Brane Universe: Global Geometry

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Abstract

The global geometries of bulk vacuum space-times in the brane-universe models are investigated and classified in terms of geometrical invariants. The corresponding Carter-Penrose diagrams and embedding diagrams are constructed. It is shown that for a given energy-momentum induced on the brane there can be different types of global geometries depending on the signs of a bulk cosmological term and surface energy density of the brane (the sign of the latter does not influence the internal cosmological evolution). It is shown that in the Randall-Sundrum scenario it is possible to have an asymmetric hierarchy splitting even with a Z_2-symmetric matching of "our" brane to the bulk.

In this talk we would like to investigate the possible global geometries of the so-called "brane universe" scenarios, in which our Universe is supposed to be a thin shell, "membrane", embedded into the space-time of larger number of dimensions, "bulk".

Our strategy is to simplify everything as much as possible and construct some exactly solvable model, because only the thorough investigation of such models is the source of the physical intuition. So, let us consider a $(N+1)$-dimensional space-time containing a N-dimensional brane (thin shell) with the metric

$$ds^2 = g_{\mu\nu}(y)dy^\mu dy^\nu. \tag{1}$$

Having in mind the very name of the Conference, we demand the brane to be time-like and have the so-called cosmological symmetry, i.e., homogeneity and isotropy. Moreover, we assume (and this is the first step in our simplification process) that outside the brane the "bulk" geometry possesses the same symmetry, in other words, locally, the bulk geometry does not depend on the place of the brane. This means, that throughout the whole $(N+1)$-dimensional manifold we can introduce the normal Gaussian coordinate system in which the metric (1) takes the form

$$\begin{aligned} ds^2 &= -dn^2 + \gamma_{ij}(n,x)dx^i dx^j \\ &= -dn^2 + B^2(n,t)dt^2 - A^2(n,t)dl_{N-1}^2, \end{aligned} \tag{2}$$

where i, j take the values $(0, 2, ...N)$, and dl_{N-1}^2 is the Robertson-Walker unit line element of a homogeneous space,

$$dl_{N-1}^2 = \frac{dr^2}{1 - kr^2} + r^2 d\Omega_{N-2}^2 \tag{3}$$

with $d\Omega_{N-2}^2$ representing the line element of the unit $(N-2)$-dimensional sphere. For $k = +1$ the homogeneous space is the unit $(N-1)$-dimensional sphere, for $k = -1$ it is the rotational hyperboloid, and $k = 0$ means that such a space is flat. For a while we suppose that there

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exists only one brane in a bulk and put it at zero value of the normal coordinate $n = 0$. Then, the general form of the energy momentum tensor $T_{\mu\nu}$ is

$$T_{\mu\nu} = S_{\mu\nu}\delta(n) \quad + \quad [T_{\mu\nu}]\,\Theta(n) + T_{-\mu\nu},$$
$$[T_{\mu\nu}] = T_{+\mu\nu} - T_{-\mu\nu}. \tag{4}$$

Here $S_{\mu\nu}$ is the surface energy-momentum tensor on the brane, square brackets $[\,]$ denote a jump of some quantity across the shell (brane), $[X] = (X_+ - X_-)$, indices " $\pm$ " indicate the $n > 0(" + ")$ and the $n < 0(" - ")$ regions outside the shell, $\delta(n)$ and $\Theta(n)$ are conventional Dirac's and step functions. If $S_{\mu\nu} \neq 0$, the hyper-surface $n = const$ is singular, otherwise it is regular. Introducing the extrinsic curvature tensor $K_{ij} = -\frac{1}{2}\frac{\partial \gamma_{ij}}{\partial n} = -\frac{1}{2}\gamma_{ij,n}$ for the N-dimensional hyper-surfaces $n = const$ (both singular and regular ones), we are able to separate the Einstein equations into three groups according to the above decomposition of the energy-momentum tensor (everywhere G is the non-renormalized $(N + 1)$-dimensional gravitational constant).

1. $\binom{n}{n}$ -equations:

$$S_j^l\left\{K_l^j\right\} \quad + \quad [T_n^n] = 0\,,$$
$$\frac{1}{2}K_{-j}^l K_{-l}^j \quad - \quad \frac{1}{2}K_-^2 - \frac{1}{2}\,^{(N)}R = 8\pi\,G\,T_{-n}^n \tag{5}$$

and analogous equations in " $+$ "-region, $K = K_j^j$ is the trace of the extrinsic curvature tensor. The parenthesis means $\{X\} = \frac{1}{2}(X_+ + X_-)$, and $^{(N)}R$ is the N-dimensional Ricci scalar on every hyper-surface $n = const$.

2. $\binom{n}{i}$-equations:

$$S_{i|l}^l \quad + \quad [T_i^n] = 0\,,$$
$$K_{-i|l}^l \quad - \quad K_{-|i} = 8\pi\,G\,T_i^n\,, \tag{6}$$

where the vertical line denotes a covariant derivative with respect to the N-dimensional metric $\gamma_{ij}(x, n)$, and for the sake of brevity we will not mention anymore the equations in " $+$ "-region. The first of this set of equations is nothing more but the continuity equation for S_i^j.

3. (ik)-equations:

$$- \quad ([K_{ik}] - \gamma_{ik}[K]) = 8\pi\,G\,S_{ik}\,,$$
$$2 \quad S_i^l\left\{K_{lk}\right\} + 2S_k^l\left\{K_{il}\right\} - \frac{3}{N-1}S\left\{K_{ik}\right\} - S_{ik}\left\{K\right\} + \gamma_{ik}S_j^l\left\{K_l^j\right\} - \frac{1}{N-1}\gamma_{ik}S\left\{K\right\} = [T_{ik}]\,,$$
$$- \quad \left(K_{-ik,n} - \gamma_{ik}K_{-,n} + 2K_{-il}K_{-k}^l - K_{-ik}K + \frac{1}{2}\gamma_{ik}\left(K_{-j}^l K_{-l}^j + K\right)\right) +\,^{(N)}G_{ik} = 8\pi\,G\,T_{-ik}\,. \tag{7}$$

The equations in the first line are known as the Israel's equations. $^{(N)}G_{ik}$ is the N-dimensional Einstein tensor on every hyper-surface $n = const$. The last equation can also be written in the form

$$^{(N)}G_{ik} = 8\pi\,G\,T_{-ik} + 8\pi\,G\,T_{-ik}^{\text{ind}} \tag{8}$$

Using the proclaimed cosmological symmetry it is easy to calculate both $^{(N)}R$ and $^{(N)}G_{ik}$. For each hyper-surface $n = const$ we introduce the cosmological time by the relation $d\tau_n = B(n, t)dt$ and the scale factor $a(\tau_n) = A(n, t)$, then, by symmetry, $^{(N)}G_2^2 =\,^{(N)}G_3^3 = ... =\,^{(N)}G_N^N$, and

$$^{(N)}R \quad = \quad -(N-1)\left(\frac{2a_{\tau\tau}}{a} + (N-2)\frac{a_\tau^2 + k}{a^2}\right),$$
$$^{(N)}G_0^0 \quad = \quad \frac{(N-1)(N-2)}{2}\frac{a_\tau^2 + k}{a^2},$$
$$^{(N)}G_2^2 \quad = \quad \frac{N-2}{2}\left(\frac{2a_{\tau\tau}}{a} + (N-3)\frac{a_\tau^2 + k}{a^2}\right) \tag{9}$$

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where $a_\tau = \frac{da}{d\tau}$, $a_{\tau\tau} = \frac{d^2 a}{d\tau^2}$

The cosmological principle allows us to use also yet another technique in investigation of the global geometry. This is the so-called $(d+2)$-decomposition, and it is not related to the singular brane, but deals exclusively with the invariants of the bulk geometry. So, let us start. The metric of any $(d+2)$-dimensional space-time which is a direct product of a d-dimensional homogeneous space (cosmological symmetry!) and a two-dimensional space-time can be written in the form

$$ds^2 = \gamma_{AB}(x)dx^A dx^B - R^2(x)dl_d^2, \tag{10}$$

where dl_d^2 - the unit Robertson-Walker line element for a homogeneous space with the curvature $d(d-1)k$, $k = \pm 1, 0$, $A = 0, 1$ ("0" for some time coordinate t, "1" for some radial coordinate q), γ_{AB} is a two-dimensional metric tensor, and $R(x)$ is the radius or, in other words, scale factor, of the d-dimensional homogeneous space. Due to the general covariance a two-dimensional geometry is locally determined actually by only one function of two variables t and q. For the $(d+2)$-dimensional manifold with cosmological symmetry we need, therefore, to know only two functions of two variables. Naturally, one of them is the radius $R(t, q)$ which is invariant under (t, q)-transformations. Surely, we want the second function to be also an invariant. Geometrically, the best choice is the squared normal to the surfaces $R = const$. So, we define our second function as

$$\Delta = \gamma^{AB} R_{,A} R_{,B} \tag{11}$$

where comma means a partial derivative, $R_{,a} = \frac{\partial R}{\partial x^A}$. Remarkably enough that, using these two invariants, we can rewrite the two-dimensional part of the Einstein equations in this case in a very convenient vector-like form

$$\left(R^{d-1} (\Delta + k) \right)_{,A} = \frac{16\pi G}{d} R^d \left(T R_{,A} - T_A^B R_{,B} \right), \tag{12}$$

where $T = T_C^C$. The third equation, for $A \neq B$ can also be obtained as an integrability condition for the above vector equation:

$$\gamma^{AC} R_{||CB} = -\frac{8\pi G}{d} R T_B^A. \tag{13}$$

The double vertical line here denotes a covariant derivative with respect to the two-dimensional metric γ_{AB}. Our invariant Δ brings a very important geometrical information. Note, first of all, that for the flat Minkowskian space-time (of any dimension) $\Delta = -1$. But in the curved space-time Δ is no more a constant and can be both negative and positive. If it is negative, we can choose the radius R as a spatial coordinate ($\dot{R} = 0$, $(R')^2 = 1 \Longrightarrow \Delta = \gamma^{11}(R')^2 = \gamma^{11} < 0$) like in the flat space-time, and the surfaces $R = const$ are time-like. Such regions are called the R-regions. Moreover, in these regions $R_{,q}$ cannot change its sign, therefore we may have either $R_{,q} > 0$ in the R_+-regions, or $R_{,q} < 0$ in the R_--regions. Analogously, if $\Delta > 0$, the surfaces $R = const$ are space-like, and the radius R can be used as a time coordinate, these regions are called the T-regions. And, again, now the sign of $R_{,t}$ cannot be changed, so, there are T_+-regions with $R_{,t>0}$ (inevitable expansion) and T_--regions with $R_{,t<0}$ (inevitable contraction). The R- and T-regions are separated by hyper-surfaces $\Delta = 0$ called the apparent horizons. The global geometry of the space-time manifolds with cosmological symmetry is, therefore, the set of $R_\pm$- and $T_\pm$-regions separated by the apparent horizons $\Delta = 0$. Of course, not all such sets are physical. As a selection rule we will use the physical principle of geodesic completeness: any null or time-like geodesics must start and end either at infinities or at singularities where the Riemann curvature tensor becomes divergent.

It is time to make the next (second) step in simplification of our model. We assume that outside the brane at $n = 0$ the space-time is a vacuum with cosmological constant Λ, thus, the energy-momentum tensor for $n \neq 0$ has the form of an invariant tensor

$$8\pi G T_\mu^\nu = \Lambda \delta_\mu^\nu, \tag{14}$$

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in particular, $8\pi\, GT^B_A = \Lambda\delta^B_A$, $8\pi\, GT^C_C = 2\Lambda$. The vector-like equations are easily integrated now to give

$$\Delta = -k + \frac{2\,G\,m}{R^{N-2}} + \frac{2}{N(N-1)}\Lambda\,R^2, \tag{15}$$

here m is an integration constant with the dimension of mass. Note that here the invariant Δ is actually a function of one variable - invariant radius R. In such a case it is easy to write explicitly the two-dimensional line element separately in R- and T-regions. If $\Delta < 0$ (R-region), we can choose the radius as the spatial coordinate q (or $-q$). Then,

$$ds_2^2 = (-\Delta)\,dt^2 - \frac{dR^2}{(-\Delta)}\,. \tag{16}$$

In what follows we will need yet another form of the line element, namely, the conformally flat one. For this let us introduce the function $R^\star(R)$ by the relation

$$dR^\star = \pm\frac{dR}{|\Delta|}\,, \tag{17}$$

then

$$ds_2^2 = (-\Delta)\left(dt^2 - dR^{\star 2}\right). \tag{18}$$

In T-regions $\Delta > 0$ and we can choose R (or $-R$) as a time coordinate, for the line element one gets

$$ds_2^2 = \frac{dR^2}{\Delta} - \Delta dq^2 = \Delta\left(dR^{\star 2} - dq^2\right), \tag{19}$$

here $R^\star\,(-R^\star)$ is a time coordinate, and q a spatial coordinate of the Minkowskian (flat) two-dimensional space-time. We assume that there can be only one singular shell in the whole $N+1$-dimensional space-time, namely, our brane. Then, to avoid a singularity at $R = 0$ we have to put $m = 0$. In this case the value $k = \pm 1, 0$ should be the same everywhere, it is a global property. For different k we obtain completely different bulk space-times. We see, that such a global feature dictates the spatial curvature on the brane. The latter can be determined, in principle, by making measurements on the brane itself. And this is the way to know something about the bulk geometry (if, of course, we have some other evidences that the brane universe hypothesis is true). The case of several branes will be briefly discussed later.

Let us go further and make use of the Eqn(13). Substituting in it the two-dimensional part of the metric (2), namely, $ds_2^2 = -dn^2 + B^2(n,t)dt^2$, we obtain

$$\Delta = \frac{1}{B^2(n,t)}R^2(n,t)_{,t} - R^2(n,t)_{,n} = f^2(t) - R^2_{,n}, \tag{20}$$

where $f(t)$ is some function of time coordinate only. From this we have

$$R_{,n} = \pm\sqrt{f^2(t) - \Delta} = \sigma\sqrt{f^2(t) - \Delta}. \tag{21}$$

We introduced new and very important sign function σ. It shows whether radii increase with $n(\sigma = +1)$, or they decrease ($\sigma = -1$). It is clear from the definition that in R_+-regions $\sigma = +1$, and $\sigma = -1$ in R_--regions. In T-regions σ may change the sign. Thus, this sign will point at the region where exactly the brane is matched to the bulk. This last equation together with the fact that the invariant Δ, Eqn.(15), depends only on the radius R allows us to obtain the solution $R(n,t)$ as an explicit function of the normal coordinate n. To have the full information we need the equations on the brane at $n = 0$. Remembering that $R(n,t) = A(n,t)$ and $f(t) = \frac{R_{,t}}{B} = a_\tau$, we are able to calculate the extrinsic curvature tensor K_{ij} and the induced energy-momentum tensor on the brane T^{ind}_{-ij}. We need also a relation between the coordinate time t and the cosmological time τ on the brane. Using the freedom (gauge) in defining the coordinate time,

we can always put $B(0,t) = 1$, in other words, $t = \tau$ for $n = 0$ on the brane. Let us remind that the Israel's equations (matching conditions) give some relations between the jump in extrinsic curvature tensor and the surface energy-momentum tensor S_i^j. It is easy to show that this tensor determines also (together with the bulk cosmological constant) the induced energy-momentum tensor. We see now that, given the surface energy-momentum tensor S_i^j, the sign of the spatial curvature k and the value and the sign of the cosmological constant Λ we can construct both the global geometry of the bulk and the trajectory of the brane. Therefore, we will know the complete geometry of the whole space-time.

Of course, in general, it is still impossible to get the solution in a closed form. Hence, we need to simplify the model further. And as the final step, we restrict ourselves to investigation of the vacuum shells. Namely, we choose the following equation of state

$$S_0^0 = S_2^2 . \tag{22}$$

From the first of Eqns.(6) we have $S_0^0 = const$, and the set of equations we need, looks as follows

$$
\begin{aligned}
R_{,n}(\pm) &= \sigma_\pm \sqrt{f^2(t) + k - \frac{2\Lambda}{N(N-1)} R^2} , \\
-\left[\frac{R_{,n}}{R}\right] &= \frac{1}{R}\left(R_{,n}(-) - R_{,n}(+)\right) = \frac{8\pi G}{N-1} S_0^0 , \\
S_0^0 &= const , \quad R(0,t) = a(\tau) , \quad \tau = t , \\
\frac{(N-1)(N-2)}{2} \frac{a_\tau^2 + k}{a^2} &= \frac{N-2}{N}\left(\Lambda + \frac{N}{2(N-1)}(4\pi G)^2 \left(S_0^0\right)^2\right) .
\end{aligned}
\tag{23}
$$

Note, first of all, that the values of $R_{,n}$ on different sides of the brane differ by their sign only, this is the consequence of our assumption to have $m = 0$ everywhere in the bulk. Therefore, $\sigma_- = -\sigma_+$, and we automatically obtain the Z_2-symmetric brane. Moreover, the signs of S_0^0 and σ_- are the same, the latter affects the matching of the brane to the bulk, but not the evolution inside the shell. Let us now solve the set of equations (23), considering all the possibilities one by one.

We begin with positive cosmological constant, $\Lambda > 0$. Introducing (for brevity) the so-called cosmological radius $R_0 = \sqrt{\frac{N(N-1)}{2\Lambda}}$ and suppressing $(\pm)$ indices we get from the first of Eqns.(23)

$$R = R_0 \sqrt{f^2(t) + k} \sin\left(\frac{\sigma n}{R_0} + \varphi(t)\right) , \tag{24}$$

where $\varphi(t)$ is an another function of time. On the brane at $n = 0$ the following equations are valid ($\sigma = \sigma_- = -\sigma_+$):

$$
\begin{aligned}
\frac{\sigma}{R_0} \cot\varphi &= \frac{4\pi G}{N-1} S_0^0 , \\
\frac{a_\tau^2 + k}{a^2} &= \frac{2\Lambda}{N(N-1)} + \left(\frac{4\pi G}{N-1}\right)^2 (S_0^0)^2 = \frac{1}{R_0^2 \sin^2\varphi} .
\end{aligned}
\tag{25}
$$

Since $S_0^0 = const$, then $\varphi = \varphi_0 = const$. We see also that the value of $\sigma(= \sigma_- = -\sigma_+)$ depends on the sign of S_0^0 and affects crucially the matching of our brane to the bulk. For different values of $k = \pm 1, 0$ only the time dependent pre-factor is changed:

$$
\begin{aligned}
R &= R_0 \sin\left(\frac{\sigma n}{R_0} + \varphi_0\right)
\begin{cases}
\cosh\frac{t}{a_0} , & \text{for} \quad k = +1 \\
e^{\frac{t}{a_0}} , & \text{for} \quad k = 0 \\
\left|\sinh\frac{t}{a_0}\right| , & \text{for} \quad k = -1
\end{cases} \\
a_0 &= R_0 \sin\varphi_0
\end{aligned}
\tag{26}
$$

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It is useful to visualize the matching by plotting the above function $R = R(n)$ for some moment of time $t = const$. We have for $\sigma = \sigma_- = +1(\sigma_+ = -1)$ (Figs.1 and 2):
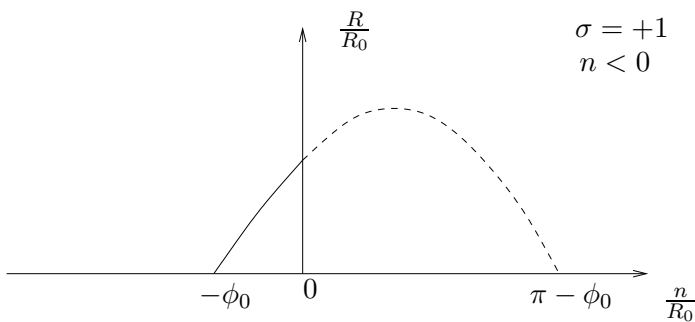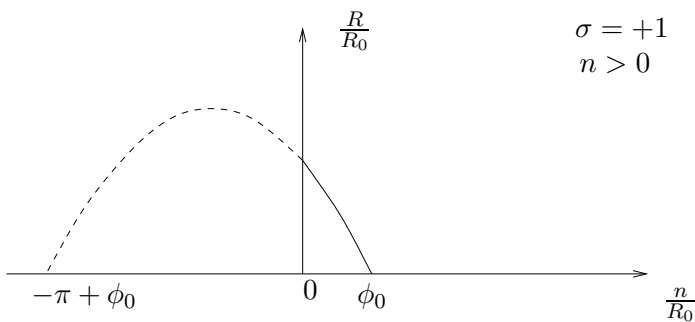

Fig. 1:

Fig. 2:

The dashed curves show a continuation of the function $R(n)$ beyond the shell. Combining these two Figures we get Fig.3:
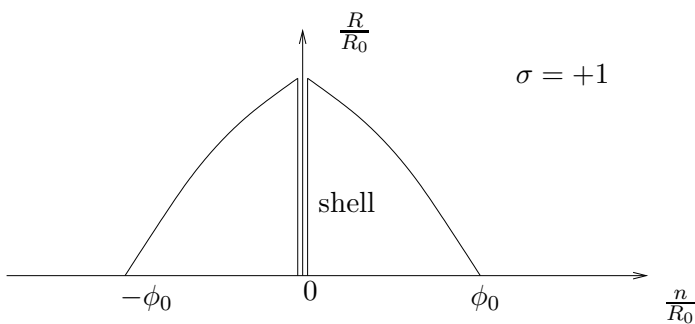

Fig. 3:

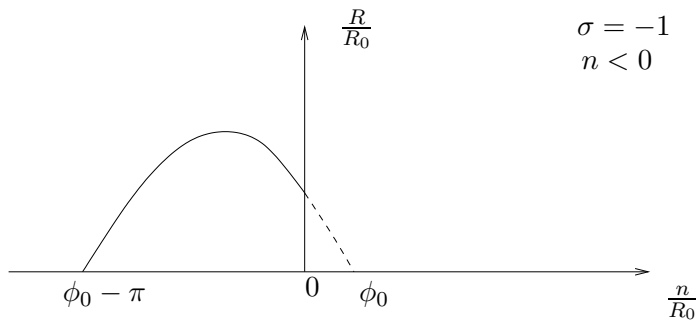For $\sigma = \sigma_- = -1(\sigma_+ = +1)$ the corresponding pictures are shown in Figs.4, 5 and 6.
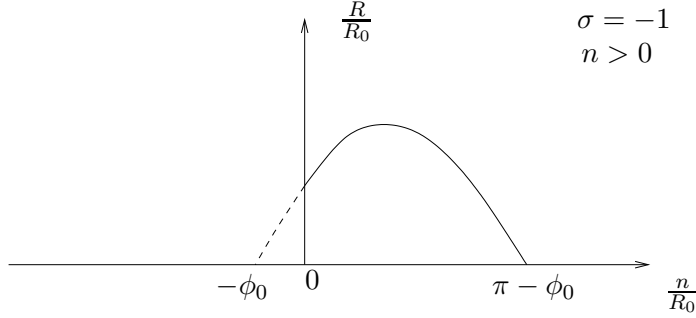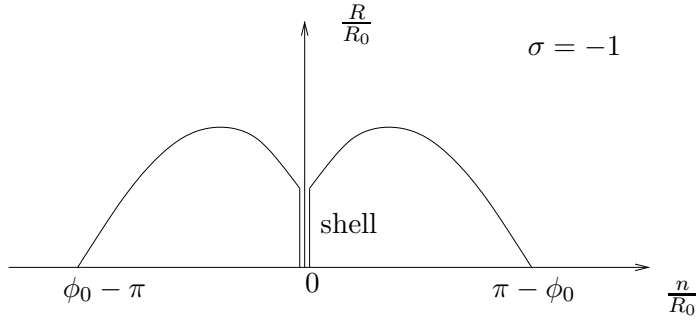
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Fig. 4:

Fig. 5:

Fig. 6:

Let us now turn to the bulk geometry. The causal structure of space-times is better seen on the so-called Carter-Penrose conformal diagrams where each point represents the $(N-1)$ dimensional homogeneous space. We suppose, everybody in the audience knows how to construct such a diagram. Below we present only the results pointing out the $R_\pm$- and $T_\pm$-regions and corresponding values of radii R and conformal radii $R^\star$ at the boundaries and horizons. Consider, first, the case $k = +1$. The relations between R and $R^\star$ are now the following

$$dR^\star = \pm \frac{dR}{1 - \frac{R^2}{R_0^2}} , \implies R^\star = \pm \frac{1}{2} \ln \frac{1 + \frac{R}{R_0}}{1 - \frac{R}{R_0}} \tag{27}$$

in $R_\pm$-regions $0 \leq R \leq R_0$, and

$$dR^\star = \pm \frac{dR}{\frac{R^2}{R_0^2} - 1} , \implies R^\star = \pm \frac{1}{2} \ln \frac{\frac{R}{R_0} + 1}{\frac{R}{R_0} - 1} \tag{28}$$

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in $T_\pm$-regions, $R_0 \leq R \leq \infty$. The Carter-Penrose diagram is the well known square for the de Sitter space-time. The time coordinate points up, while the radial coordinate goes from left to right, and the null curves are straight lines with $\pm 45°$ (Figs.7 and 8.
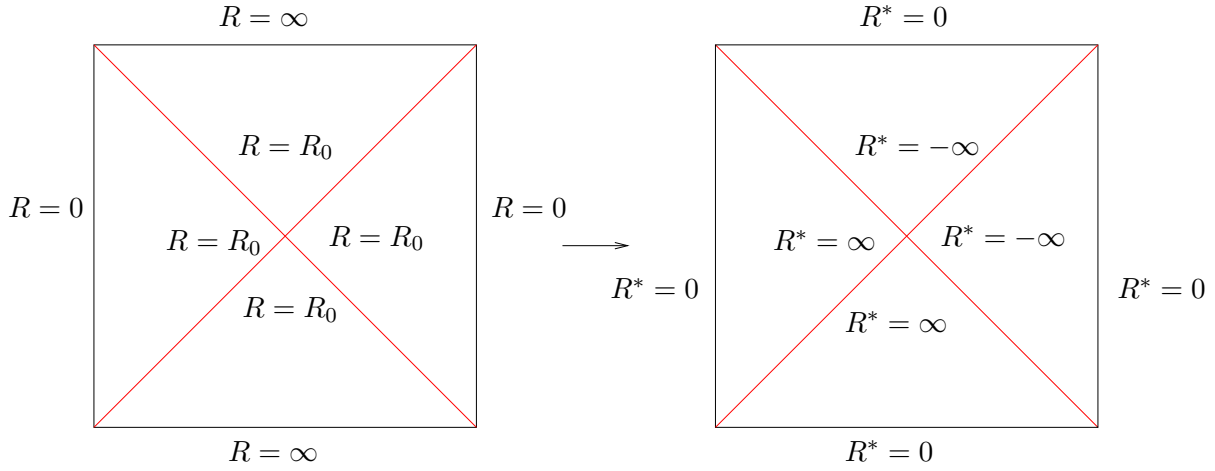
Fig. 7:
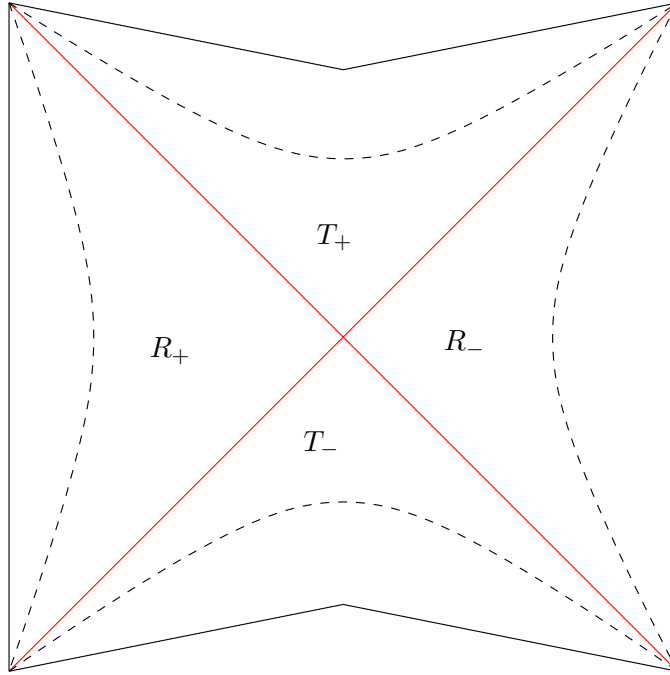
Fig. 8:

In Fig.8 the dashed curves represent the surfaces $R = const$ (time-like in R-regions and space-like in T-regions), and we slightly distorted the $R = \infty$ space-like boundaries in order to make the matchings of the brane to the bulk more visual. And, finally, the conformal diagrams for the complete geometry of the space-time with the brane in the case $\Lambda > 0$, $k = +1$ are shown in Figs.9 and 10. Clearly, they are different for $S_0^0 > 0$ ($\sigma = \sigma_- = +1$) and for $S_0^0 < 0$ ($\sigma = \sigma_- = -1$).
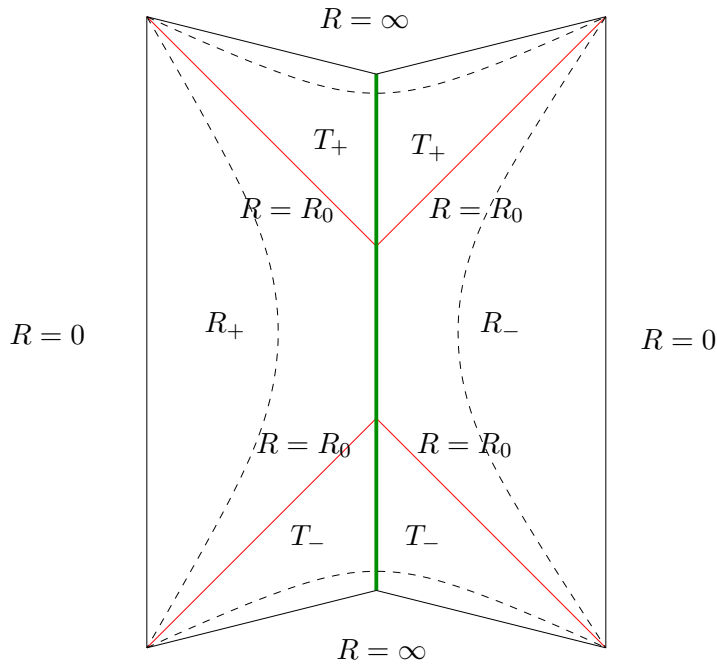
Fig. 9:
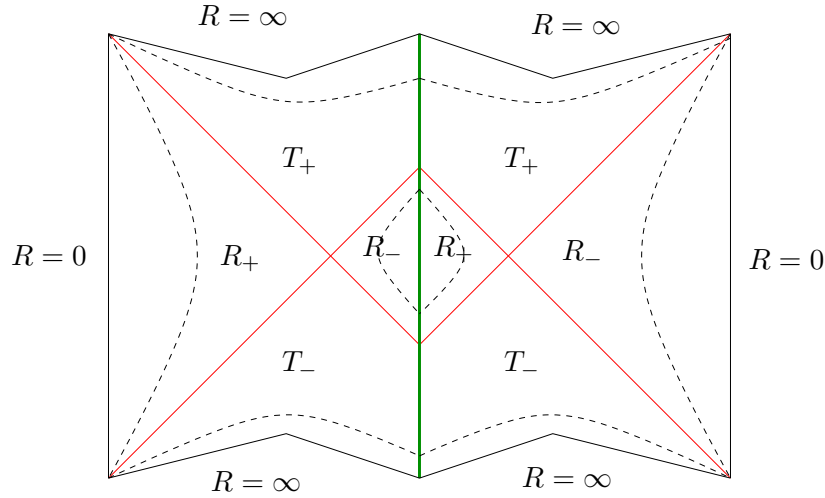
Fig. 10:

Dashed curves are hyper-surfaces $R = const$. The case of negative surface energy density $S_0^0 < 0$ is much more interesting from the physics point of view. The bulk geometry on both sides of the brane has the Einstein-Rosen bridge, or a throat, at the intersection of horizons $R = R_0$ (this is the so-called bifurcation point). In this sense such a geometry reminds that of non-traversable wormhole. The interesting physics begins if there are several more branes (say, two) in the space-time. Let us imaging that one of the additional branes is located on the same side of the Einstein-Rosen bridge as "our" brane is (to the left on the diagram), while the second one is on the other side (to the right). In classical theory their existence does not affect the dynamics of "our" shell or destroy the Z_2-symmetry of the matching. But in quantum theory these additional shells will cause the energy level splitting and such a splitting will inevitably be asymmetric, resulting in an asymmetric hierarchy of fundamental interactions.

Consider now the more simple case $k = 0$. Evidently, we have only the $T_\pm$-regions everywhere

except the hyper-surfaces $R = 0$ that serve as the apparent horizons. The R and $R^\star$ (time-like) are reciprocal,

$$R^\star = \mp \frac{1}{R} \tag{29}$$

in $T_\pm$-regions, and the conformal diagrams for the bulk are simple orthogonal triangles, Fig.11.
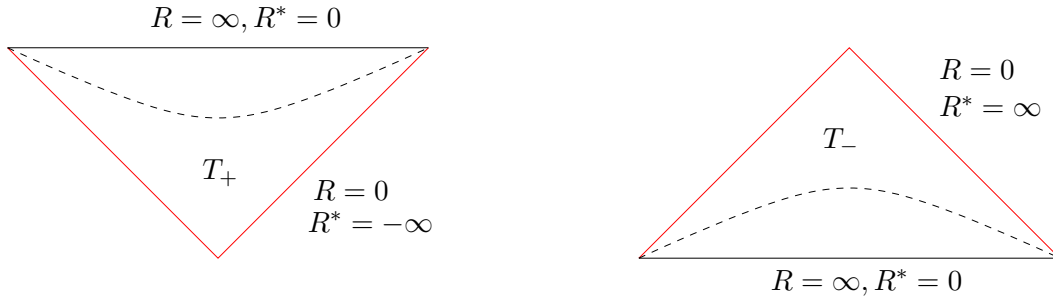
Fig. 11:

The complete geometries with the brane are Figs.12 and 13.
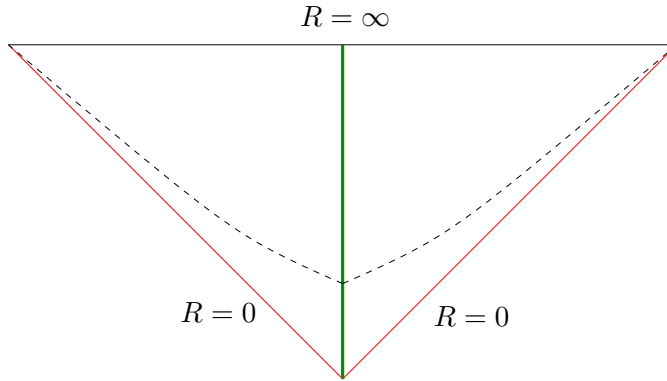
Fig. 12:
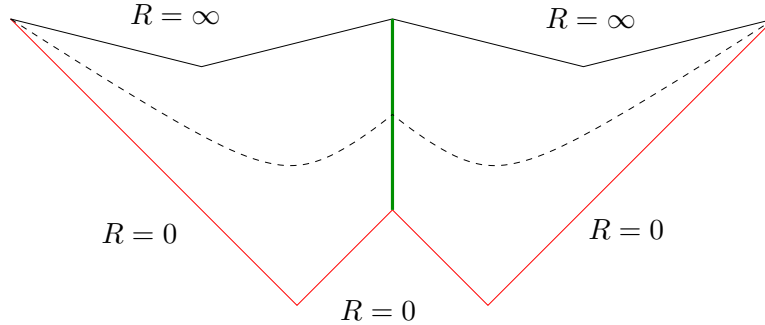
Fig. 13:

plus their time reversals. Again, the dashed curves represent constant radii.

The case $k = -1$ is a little bit more complex. The whole region $0 \leq R < \infty$ is now the T-region without horizons. For the conformal time $R^\star$ we have

$$R^\star = \pm \arctan \frac{R}{R_0}, \implies R = \pm R_0 \tan \frac{R^\star}{R_0}, \tag{30}$$

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where the signs " $\pm$ " stand for $T_\pm$-regions. when R increases from zero to infinity, $0 \leq R < \infty$, the cosmological time $R^\star$ changes from 0 to $\frac{\pi}{2} R_0$ in T_+-region, the region of inevitable expansion (in T_--region $-\frac{\pi}{2} R_0 \leq R^\star \leq 0$ when R decreases from ∞ to 0, this is the case of inevitable contraction). Formally, we can extend the conformal time $R^\star$ to run from $-\infty$ to ∞ and, thus, arrive at the so-called unfolded description. In the purely vacuum space-time there are no physical observers, but in more realistic brane universe scenarios everything depends on the physical conditions inside the shell (possible appearance of real singularities and so on). Further, it is easy to notice that the two-dimensional metric for $\Lambda < 0$, $k = +1$ differs from that one for $\Lambda > 0$, $k = -1$ only by the signature: $(+-) \to (-+)$. This means that the corresponding Carter-Penrose conformal diagram can be obtained from that of conventional anti-de Sitter space-time by interchanging $R_\pm$- and $T_\pm$-regions,the horizontal lines being replaced by vertical ones. With this in mind, we get for the bulk geometry (Fig.14).
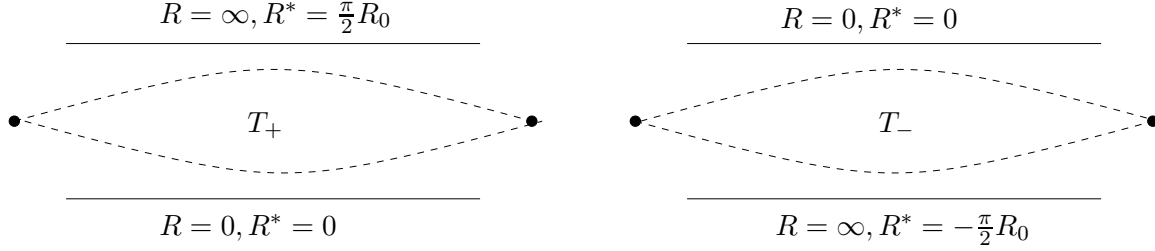
Fig. 14:

Here two isolated points on each diagram are spatial infinities ($-\infty$ on the left and $+\infty$ on the right), and the dashed curves are for $R = const$. The complete geometries with the brane for $S_0^0 > 0$ and $S_0^0 < 0$ look on the conformal diagrams as in Fig.15.
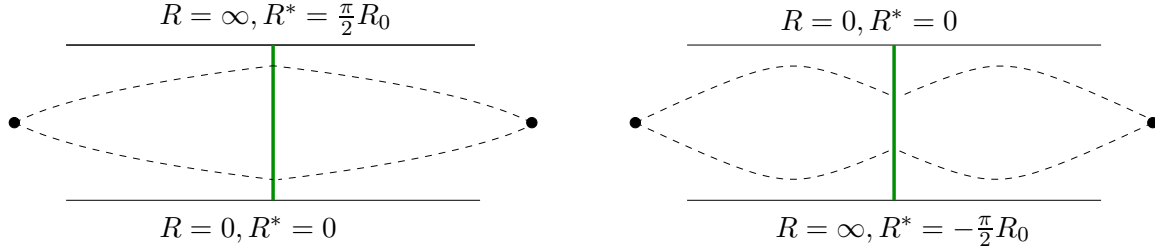
Fig. 15:

This is the case of inevitable expansion. For inevitable contraction the diagrams are essentially the same.

And now we will describe all possible global geometries when the bulk is a vacuum $(N + 1)$-dimensional space-time with negative cosmological constant, $\Lambda < 0$. First of all, let us have a look at the differential equation for $R(n, t)$,

$$R_{,n} = \sigma \sqrt{f^2(t) + k - \frac{2\Lambda}{N(N-1)} R^2} = \sigma \sqrt{f^2(t) + k + \frac{R^2}{R_0^2}}, \tag{31}$$

where we introduced the cosmological radius $R_0 = \sqrt{\frac{N(N-1)}{2|\Lambda|}}$. In contrast to the case of positive Λ, there are two possibilities: either $f^2(t) + k > 0$, or $f^2(t) + k < 0$. We start with the case $f^2(t) + k > 0$, the solution to the Eqn.(31) is

$$R = R_0 \sqrt{f^2(t) + k} \, \sinh\left(\frac{\sigma n}{R_0} + \varphi(t)\right). \tag{32}$$

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The Einstein equations on the brane ($n = 0$) take now the form

$$\frac{\sigma n}{R_0} \coth \varphi(t) = \frac{4\pi G}{N-1} S_0^0,$$

$$\frac{f^2(t)+k}{a^2} = \frac{a_\tau^2+k}{a^2} = \frac{2\Lambda}{N(N-1)} + \left(\frac{4\pi G}{N-1}\right)^2 S_0^{0^2} = \frac{1}{R_0^2 \sinh^2 \varphi(t)}. \quad (33)$$

Again, for vacuum shells ($S_0^0 = S_2^2 = const$) we have $\varphi(t) = \varphi_0 = const$. We see that the transition from the positive cosmological constant to the negative one results in replacing the trigonometric by corresponding hyperbolic functions. Moreover, in this case ($f^2(t)+k > 0$) the induced energy density inside the shell is positive, so qualitatively, the inner evolution is exactly the same as for positive Λ. Since the absolute value of the surface energy density is bounded from below, $|S_0^0| > \sqrt{\frac{(N-1)|\Lambda|}{2\pi G N}}$, we may call such a brane "the heavy shell". The plots of the functions $R(n)$ for different $\sigma = \pm 1$ are shown in Figs.16 and 17.
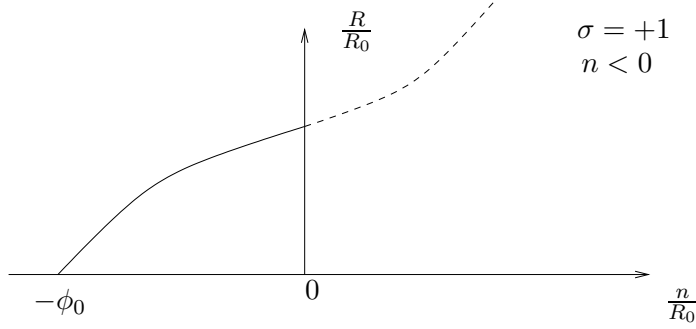
Fig. 16:
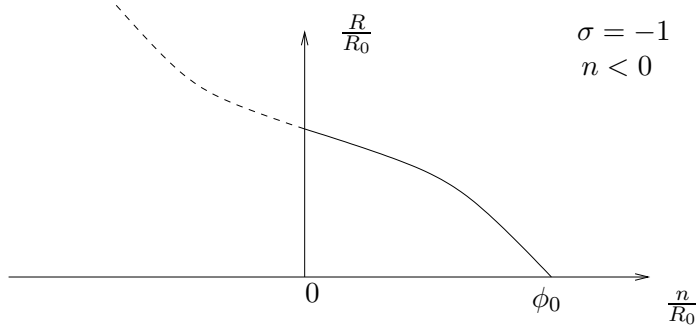
Fig. 17:

With the brane the pictures are Figs.18 and 19.
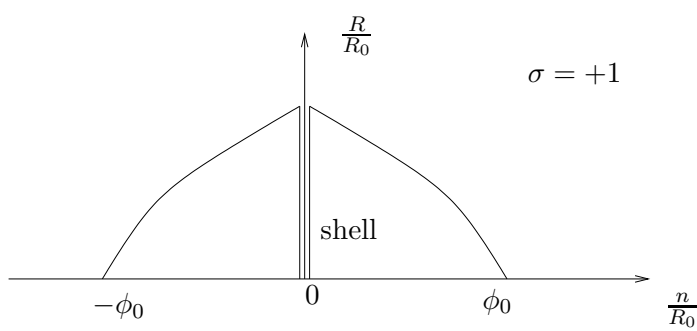
Fig. 18:
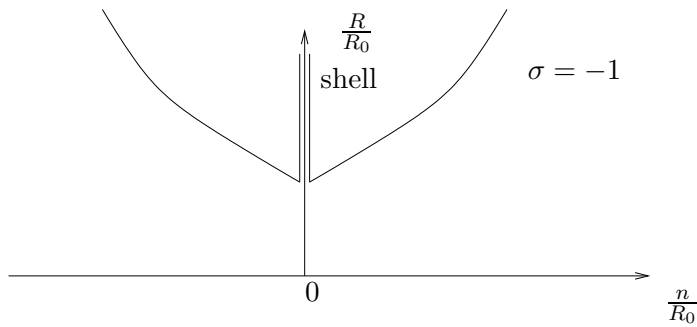
Fig. 19:

Despite of the similar behavior inside the brane, the bulk geometries are completely different from that for positive cosmological term.

The case $k = +1$ is the conventional anti-de Sitter space-time, and the conformal Carter-Penrose diagram is the same as for $\Lambda > 0$, $k = -1$, but it becomes vertical, because instead of T-regions we have now the R-regions everywhere, see Fig.20.
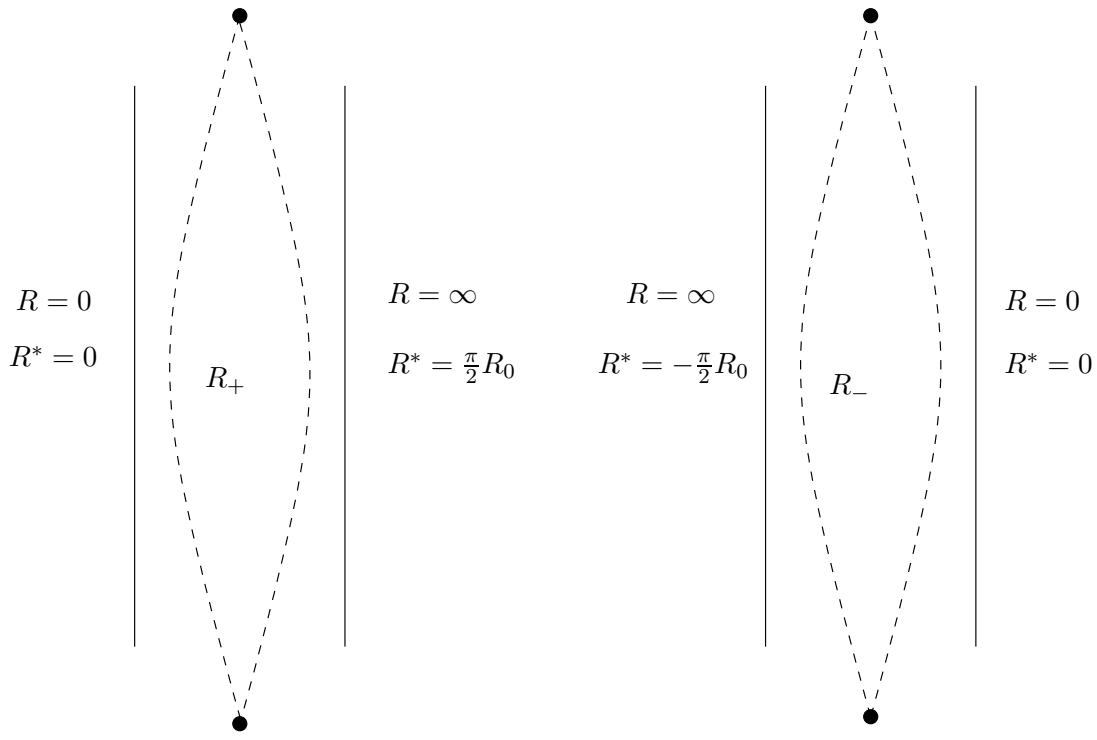
13

Fig. 20:

The isolated points are the future and past time infinities. Of course, we can consider a variety of unfolded version (with different identifications) of this AdS space-time, and such extensions are even more natural than before, because due to the negative curvature the light rays reach the "boundary" at $R = \infty$ in finite coordinate time interval. But, again, everything depends on the specific properties of the matter inside the brane. Remembering that the scale factor of the brane evolution for $k = +1$ is bounded from below we obtain the following two types of global geometries for $S_0^0 > 0$ and $S_0^0 < 0$, up to possible unfoldings, Figs.21 and 22.

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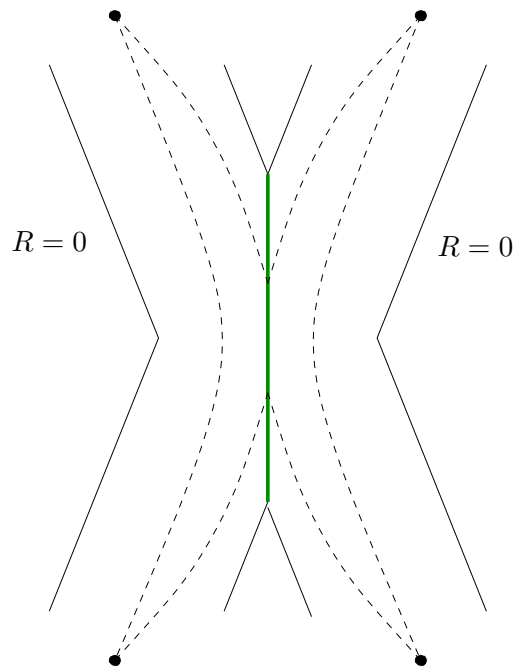
Fig. 21:
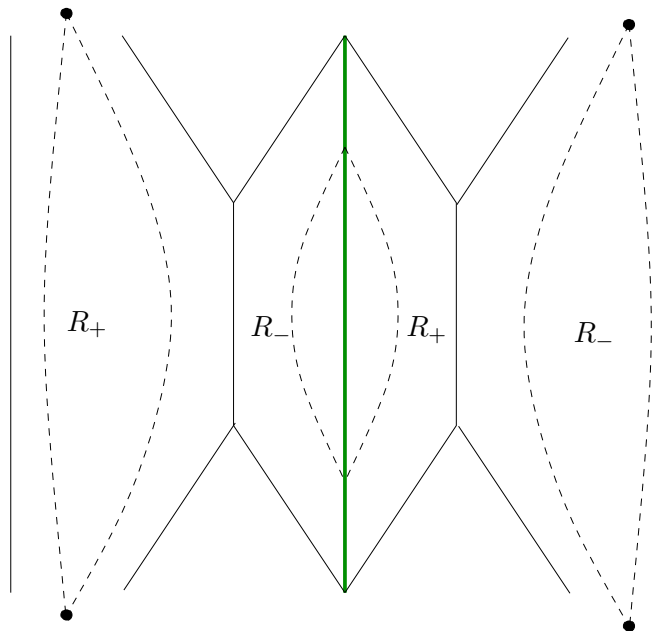
Fig. 22:

The dashed curves represent the hyper-surfaces $R = const.$

In the case $k = 0$, $\Lambda < 0$ everything is similar to that for $k = 0$, $\Lambda > 0$. Again, the T-region is replaced by the R-region, and the vertical orthogonal triangle becomes horizontal. The surfaces $R = 0$ are the apparent horizons. For the bulk geometry we have Fig.23.
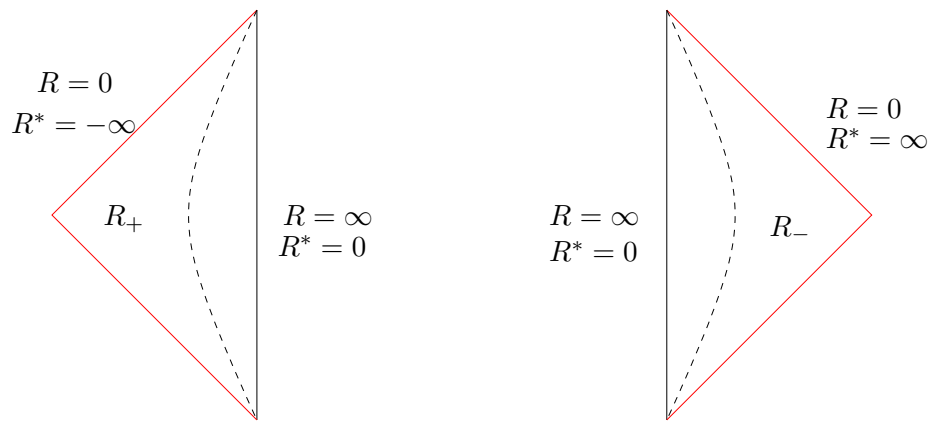
Fig. 23:

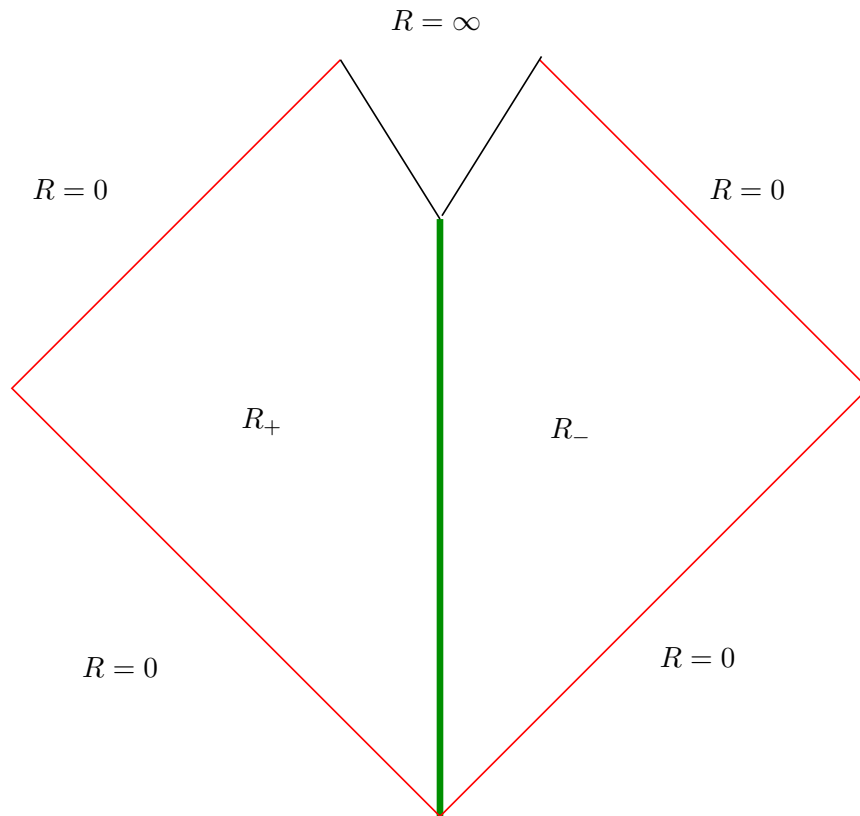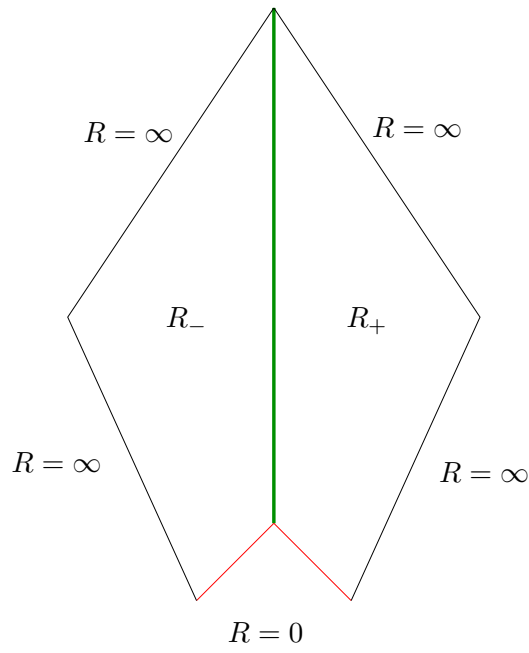After inclusion of the brane we get, Figs.24 and 25.

Fig. 24:

Fig. 25:

The most unusual is the case $\Lambda < 0$, $k = -1$. It admits both the "heavy shells" with $\left|S_0^0\right| > \sqrt{\frac{(N-1)|\Lambda|}{2\pi\,GN}}$, and the "light shells" for which $\left|S_0^0\right| < \sqrt{\frac{(N-1)\,|\Lambda|}{2\pi\,GN}}$. For the "heavy" shells

$$R = R_0 \sinh\left(\frac{t}{R_0 \sinh\varphi_0}\right) \sinh\left(\frac{\sigma n}{R_0} + \varphi_0\right), \tag{34}$$

and the shell infinitely expands from zero radius to infinity. For the "light" shells

$$
\begin{aligned}
R &= R_0 \sin\left(\frac{t}{R_0 \cosh\varphi_0}\right) \cosh\left(\frac{\sigma n}{R_0} + \varphi_0\right), \\
R &= \pm R_0 \tanh\frac{R^\star}{R_0}, \qquad 0 \le R \le R_0, \\
R &= R_0 \coth\frac{R^\star}{R_0}, \qquad R_0 \le R < \infty.
\end{aligned}
\tag{35}
$$

The "light" shells, first expands from zero radius $R = 0$ to the maximum at $R = R_0 \cosh\varphi_0$ and then contracts back to $R = 0$. The curves $R(n,)$ for $t = const$ look in this case as shown in Figs.26 and 27.

Fig. 26:
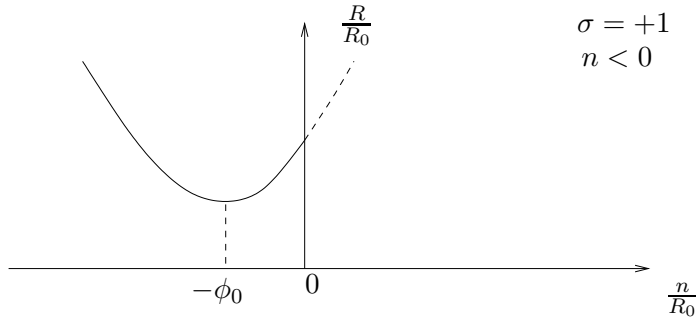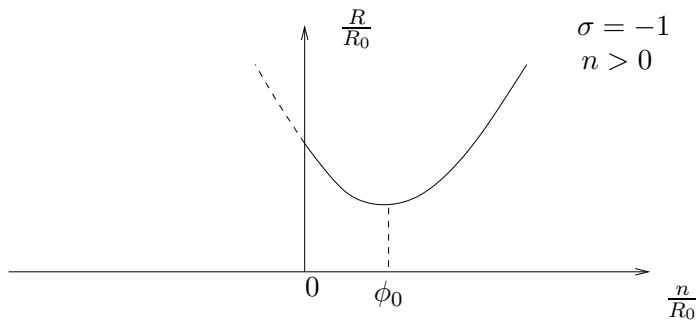
17

Fig. 27:

With inclusion of the shell, the complete pictures are ($\sigma = +1$ for $S_0^0 > 0$, $\sigma = -1$ for $S_0^0 < 0$) Figs.28 and 29.
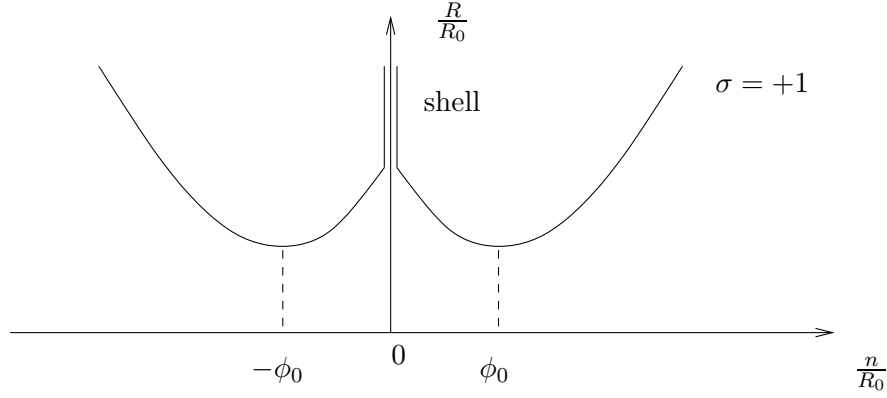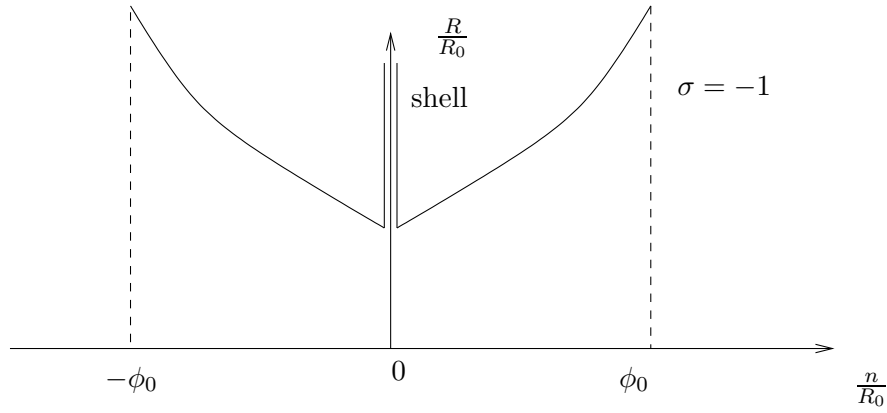

Fig. 28:

Fig. 29:

The Carter-Penrose diagram is also unusual. Formally, it can be obtained from that of $\Lambda > 0$, $k = +1$ by interchanging R- and T-regions. We get Fig.30.
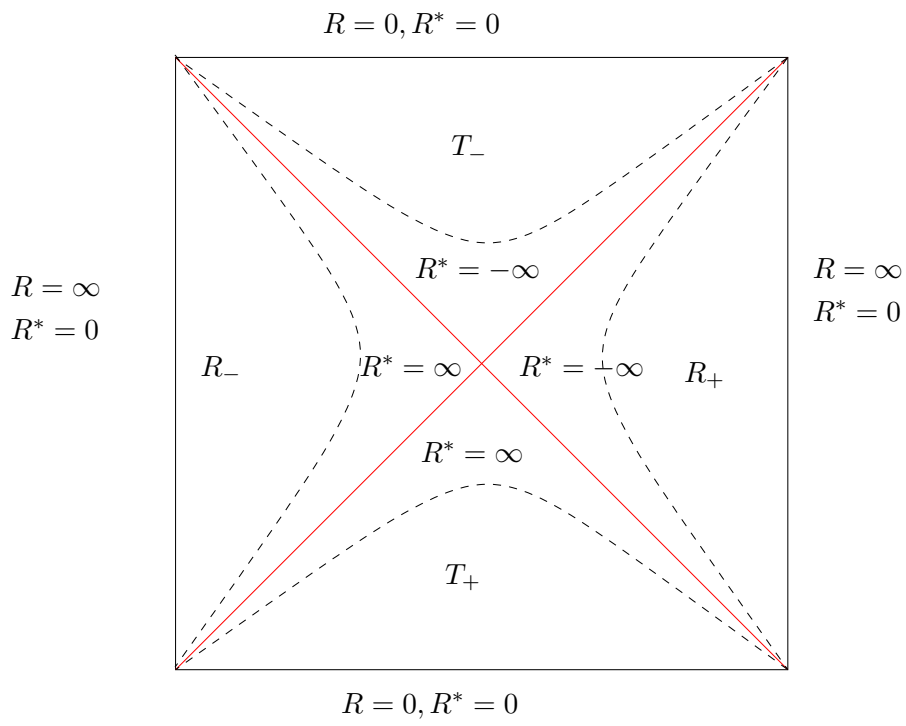
Fig. 30:

This diagram is the same as for the Schwarzschild black hole with the only difference that instead of null infinities we have now the time-like infinities on both sides of the Einstein-Rosen bridge. But this black hole is strange: it has zero mass (!), and the horizon radius $R = R_0$ is connected to the negative (!) cosmological constant. Besides, there are no singularities at $R = 0$. The complete Carter-Penrose diagrams for the "heavy" shell with $S_0^0 > 0$ and $S_0^0 < 0$ look as in Figs.31 and 32.
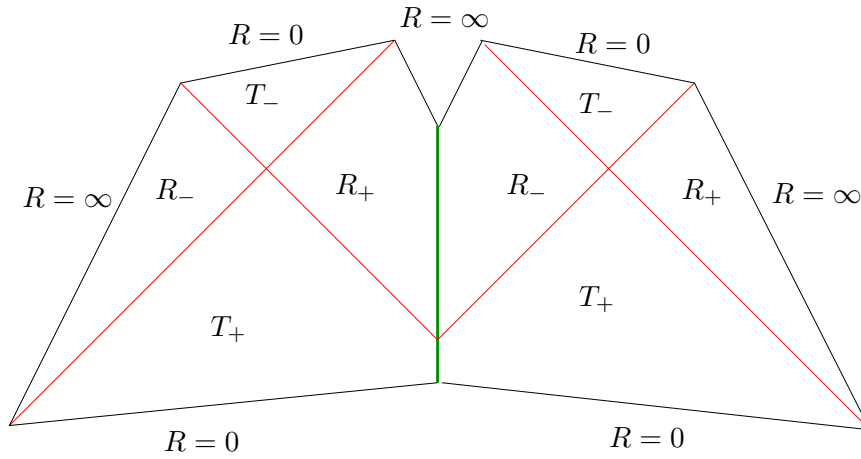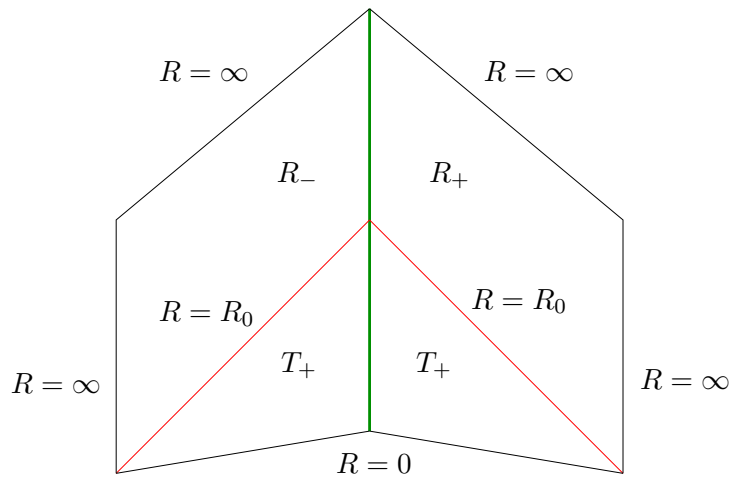

Fig. 31:

Fig. 32:

We see that in the case $S_0^0 > 0$ there zero mass black holes on both sides of the shell and, consequently, two Einstein-Rosen bridges. The latter property allows to have asymmetric hierarchy without destroying the Z_2-symmetry of the matching. For the "light" shell the complete Carter-Penrose diagrams are Figs.33 and 34.
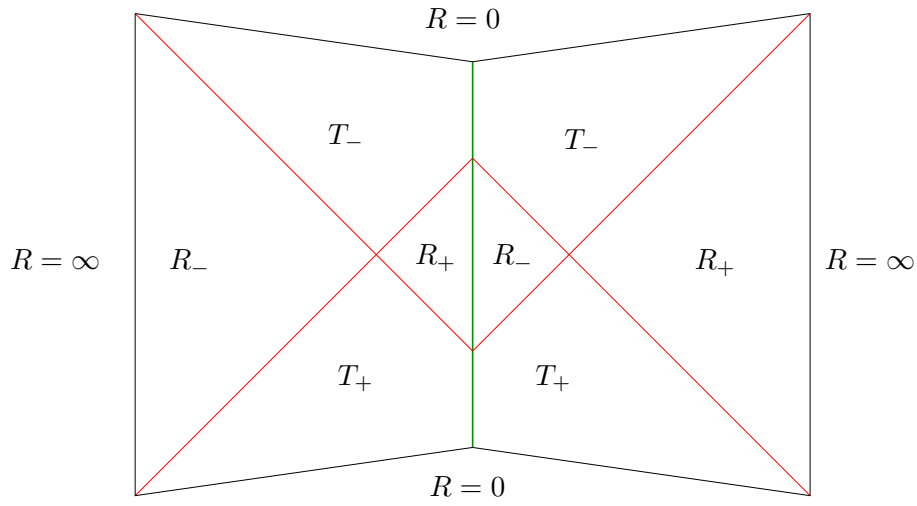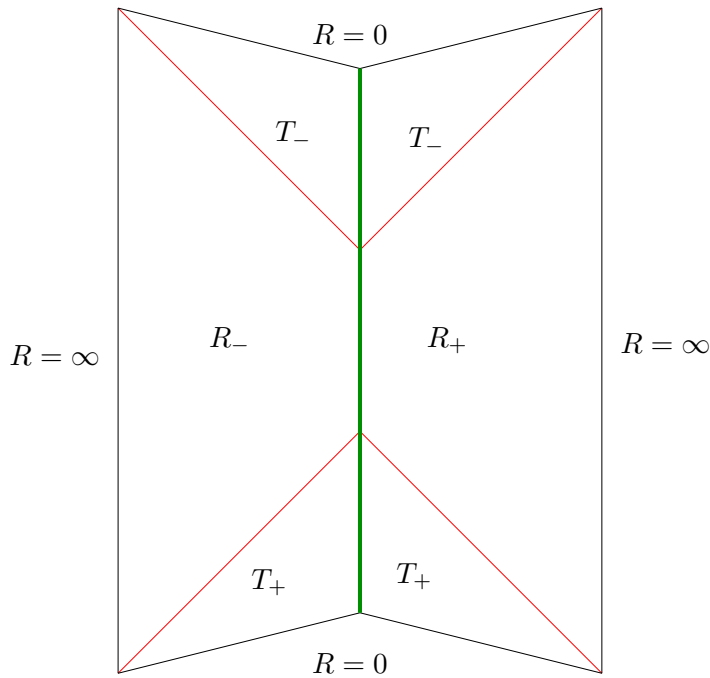

Fig. 33:

Fig. 34:

For both signs of S_0^0 our shell undergoes a bound motion and, thus, can be quantized in the same way as the bound states, say, of hydrogen atom. And the last note: because the proper time is finite when traveling from an initial state at $R = 0$ to the final state at $R = 0$, we can use the unfolded description in the time direction (both past and future) as well.

We would like now to summarize what we learned studying the global geometry of the brane universe cosmological models. About the assumption. The most severe one is that about the existence of cosmological symmetry throughout the whole space-time. This means that the brane does not affect the local bulk geometry, in other words, the latter does not depend on the place of brane matching (in terms of invariant radius R). Thus, the singular shell does not send any (gravitational) signal about its existence. The very interesting result is the connection between the spatial curvature of a homogeneous space on the brane and the global geometry. This is rather unexpected and contradicts, in a sense, our four-dimensional experience and intuition. We are used ti think of the thin shells as the two-dimensional bubble walls embedded into a three-dimensional space. Because these walls are spherically symmetric (assuming "cosmological" symmetry) we get automatically $k = 1$. It is appeared unexpected also the possibility of non-symmetric (from the global geometry point of view) inclusion into consideration of several additional branes without destroying the Z_2-symmetric matching of "our" brane to the bulk. This may be useful in attempts to understand the observed non-symmetric hierarchy of the fundamental interactions. We saw that such a property exists (even for zero Schwarzschild mass) if we adopt some unfolded description of the space-times with negative cosmological constant, and also in the case $\Lambda < 0\, k = -1$, where we found the Einstein-Rosen bridge which exactly like in the Schwarzschild black hole space-time, but with the cosmological constant playing the role of the nonzero mass. This last case is also interesting for constructing more realistic models because it allows a transition from the "light" shells (with the surface energy density bounded from above) to the "heavy" shells (surface energy density bounded from below). Besides, the quantized "light" shells will form the bound states and this may become very important in investigating the quantum models with several branes which could exhibit the hierarchy features.