2}\right)\phi(x-1) - i\frac{M^2 - Q_{\rm out}^2 + Q_{\rm in}^2}{x}\phi(x) = 2E\phi(x). \tag{12}$$

Consider a simple case, when $m_{\rm in} = Q_{\rm in} = 0$. Solution of a wave equation (12) expresses over the Meiksner polynoms

$$\phi_n(x) = C(x)\frac{\beta^x x}{\beta^{2x+2n}}\triangle^n\left[\frac{\beta^{2x}\Gamma(x)}{\Gamma(x+1-n)}\right], \tag{13}$$

where

$$\beta = E + \sqrt{E^2 - 1}, \quad \triangle f(x) = f(x+1) - f(x), \quad C(x) = C(x+1), \tag{14}$$

$\Gamma(x)$ is a gamma function and $C(x)$ is a periodical function with a time period equals to 1. We expand the periodical function $C(x)$ in Fourier series:

$$C(x) = \sum_{k=-\infty}^{\infty} c_k \exp(2\pi i k x). \tag{15}$$

The same factor $C(x)$ is appeared, if we will find solution of wave equation in the impulse presentation [13, 16]. The coefficients c_k may be found from boundary conditions (10).

The wave function $\phi_n(x)$ is orthogonal in the following meaning. If we put $x_i = x$ and $x_{i+1} = x_i + 1$, and use an orthogonal relation for the Meiksner polynoms, then, then we obtain

$$\sum_{x_i=0}^{\infty} \phi_n(x_i)\phi_m(x_i)\rho(x_i) = \delta_{nm} d_n^2, \tag{16}$$

where a weight function $\rho(x) = \left[xC^2(x)\right]^{-1}$ and normalization coefficients

$$d_n^2 = \frac{\Gamma(n)\Gamma(n+1)}{\beta^{2n}}.$$

From equation (16) it follows that a wave function $\phi_n(x)$ for $n \neq m$ is orthogonal if $0 < \beta < 1$ [15]. It is easy to see that this function is also the orthonormal.

A discrete mass spectrum E_n, corresponding to a wave function $\phi_n(x)$ in the form of Meiksner polynom [15], satisfies the equation

$$i(M^2 - Q_{\rm out}^2) = 2n\sqrt{E_n^2 - 1}. \tag{17}$$

The resulting mass spectrum is

$$E_n^2 = \left(\frac{m_{\rm out}}{M}\right)^2 = 1 - \frac{(M^2 - Q_{\rm out}^2)^2}{4n^2}. \tag{18}$$

The hydrogen-like mass spectrum mass (18) is a generalization of a similar spectrum in work [11], and coincides with the spectrum obtained by different method in [13, 16, 17]. A discrete mass spectrum (18) is realized if $E_n < 1$. In the opposite case $E_n > 1$, a mass spectrum mass would be continuous [14]. We see that according to expression (18), a total gravitational mass $m_{\rm out}$ is less then a classical naked mass M due to a quantum correction.

In the case $(M^2 - Q_{\rm out}^2)^2/4 > 1$, it can be written a condition for the applicability of quasiclassical approximation. Really, there is minimal value of quantum number $n_{\rm min} = [(M^2 - Q_{\rm out}^2)/2]$, where the bracket symbol "[ ]" denotes the integer part of number. A quasiclassical approximation will be true if $n \gg n_{\rm min}$. Respectively, if the opposite condition is satisfied, $(M^2 - Q_{\rm out}^2)^2/4 < 1$, a quasiclassical approximation will be true for any number n, .

To write a solution for the original wave equation (9), it is necessary to made a reverse transformation of variable, $x \to ix$. For example, for the first two Meiksner polynoms, we obtain

$$\begin{aligned}
\phi(x)_{n=1} &= ix\beta^{ix}\frac{\beta^2 - 1}{\beta^2}\sum_{k=-\infty}^{\infty} c_k \exp(-2\pi kx), \\
\phi(x)_{n=2} &= ix\beta^{ix}\left[ix\left(1 - \frac{1}{\beta^2}\right)^2 + 1 - \frac{1}{\beta^4}\right]\sum_{k=-\infty}^{\infty} c_k \exp(-2\pi kx).
\end{aligned}$$

For finiteness of wave function at the space infinity, it is requested that constants $c_k = 0$ for $k < 0$. The remaining integration constant may be found from the boundary condition at $x = 0$. It was found in [13, 16] for $n = 1$, and we will not present it here. Making the formal replacement $E = \cos\lambda$, the solution for wave function may be presented in the form $\phi(x)_n = P_n(x)\exp(-\lambda x)C(x)$, where $P_n(x)$ are some polynoms of the order n (for details see [13]).

In the extreme case, when $m_{\rm out} = Q_{\rm out}$ the mass spectrum mass is degenerated (does not dependent on n), and $E = 1$. The absence of extreme state in the quantum spectrum is in a full agreement with a similar result for the mass spectrum mass of Reissner-Nordström black hole [18]. This result also corresponds to the third law of black hole thermodynamics, which states that the black hole extreme state is unreachable. On the quantum level this means the absence of possibility for a black hole decay into the extreme state [19, 20, 21]. A formal solution of wave equation (12) in the extreme state is

$$\phi(x) = C_1(x) + C_2(x)x,$$

where C_1, C_2 are periodical functions with a time period equals 1. This periodical function is expanded in the Fourier series

$$\phi(x) = \sum_{k=0}^{\infty} c_k \exp(-2\pi kx) + ix\sum_{k=0}^{\infty} d_k \exp(-2\pi kx), \tag{19}$$

where it is used a reverse transformation $x \to ix$ and the boundary condition at the space infinity. The corresponding coefficients c_k and d_k may be found by using the boundary conditions (10).

In the case, when $M^2 \le Q^2_{\rm out}$, the signs of $\sigma_{\rm in}$ can be as a plus and a minus one, $\sigma_{\rm in} = \pm 1$. For radius $\rho < (Q^2_{\rm out} - M^2)/2m_{\rm out} < 2m_{\rm out}$ the sign of $\sigma_{\rm in} = -1$. A corresponding solution for the wave equation will have a form $\hat\phi(x) = (-1)^x\phi(x)$, where $\phi(x)$ is the solution (13) with the same mass spectrum.

In order to find a general solution of wave equation (9) in the more general case, when only $Q_{\rm in} = 0$, we made an another transformation of coordinate variable, namely, $x \to ix$. This transformation means a rotation on the $\pi/2$ degrees in the complex plane. Equation (9) now are rewritten in the form

$$\phi(x-1) + \left(1 + \frac{2m_{\rm in}Mi}{x}\right)\phi(x+1) + i\frac{M^2 - Q^2_{\rm out}}{x}\phi(x) = 2E\phi(x). \tag{20}$$

Solution of the this equation is expressed through the Meiksner polynoms [15]:

$$\phi_n(x) = C(x)\frac{\tilde\beta^x\Gamma(x+1)}{\tilde\beta^{2x+2n}\Gamma(\gamma+x)}\triangle^n\left[\frac{\tilde\beta^{2x+2n}\Gamma(\gamma+x)}{\Gamma(x+1-n)}\right], \tag{21}$$

where

$$\tilde\beta = E - \sqrt{E^2-1}, \quad \gamma = i2m_{\rm in}M, \quad C(x) = C(x+1), \tag{22}$$

and $C(x)$ is a periodical function with a time period equals 1. The corresponding first two polynoms are

$$\begin{aligned} \phi(x)_{n=1} &= [\tilde\beta^2(\gamma+x) - x]\tilde\beta^x C(x), \\ \phi(x)_{n=2} &= \tilde\beta^x[\tilde\beta^4(\gamma+x+1)(\gamma+x) - 2\tilde\beta^2 x(\gamma+x) + x(x-1)]C(x). \end{aligned} \tag{23}$$

As in the previous case, we put $x = x_i$, $x_{i+1} = x_i + 1$ and will sum the wave function with a weight

$$\rho(x) = \frac{\Gamma(\gamma+x)}{\Gamma(1+x)\Gamma(\gamma)C^2(x)}, \tag{24}$$

In result, the considered polynoms will be orthogonal [15]:

$$\sum_{x_i=0}^{\infty}\phi_n(x_i)\phi_m(x_i)\rho(x_i) = \delta_{nm}d_n^2, \tag{25}$$

where

$$d_n^2 = \frac{n!\Gamma(n+\gamma)}{\tilde\beta^{2n}(1-\tilde\beta^2)^\gamma\Gamma(\gamma)}. \tag{26}$$

By making the reverse replacement $x \to -ix$, polynoms may be written as

$$\begin{aligned} \phi(x)_{n=1} &= [\tilde\beta^2(\gamma - ix) + ix]\tilde\beta^{-ix}\sum_{k=-\infty}^{\infty} c_k\exp(-2\pi kx), \\ \phi(x)_{n=2} &= \tilde\beta^{-ix}[\tilde\beta^4(\gamma-ix+1)(\gamma-ix) + 2\tilde\beta^2 ix(\gamma-ix) + ix(ix+1)]\sum_{k=-\infty}^{\infty} c_k\exp(-2\pi kx). \end{aligned} \tag{27}$$

It is easy to show, that for $\gamma = 0$, i. e. for $m_{\rm in} = 0$, the solution (27) for wave equation reduces to the solution (19). A resulting discrete mass spectrum is

$$i(M^2 - Q^2_{\rm out} + 2m_{\rm in}M\tilde\beta) = 2n\sqrt{E_n^2 - 1}. \tag{28}$$

This mass spectrum is an imaginary because the Hamiltonian (5) is not hermitian for $m_{\rm in} \neq 0$. There only case, when a mass spectrum is real, is when $E_n^2 = 1$ and $M^2 - Q^2_{\rm out} + 2m_{\rm in}M\tilde\beta = 0$. From these two equations there are follow corresponding conditions on parameters: $m_{\rm out} =$

$m_{\rm in} + M$ and $M = -m_{\rm in} + \sqrt{m_{\rm in}^2 + Q_{\rm out}^2}$. This as a limiting case, corresponding to transition from the discrete spectrum to continuous one. In order to made the Hamiltonian to be the Hermitian one for $m_{\rm in} \neq 0$, it is necessary to made the replacement in operators $A(x)B(p) \to \frac{1}{2}[A(x)B(p) + B^*(p^*)A^*(x)]$, which in our case is

$$\frac{1}{x}\exp\left(i\frac{\partial}{\partial x}\right) \to \frac{1}{2}\left[\frac{1}{x}\exp\left(i\frac{\partial}{\partial x}\right) + \exp\left(-i\frac{\partial}{\partial x}\right)\frac{1}{x}\right]. \tag{29}$$

After this replacement of operator, the wave equation for the Hermitian Hamiltonian has a form

$$\phi(x+i) + \phi(x-i) - m_{\rm in}M\left[\frac{\phi(x+i)}{x} + \frac{\phi(x-i)}{x-i}\right] - \frac{M^2 - Q_{\rm out}^2}{x}\phi(x) = 2E\phi(x). \tag{30}$$

This equation is a very complex. Solution of this equation and energy spectrum mass must depended on two quantum numbers, the inner $m_{\rm in}$ and the outer $m_{\rm out}$ mass of black hole. We did not find the corresponding exact solution.

In summary, we considered a simple model of quasi-classical quantization of thin shell in the Reissner - Nordström geometry. It was used an ansatz, based on mass spectrum for a more simple model, when the inner and outer space-times of the shell are the Schwarzschild geometries. The corresponding mass spectrum of the shell depends on two quantum numbers. It has a simple explanation. On the Carter-Penrose diagram for Schwarzschild black hole, there are two space-time regions, R_- and R_+ (see Fig. 1). According to [1, 2], for the every region $R_\pm$, a mass spectrum depends on one quantum number. Respectively, the mass spectrum in general depends on two quantum numbers, due to existence of two asymptotically flat regions in the global Schwarzschild space-time.

Authors would like to thank V. A. Berezin for useful discussion. This work was supported in part by the Russian Foundation for Basic Research grant 10-02-00635 and by the grant of the Leading Scientific Schools 3517.2010.2. The work of S. V. Chernov was supported in part by the Dynasty Foundation.

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