Local curvature expansion in brane induced gravity models

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Abstract

We report on the Schwinger-DeWitt technique for the covariant curvature expansion of the quantum effective action for brane induced gravity models in curved spacetime. This expansion has a part nonanalytic in DGP type scale parameter m, leading to the cutoff scale which is given by the geometric average of the mass of the quantum field in the bulk M and m. This cutoff $M_{\rm cutoff} = \sqrt{Mm}$ is much higher than the analogous strong coupling scale of the DGP model treated by weak field expansion in the tree-level approximation. The lowest orders of this curvature expansion are presented for the case of the scalar field in the $(d{+}1)$-dimensional bulk with the brane carrying the d-dimensional kinetic term of this field. The effective potential of this model is presented for any d along with its ultraviolet divergences in a particular case of $d = 4$.

1 Introduction

The purpose of this paper is to report on the technique of covariant local gradient and curvature expansion for brane induced gravity models [1, 2] which are expected to be promising candidates for the resolution of dark energy [3], ghosts [4] and hierarchy problems [5, 6, 7] in the present day astroparticle cosmology and high-energy theory. The peculiarity of these models is that, unlike in theories with local Dirichlet or Neumann boundary conditions on boundaries/branes, calculations do not reduce to a simple bookkeeping of known bulk and boundary contributions to the Schwinger-DeWitt coefficients of the relevant heat kernel [8, 9, 10, 11]. In such theories with the second order covariant operator of a general form

$$F(\nabla_X) = -\Box^{(d+1)} + M^2 + P = -G^{AB}\nabla_A\nabla_B + M^2 + P \qquad (1)$$

acting on a scalar field $\Phi(X)$ in a $(d+1)$-dimensional curved spacetime with the metric G_{AB}, $X = X^A$, $A = 1, 2, ...d+1$, ($\Box^{(d+1)} = G^{AB}\nabla_A\nabla_B$ is the covariant d'Alembertian in terms of covariant derivatives ∇_A, $P = P(X)$ is a generic X-dependent potential term and M is a large mass parameter), the one-loop effective action is expandable in $1/M$-series with the coefficients $A_n^{D/N}$

$$\begin{aligned}\frac{1}{2}\,{\rm Tr}^{(d+1)}_{D/N}\ln F &= -\frac{1}{2}\int\limits_0^\infty \frac{ds}{s}\frac{1}{(4\pi s)^{(d+1)/2}}\,e^{-sM^2}\sum_{n=0}^\infty s^{n/2}A_n^{D/N}\\ &= -\frac{1}{2}\frac{M^d}{(4\pi)^{(d+1)/2}}\sum_{n=0}^\infty\frac{\Gamma\big(\frac{-d-1+n}{2}\big)}{M^{n-1}}\,A_n^{D/N}, \end{aligned} \qquad (2)$$

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labeled for the Dirichlet/Neumann boundary conditions respectively by D/N-superscripts.

For the Dirichlet boundary condition on the quantum field $\Phi|_{\mathbf{b}}=0$ these coefficients in few lowest orders read as the following sums of the integrals over the bulk $\boldsymbol{B}$ and the relevant brane/boundary $\mathbf{b}$ domains [10, 11]

$$\begin{aligned}
A_0^D &= \int\limits_{\boldsymbol{B}} d^{d+1}X\,\sqrt{G}\,, \\
A_1^D &= -\frac{\sqrt{\pi}}{2}\int\limits_{\mathbf{b}} d^dx\,\sqrt{g}\,, \\
A_2^D &= \int\limits_{\boldsymbol{B}} d^{d+1}X\,\sqrt{G}\,\left(-P+\frac{1}{6}\,{}^{\boldsymbol{B}}\!R\right)+\int\limits_{\mathbf{b}} d^dx\,\sqrt{g}\,\left(-\frac{1}{3}\,k\right)\,, \\
A_3^D &= \sqrt{\pi}\int\limits_{\mathbf{b}} d^dx\,\sqrt{g}\,\left(+\frac{1}{2}P-\frac{1}{12}\,{}^{\boldsymbol{B}}\!R+\frac{1}{24}\,{}^{\boldsymbol{B}}\!R_{nn}-\frac{7}{192}\,k^2+\frac{5}{96}\,k_{\alpha\beta}^2\right),
\end{aligned} \tag{3}$$

where $g_{\alpha\beta}$ denotes the induced metric on the boundary/brane $\mathbf{b}$, $k_{\alpha\beta}$ is the extrinsic curvature of the latter, $k=g^{\alpha\beta}k_{\alpha\beta}$ is its trace, the curvature objects labeled by B are the bulk quantities, like ${}^{\boldsymbol{B}}\!R=G^{AB}\,{}^{\boldsymbol{B}}\!R_{AB}$ is the scalar Ricci curvature of the bulk and ${}^{\boldsymbol{B}}\!R_{nn}=n^An^B\,{}^{\boldsymbol{B}}\!R_{AB}$ is the normal-normal projection of the bulk Ricci tensor onto the boundary.

For the Neumann (Robin) boundary conditions involving the normal derivative, $(\partial_n-S)\Phi|_{\mathbf{b}}=0$, the same coefficients have the form including the coefficient of the non-derivative part of this boundary condition S,

$$\begin{aligned}
A_0^N &= \int\limits_{\boldsymbol{B}} d^{d+1}X\,\sqrt{G}\,, \\
A_1^N &= \frac{\sqrt{\pi}}{2}\int\limits_{\mathbf{b}} d^dx\,\sqrt{g}\ , \\
A_2^N &= \int\limits_{\boldsymbol{B}} d^{d+1}X\,\sqrt{G}\,\left(-P+\frac{1}{6}\,{}^{\boldsymbol{B}}\!R\right)+\int\limits_{\mathbf{b}} d^dx\,\sqrt{g}\,\left(-\frac{1}{3}k-2S\right)\,, \\
A_3^N &= \sqrt{\pi}\int\limits_{\mathbf{b}} d^dx\,\sqrt{g}\,\left(-\frac{1}{2}P+\frac{1}{12}\,{}^{\boldsymbol{B}}\!R-\frac{1}{24}\,{}^{\boldsymbol{B}}\!R_{nn}\right.\\
&\qquad\qquad\left.+\frac{13}{192}\,k^2+\frac{1}{96}\,k_{\alpha\beta}^2+\frac{1}{2}\,kS+S^2\right).
\end{aligned} \tag{4}$$

Analogous expressions exist or can be derived for higher order coefficients, and can be generalized to fields Φ of arbitrary spin-tensor structure. The integrands of the volume (bulk) parts – the Schwinger-DeWitt coefficients – can be obtained by a regular recurrent procedure [8, 9], whereas the surface parts follow from a less systematic but generally well-known technique based on a combination of various methods [10, 11].

The Schwinger-DeWitt technique cannot however be directly applied to brane induced gravity models. The main peculiarity of these models is that due to quantum field fluctuations on the branes the field propagator is subject to generalized Neumann boundary conditions involving not only normal but also tangential derivatives on the brane/boundary surfaces. This presents both technical and conceptual difficulties, because such boundary conditions are much harder to handle than the simple Dirichlet ones. A general method for treating quantum effective actions in this case was recently suggested in [1, 12, 2, 13]. This method provides a systematic reduction of the generalized Neumann boundary conditions to Dirichlet conditions. As a byproduct it disentangles from the quantum effective action the contribution of the surface modes mediating the brane-to-brane propagation, which play a very important role in the zero-mode localization mechanism of the Randall-Sundrum type [15]. The purpose of this work is to make the next step — to extend a well-known Schwinger-DeWitt technique of the above type [8, 9, 10, 11] to the calculation of this contribution in the DGP model in the form of the *covariant* curvature expansion.

Briefly the method of [1] looks as follows. The action of a (free field) brane model generally contains the bulk and the brane parts,

$$S[\Phi]=\frac{1}{2}\int\limits_{\mathbf{B}} d^{d+1}X\sqrt{G}\;\Phi(X)\,F(\nabla_X)\,\Phi(X)+\frac{1}{2}\int\limits_{\mathbf{b}} d^dx\,\sqrt{g}\;\;\varphi(x)\,\varkappa(\nabla_x)\,\varphi(x), \tag{5}$$

where the $(d+1)$-dimensional bulk and the d-dimensional brane coordinates are labeled respectively by $X=X^A$ and $x=x^\mu$, and the boundary values of bulk fields $\Phi(X)$ on the brane/boundary $\mathbf{b}=\partial\boldsymbol{B}$ are denoted by $\varphi(x)$,

$$\begin{aligned}&\Phi(X)\Big|_{\mathbf{b}}=\varphi(x),\;\;\mathbf{b}=\partial\boldsymbol{B},\\&X=X^A,\;\;\;A=1,2,...d+1,\\&x=x^\mu,\;\;\;\mu=1,...d,\\&X=e(x),\end{aligned} \tag{6}$$

where the last relation implies the embedding of the brane/boundary into the bulk.

The kernel of the bulk Lagrangian is given by the second order differential operator $F(\nabla_X)$, whose covariant derivatives ∇_X in (5) are integrated by parts in such a way that they form bilinear combinations of first order derivatives acting on two different fields. Integration by parts in the bulk gives nontrivial surface terms on the brane/boundary. In particular, this operation results in the Wronskian relation for generic test functions $\Phi_{1,2}(X)$,

$$\int\limits_{\boldsymbol{B}} d^{d+1}X\,\sqrt{G}\,\Big(\,\Phi_1\overrightarrow{F}(\nabla_X)\,\Phi_2-\Phi_1\overleftarrow{F}(\nabla_X)\,\Phi_2\Big)=-\int\limits_{\mathbf{b}} d^dx\,\sqrt{g}\;\Big(\varphi_1\overrightarrow{W}\Phi_2-\Phi_1\overleftarrow{W}\varphi_2\Big) \tag{7}$$

Arrows everywhere here indicate the direction of action of derivatives either on Φ_1 or Φ_2.

The variational procedure for the action (5) with dynamical (not fixed) fields on the boundary $\varphi(x)$ naturally leads to generalized Neumann boundary conditions of the form

$$\Big(\overrightarrow{W}(\nabla_X)+\varkappa(\nabla_x)\Big)\,\Phi\,\Big|_{\mathbf{b}}=0 \tag{8}$$

which uniquely specify the propagator of quantum fields and, therefore, a complete Feynman diagrammatic technique for the system in question. The method of [1] allows one to systematically reduce this diagrammatic technique to the one subject to the Dirichlet boundary conditions $\Phi|_{\mathbf{b}}=0$. The main additional ingredient of this reduction procedure is the brane operator $\boldsymbol{F}^{\,\rm brane}(x,x')$ which is constructed from the Dirichlet Green's function $G_D(X,X')$ of the operator $F(\nabla)$ in the bulk,

$$\begin{aligned}&\boldsymbol{F}^{\,\rm brane}(x,x')=-\overrightarrow{W}(\nabla_X)\,G_D(X,X')\,\overleftarrow{W}(\nabla_{X'})\,\Big|_{X=e(x),\,X'=e(x')}+\varkappa(\nabla)\,\delta(x,x'),\\&F(\nabla_X)\,G_D(X,X')=\delta(X,X')\\&G_D(X,X')\,|_{\,X=e(x)}=0.\end{aligned} \tag{9}$$

This operator determines the brane-to-brane propagation of the physical modes in the system with the classical action (5) (its inverse is the brane-to-brane propagator) and additively contributes to its full one-loop effective action according to

$$\boldsymbol{\varGamma}_{1-{\rm loop}}\;\equiv\;\frac{1}{2}\,{\rm Tr}^{(d+1)}_N\ln F\;=\;\frac{1}{2}\,{\rm Tr}^{(d+1)}_D\ln F+\frac{1}{2}\,{\rm Tr}^{(d)}\ln\boldsymbol{F}^{\,\rm brane}, \tag{10}$$

where $\mathrm{Tr}^{(d+1)}_{D,N}$ denotes functional traces of the bulk theory subject to Dirichlet and Neumann boundary conditions, respectively, while $\mathrm{Tr}^{(d)}$ is a functional trace in the boundary d-dimensional theory. The full quantum effective action of this model is obviously given by the functional determinant of the operator $F(\nabla_X)$ subject to the generalized Neumann boundary conditions (8), and the above equation reduces its calculation to that of the Dirichlet boundary conditions plus the contribution of the brane-to-brane propagation.

Here we apply (10) to a simple model of a scalar field which mimics in particular the properties of the brane-induced gravity models and the DGP model [16]. This is the $(d+1)$-dimensional massive scalar field $\Phi(X) = \Phi(x, y)$ with mass M living in the *curved* half-space $y \geq 0$ with the additional d-dimensional kinetic term for $\varphi(x) \equiv \Phi(x, 0)$ localized at the brane (boundary) at $y = 0$,

$$S[\phi] = \frac{1}{2} \int\limits_{y\geq 0} d^{d+1}X\,\sqrt{G}\,\Big((\nabla_X\Phi(X))^2 + M^2\Phi^2(X) + P(X)\Phi^2(X)\Big)$$
$$+\frac{1}{4m}\int d^dx\,\sqrt{g}\,\Big(\big(\nabla_x\varphi(x)\big)^2 + \mu^2\varphi^2(x) + p(x)\varphi^2(x)\Big), \tag{11}$$

which corresponds to the following choice of the bulk (1) and brane $\varkappa(\nabla)$ operators along with the generalized Neumann boundary conditions (8) with $\overrightarrow{W}(\nabla_X) = -\partial_y$ and

$$\varkappa(\nabla) = \frac{1}{2m}\left(-\Box + \mu^2 + p\right), \quad \Box = \Box^{(d)} \equiv g^{\mu\nu}\nabla_\mu\nabla_\nu, \tag{12}$$

$$\Big(\partial_y - \varkappa(\nabla)\Big)\,\Phi(X)\,\Big|_{\mathbf{b}} = 0. \tag{13}$$

As was shown [13], the flat space brane-to-brane operator for such a model without potential terms has the form of the pseudodifferential operator with the flat-space $\Box$,

$$\boldsymbol{F}^{\,\mathrm{brane}}(\nabla) = \frac{1}{2m}\,(-\Box + \mu^2 + 2m\sqrt{M^2-\Box}\,). \tag{14}$$

In the massless case of the DGP model [16], $M = 0$, this operator is known to mediate the gravitational interaction on the brane, interpolating between the four-dimensional Newtonian law at intermediate distances and the five-dimensional law at the horizon scale $\sim 1/m$ [6].

Here we generalize this construction to a curved spacetime and expand the brane-to-brane operator and its effective action in covariant curvature series. This is the expansion in powers of the bulk curvature ${}^BR$, extrinsic curvature of the brane $k_{\alpha\beta}$, the potential terms of the bulk P and brane p operators and their covariant derivatives — all taken at the location of the brane. The expansion starts with the approximation (14) based on the *full covariant* d'Alembertian on the brane. We present a systematic technique of calculating curvature corrections in (10) and rewrite their nonlocal operator coefficients — functions of the covariant $\Box$ – in the form of the generalized (weighted) proper time representation.

The result of this calculation is peculiar. Unlike the usual Schwinger-DeWitt expansion (2) the brane effective action takes the form

$$\frac{1}{2}\,\mathrm{Tr}\ln\boldsymbol{F}^{\,\mathrm{brane}} = \left(\frac{Mm}{4\pi}\right)^{d/2}\sum_{N=0}^{\infty}\frac{1}{M^N}\sum_{i\leq N}\frac{O(\mathfrak{M}^{2N-i})}{m^{N-i}}$$
$$+\frac{M^d}{(4\pi)^{d/2}}\sum_{N=0}^{\infty}\frac{1}{M^N}\sum_{i\leq N} m^i\,O(\mathfrak{M}^{N-i})\,, \tag{15}$$

where $O(\mathfrak{M}^k)$ represent the integrals over the brane/boundary space of local invariants of dimensionality k in units of mass or inverse length. With this notation, in particular, $A^{D/N}_n = O(\mathfrak{M}^n)$.

More generally, these invariants (or spacetime covariant higher-dimensional operators) are composed of the powers of the bulk and brane curvature, extrinsic curvature of the brane/boundary, the potential terms of the bulk and brane operators and their covariant derivatives.

The main difference of (15) from (2) is that in addition to a usual part analytic in m with a typical M-dependence (second series in (15)) we also have the part singular in $m \to 0$ with a qualitatively different analytic dependence on the bulk mass ($M^{d/2-N}$ instead of M^{d-N}). This property was recently discovered for the effective potential in the toy model of the DGP type [13]. Physically this leads to an essential modification of the perturbation theory cutoff — the domain of validity of the local expansion $\mathfrak{M} \ll M_{\rm cutoff}$. It reduces this cutoff from $M_{\rm cutoff} = M$ to

$$M_{\rm cutoff} = \sqrt{Mm}. \tag{16}$$

In physically interesting brane models with $m \ll M$ this implies essential reduction of $M_{\rm cutoff}$ and signifies the problem of a low strong coupling scale [6]. While in [6] this phenomenon was observed in the tree-level theory, here we extend it to the quantum one-loop approximation.

As an application of this generalized Schwinger-DeWitt expansion we calculate the one-loop brane effective action of the quantum scalar field with the accuracy $O(\mathfrak{M}^2)$. In this approximation the basis of local curvature invariants includes one structure as a cosmological term, two structures linear in the extrinsic curvature and the potential term of the brane operator (12) and seven structures of dimensionality $(\mathfrak{M}^2)$,

$$O(\mathfrak{M}^0) = \int\limits_{\mathbf{b}} d^dx\,\sqrt{g}\;, \tag{17}$$

$$O(\mathfrak{M}^1) = \int\limits_{\mathbf{b}} d^dx\,\sqrt{g}\,k,\;\; \int\limits_{\mathbf{b}} d^dx\,\sqrt{g}\,\frac{p}{2m}, \tag{18}$$

$$\begin{aligned} O(\mathfrak{M}^2) = &\int\limits_{\mathbf{b}} d^dx\,\sqrt{g}\;{}^B\!R,\;\; \int\limits_{\mathbf{b}} d^dx\,\sqrt{g}\;{}^B\!R_{nn},\;\; \int\limits_{\mathbf{b}} d^dx\,\sqrt{g}\,k^2_{\alpha\beta},\;\; \int\limits_{\mathbf{b}} d^dx\,\sqrt{g}\,k^2,\\ &\int\limits_{\mathbf{b}} d^dx\,\sqrt{g}\,P,\;\; \int\limits_{\mathbf{b}} d^dx\,\sqrt{g}\,\Big(\frac{p}{2m}\Big)^2,\;\; \int\limits_{\mathbf{b}} d^dx\,\sqrt{g}\,k\frac{p}{2m}. \end{aligned} \tag{19}$$

Below we derive the coefficients of these structures in (15) as nontrivial functions of mass parameters M, m and μ, present explicit answer for the effective potential – the coefficient of (17) and find 4-dimensional UV divergences in this model.

2 Perturbation theory for the bulk Green's function and brane effective action

The first step of the calculational procedure is the perturbation theory for the Dirichlet Green's function in the bulk. It begins with decomposing the full bulk operator into the unperturbed F_0 and perturbation parts, which in the Gaussian normal coordinates of the brane $X^A = (x^\mu, y)$, $y \equiv X^{d+1}$, takes the form

$$\begin{aligned} F(\nabla) &= -\Box_X^{(d+1)} + M^2 + P(X)\\ &= -\partial_y^2 - \Box + M^2 - V(X\,|\,\partial_y, \nabla) \;\equiv\; F_0 - V, \quad \Box \equiv \Box_x(0). \end{aligned} \tag{20}$$

It is important that $\Box \equiv \Box_x(0)$ in $F_0 = -\partial_y^2 - \Box + M^2$ is the covariant d-dimensional d'Alembertian defined relative to the curved induced metric $g_{\mu\nu}(x)$ on the brane at $y = 0$.

Then the Green's function of this operator in the bulk can be obtained as a formal series in the perturbation V

$$G_D = G_D^0 + G_D^0 V G_D^0 + ... = G_D^0 \sum_{n=0}^{\infty} \big(V\, G_D^0\big)^n, \tag{21}$$

$$G_D^0(y,y') = \frac{e^{-|\,y-y'|\,\sqrt{M^2-\Box}} - e^{-(y+y')\,\sqrt{M^2-\Box}}}{2\sqrt{M^2-\Box}}, \tag{22}$$

where the zeroth order Dirichlet Green's function is explicitly known as the elementary function of $\sqrt{M^2-\Box}$ because of y-independence of the coefficients of the operator $F_0 = -\partial_y^2 - \Box + M^2$. With the expansion of the perturbation operator V in the Taylor series in powers of the coordinate y this operator takes the form

$$V = k(x,y)\partial_y + \Box(y) - \Box - P(x,y) \equiv \sum_{k=0}^{\infty} \frac{1}{k!} u_k\, y^k\, \partial_y - \sum_{k=0}^{\infty} \frac{1}{k!} v_k\, y^k$$

(here $k(x,y)$ is the trace of the extrinsic curvature of the surfaces y = const), where $u_k(\nabla)$ and $v_k(\nabla)$ form a set of y-independent *local d-dimensional covariant* operators of maximum second order in ∇_x. The coefficients of these operators are given by the powers of the bulk and brane curvature, the extrinsic curvature of the brane, the potential term P and the covariant derivatives of all these quantities — all of them taken at the brane. Several lowest order coefficients are

$$\begin{aligned}
u_0 &= k(x,y)\big|_{y=0} \equiv k,\\
u_1 &= \partial_y k(x,y)\big|_{y=0} = -{}^B\!R_{nn} - k_{\mu\nu}^2,\\
v_0 &= P(x,y)\Big|_{y=0} \equiv P,\\
v_1 &= \partial_y\big(-\Box_x(y) + P(x,y)\big)\Big|_{y=0} = 2k^{\alpha\beta}\,\nabla_\alpha\nabla_\beta + 2(\nabla_\alpha k^{\alpha\beta})\,\nabla_\beta - (\nabla^\beta k)\,\nabla_\beta + O(\mathfrak{M}^3),\\
v_2 &= \partial_y^2\big(-\Box_x(y) + P(x,y)\big)\Big|_{y=0} = \Big(-2\,{}^B\!R^{\alpha\ \,\beta}_{\ \,n\ \,n} - 6\,k^{\alpha\mu}k_\mu^{\ \beta}\Big)\,\nabla_\alpha\nabla_\beta + O(\mathfrak{M}^3).
\end{aligned}$$

With this expansion the inner integrals over the extra-dimensional coordinate y in the composition law of (21)

$$G_D^0 V G_D^0(y,y') = \int_0^\infty dy''\, G_D^0(y,y'')V(y'',\partial_{y''})G_D^0(y'',y')$$

can be term by term explicitly calculated to yield the inverse powers of $\sqrt{M^2-\Box}$ increasing with the growing power of y, $y^k \to 1/(\sqrt{M^2-\Box}\,)^k$. This, of course, signifies the efficiency of this Taylor expansion in y for the sake of the expansion in powers of $1/M$.

Thus, the substitution of the resulting (21) in (9) after the integration over y leads to a nonlocal series in inverse powers of $\sqrt{M^2-\Box}$. Each k-th order of this series arises in the form of the following nonlocal chain of square root "propagators",

$$\frac{1}{(\sqrt{M^2-\Box}\,)^{l_1}}\, V_1 \frac{1}{(\sqrt{M^2-\Box}\,)^{l_2}}\, V_2 ... V_{p-1} \frac{1}{(\sqrt{M^2-\Box}\,)^{l_p}},$$

with some differential operators V_i as its vertices. All these propagators can be systematically commuted to the right of the expression by the price of extra commutator terms originating from the commutation of $\Box$ with these vertex operators, and the perturbation expansion for the brane-to-brane operator (9) finally takes the form

$$\boldsymbol{F}^{\,\rm brane}(\nabla) = \boldsymbol{F}_0 - \sum_{k=1}^{\infty} U_k(\nabla) \frac{1}{(\sqrt{M^2-\Box}\,)^{k-1}}, \tag{23}$$

$$\boldsymbol{F}_0 = \frac{1}{2m}\Big(-\Box + \mu^2 + 2m\sqrt{M^2-\Box}\Big) \tag{24}$$

with the operators $U_k(\nabla)$ which are polynomial in the above mentioned operator coefficients u_k, v_k and their covariant derivatives.

The corresponding perturbation theory for the brane part of the effective action (10)

$$\frac{1}{2}\,\mathrm{Tr}\ln \boldsymbol{F}^{\,\mathrm{brane}} = \frac{1}{2}\mathrm{Tr}\ln \boldsymbol{F}_0 + \frac{1}{2}\mathrm{Tr}\ln\left(1-\sum_{k\geq 1}U_k(\nabla)\frac{1}{(\sqrt{M^2-\Box}\,)^{k-1}}\,\frac{1}{\boldsymbol{F}_0}\right)$$

is straightforward. By expanding the logarithm in powers of the nonlocal perturbation term and commuting all nonlocal factors $1/(\sqrt{M^2-\Box}\,)^l$ and $1/\boldsymbol{F}_0$ to the right we obtain

$$\frac{1}{2}\,\mathrm{Tr}\ln \boldsymbol{F}^{\,\mathrm{brane}} = \frac{1}{2}\mathrm{Tr}\ln \boldsymbol{F}_0 - \frac{1}{2}\sum_{k\geq 0,\,l\geq 1}\mathrm{Tr}\;W_{kl}(\nabla)\frac{1}{(\sqrt{M^2-\Box}\,)^{k}}\,\frac{1}{\boldsymbol{F}_0{}^l}, \qquad (25)$$

where the new set of coefficients – local differential operators $W_{kl}(\nabla)$ – arises from the products of $U_k(\nabla)$ and their covariant derivatives. One can show that these operators are polynomials in $1/m$ of the highest order $\max\{0,\,l-2\}$ [2] which is the extra source of non-analyticity in the DGP scale m at $m\to 0$ and the problem of low strong coupling scale (in addition to $p/2m$ factors in the set of invariants (17)-(19)). Several lowest order coefficients of this type read

$$\begin{aligned}
&W_{01} = U_1 = -\frac{1}{2}k - \frac{p}{2m},\\
&W_{11} = U_2 = \frac{1}{4}\,{}^B\!R_{nn} + \frac{1}{4}\,k_{\mu\nu}^2 - \frac{1}{2}\,P - \frac{1}{8}\,k^2,\\
&W_{02} = \frac{1}{2}\,U_1^2 = \frac{1}{2}\left(\frac{1}{2}k + \frac{p}{2m}\right)^2,\\
&W_{21} = U_3 = -\frac{1}{2}\,k^{\alpha\beta}\,\nabla_\alpha\nabla_\beta - \frac{1}{2}\,(\nabla_\alpha k^{\alpha\beta})\nabla_\beta + O(\mathfrak{M}^3),\\
&W_{31} = U_4 = \left(\frac{1}{4}\,{}^B\!R^{\alpha\ \beta}_{\ n\ n} + \frac{3}{4}\,k^{\alpha\mu}k_\mu^{\ \beta}\right)\nabla_\alpha\nabla_\beta + O(\mathfrak{M}^3).
\end{aligned} \qquad (26)$$

3 The generalized proper time method

The success of further calculations depends on the possibility of inverse mass expansion for $\mathrm{Tr}\ln\boldsymbol{F}_0$ and the functional traces of the form

$$\mathrm{Tr}\,\left(W_{kl}(\nabla)\frac{1}{(\sqrt{M^2-\Box}\,)^{k}}\,\frac{1}{\boldsymbol{F}_0{}^l}\right).$$

– the generalization of the *universal functional traces* introduced in [9]. Their calculation is based on the generalized proper time representation in terms of the heat kernel $\exp(s\Box)$

$$\begin{aligned}
\ln\boldsymbol{F}_0 &= -\ln(2m) - \int\limits_0^\infty \frac{ds}{s}\,w_{00}(s)\,e^{-s(M^2-\Box)}\,,\\
\frac{1}{\left(\sqrt{M^2-\Box}\right)^k}\,\frac{1}{\boldsymbol{F}_0{}^l} &= \int\limits_0^\infty \frac{ds}{s}\,w_{kl}(s)\,e^{-s(M^2-\Box)}\,, \qquad (l\geq 1),
\end{aligned}$$

differing from the well-known Schwinger representation [8] by nontrivial weight functions $w_{kl}(s)$. By using the decomposition of the zeroth-order operator

$$\begin{aligned}
&\boldsymbol{F}_0 = \frac{1}{2m}(\sqrt{M^2-\Box}\, - m_+)(\sqrt{M^2-\Box}\, - m_-),\\
&m_+ = -m + \sqrt{m^2+M^2-\mu^2}, \qquad m_- = -m - \sqrt{m^2+M^2-\mu^2}
\end{aligned} \qquad (27)$$

these weight functions can be obtained by reducing the full set of nonlocal expressions to the linear combinations of the structures [2]

$$\frac{1}{(\sqrt{M^2-\Box}\,)^a} = \frac{1}{\Gamma(a/2)}\int\limits_0^\infty ds\; s^{a/2-1}\, e^{-s(M^2-\Box)},$$

$$\ln(\sqrt{M^2-\Box}-m) = -\frac{1}{2}\int\limits_0^\infty \frac{ds}{s}\, w(-m\sqrt{s})\, e^{-s(M^2-\Box)},$$

$$\frac{1}{(\sqrt{M^2-\Box}-m)^a} = \frac{1}{2\,\Gamma(a)}\int\limits_0^\infty ds\; s^{a/2-1}\, \frac{d^a w(-\sigma)}{d\sigma^a}\Big|_{\sigma=m\sqrt{s}} e^{-s(M^2-\Box)}. \qquad (28)$$

The latter have weighted proper time representations in terms of the main weight function $w(-\sigma)$ expressible in terms of the error function

$$w(-\sigma) = e^{\sigma^2}\,\mathrm{erfc}\,(-\sigma),$$

$$\mathrm{erfc}(z) = \frac{2}{\sqrt{\pi}}\int\limits_z^\infty dt\; e^{-t^2}.$$

Thus finally the expansion (25) takes the form in terms of the universal functional traces with *weighted proper time representation*

$$\tfrac{1}{2}\,\mathrm{Tr}\ln \boldsymbol{F}^{\,\mathrm{brane}} \equiv -\frac{1}{2}\sum_{k,\,l=0}^{\infty}\int\limits_0^\infty \frac{ds}{s}\, w_{kl}(s)\, e^{-sM^2}\,\mathrm{Tr}\Big(W_{kl}(\nabla)\, e^{s\Box}\Big). \qquad (29)$$

Here we introduced additional coefficients $W_{00}=1,\ W_{k0}=0,\ k\geq 1$.

4 Large mass expansion and its cutoff scale

Using in (29) the proper time expansion of the d-dimensional heat kernel [8, 9]

$$e^{\,s\Box}\delta(x,x') = \frac{1}{(4\pi s)^{d/2}} D^{1/2}(x,x')\, e^{-\sigma(x,x')/2s}\sum_{n=0}^{\infty} s^n\, a_n(x,x')$$

and the well-known coincidence limits of the covariant derivatives of the Schwinger-DeWitt coefficients $a_n(x,x')$, the world function $\sigma(x,x')$ and the relevant Pauli-van Vleck determinant $D(x,x')$ which are local invariant of growing power in the brane curvatures and their derivatives

$$\nabla_{\mu_1}...\nabla_{\mu_p}\, a_n(x,x')\Big|_{x'=x} = O(\mathfrak{M}^{2n+p}),\quad \nabla_{\mu_1}...\nabla_{\mu_p}\sigma(x,x')\Big|_{x'=x} = O(\mathfrak{M}^{2n-2}),$$
$$\nabla_{\mu_1}...\nabla_{\mu_p} D(x,x')\Big|_{x'=x} = O(\mathfrak{M}^{2n})$$

we have

$$\frac{1}{2}\,\mathrm{Tr}\ln \boldsymbol{F}^{\,\mathrm{brane}} = -\frac{1}{2}\frac{1}{(4\pi)^{d/2}}\sum_{k,\,l,\,n=0}^{\infty}\ \sum_{p=0}^{\max\{0,\,l-2\}}\ \sum_{c=0}^{[\frac{2k+2l+p}{6}]}$$
$$\times\int\limits_0^\infty \frac{ds}{s}\, s^{n-c-d/2}\, w_{kl}(s)\, e^{-sM^2}\, \frac{O\big(\mathfrak{M}^{k+l+p+2n-2c}\big)}{m^p}. \qquad (30)$$

Here extra negative powers of the proper time s^{-c} follow from the differentiation of $\sigma/2s$ in the exponential of (30). The highest power of $1/s$ is determined by (the integer part of) half the order of $W_{kl}(\nabla)$ in derivatives. As local differential operators W_{kl} are polynomial in the inverse DGP mass scale, we also have negative powers of $1/m^p$. Careful bookkeeping of the summation ranges of these powers is made in [2] and leads to (30).

One can show [2]) that in the domain of convergence of the proper time integral of Eq.(30) in the complex plane of d has two parts analytic and non-analytic in m

$$\int\limits_0^\infty \frac{ds}{s}\, s^{j-d/2}\, w_{kl}(s)\, e^{-sM^2} = \frac{1}{\Gamma(d/2-j)} \int\limits_0^\infty \frac{d\lambda}{\lambda}\, \lambda^{d/2-j}\, \frac{1}{(\sqrt{M^2-\Box})^k}\, \frac{1}{\boldsymbol{F}_0{}^l}\Big|_{\Box=-\lambda}$$

$$= C_1 \frac{(mM)^{d/2}}{m^j\, M^{j+k+l}} + C_2 \frac{M^d m^l}{M^{2j+k+2l}}, \quad M \to \infty.$$

This asymptotics finally leads to the expansion (15) advocated in Introduction, which we repeat here

$$\frac{1}{2}\,\mathrm{Tr}\ln \boldsymbol{F}^{\,\mathrm{brane}} = \left(\frac{Mm}{4\pi}\right)^{d/2} \sum_{N=0}^{\infty} \frac{1}{M^N} \sum_{i\leq N} \frac{O(\mathfrak{M}^{2N-i})}{m^{N-i}} + \frac{M^d}{(4\pi)^{d/2}} \sum_{L=0}^{\infty} \frac{1}{M^L} \sum_{i\leq L} m^i\, O(\mathfrak{M}^{L-i}) \tag{31}$$

As shown in [2] this expansion is efficient, because the coefficient of any given inverse power of M is given by a finite sum (over i) of curvature invariants. Also this expansion contains the two series having qualitatively different analytic behavior in M and m, — the property recently discovered for the effective potential of the toy DGP model [13]. Whereas the second part of (31) is analytic in a small DGP scale $m \to 0$, the first nonanalytic part is formally singular in this limit, and this leads to the redefinition of the cutoff $M_{\rm cutoff}$ of the theory, below which $\mathfrak{M} \ll M_{\rm cutoff}$ the local expansion remains valid.

Indeed, despite the efficiency of the obtained expansion, in the first series of (31) it contains negative powers of the DGP scale m and blows up for small $m \to 0$. This is a typical situation of the presence of a strong-coupling scale [6]. In fact, for $m < M$ the actual cutoff is lower than M and is given by the expression (16), $M_{\rm cutoff} = \sqrt{Mm}$, presented in Introduction (the condition of smallness of the strongest $i = 0$ term in the first series of (31), $\mathfrak{M}^2/Mm \ll 1$).

In fact the coefficients (31) of the curvature invariants of a given background dimensionality in (30) are calculable in terms of the hypergeometric function. They represent exact nontrivial functions of m and M and express in terms of

$$\Phi_{(0)}(a,\, b) = \left(\frac{\varepsilon}{2}\right)^{\frac{d+1-a-b}{2}} \Gamma\Big(\frac{a+b-1-d}{2}\Big) \times {}_2F_1\left(\frac{d+b-a+1}{2},\, \frac{a-b+1-d}{2};\, \frac{d+3-a-b}{2};\, \frac{\varepsilon}{2}\right) + \frac{\Gamma(a-d)\,\Gamma(b)\,\Gamma\big(\frac{d-a-b+1}{2}\big)}{\Gamma\big(\frac{d+b-a+1}{2}\big)\,\Gamma\big(\frac{a-b+1-d}{2}\big)}\, {}_2F_1\left(a-d,\, b;\, \frac{a+b-1-d}{2};\, \frac{\varepsilon}{2}\right), \tag{32}$$

for integer a and b as certain sums over $\varepsilon = \varepsilon_\pm \equiv 1 - m_\pm/M$ with $m_\pm$ given by (27). In view of $\varepsilon_+ \sim m/M$ the first term here generates the first nonanalytic in m series of (31) and the second term is responsible for the analytic part because the hypergeometric function is expandable in Taylor series in $\varepsilon_+/2 \to 0$. The ε_- part of the action contributes only to the analytic part, because $m_-/2M \to -1/2$ and the relevant large M expansion does not give rise to nonanalytic terms.

As we see these coefficients are rather complicated – for the invariants (17)-(19) they are explicitly given in [2]). Even the coefficient of the zeroth power in spacetime curvature – effective potential of the brane in brane induced gravity model – is a very nontrivial function of m and M given by [13] (see also [14] for its logarithmic part)

$$V_{\rm eff} = \frac{1}{d\,\Gamma(d/2)}\,\frac{\pi}{\sin\frac{\pi d}{2}}\left(\frac{Mm}{4\pi}\right)^{d/2}\left(\sqrt{1+\frac{m^2}{M^2}}-\frac{m}{M}\right)^{d/2}$$
$$-\left(\frac{M^2}{4\pi}\right)^{d/2}\frac{\Gamma\left(\frac{1-d}{2}\right)}{2d\,\sqrt{\pi}}\sum_{v=\frac{m_\pm}{M}} v\;{}_2F_1\Big(1,\,\frac{1-d}{2};\,1-\frac{d}{2};\,1-v^2\Big)$$

Finally we present the ultraviolet divergences of the brane action for the four-dimensional case in the dimensional regularization $d\to 4$. They read

$$\begin{aligned}
\tfrac12\,{\rm Tr}\ln \boldsymbol{F}^{\rm brane}\,\Big|^{\rm div} = {}& \frac{1}{32\pi^2(4-d)}\int\limits_{\mathbf b} dx\sqrt{g}\,\Big(-4m^2(M^2+2m^2-2\mu^2)-\mu^4\Big)\\
&+\frac{1}{32\pi^2(4-d)}\int\limits_{\mathbf b} dx\sqrt{g}\,\Big(\frac12 m\big(M^2+12m^2-3\mu^2\big)\,k+2\big(\mu^2-4m^2\big)p\Big)\\
&+\frac{1}{32\pi^2(4-d)}\int\limits_{\mathbf b} dx\sqrt{g}\,\Big(\frac13(-2m^2+\mu^2)\,{}^B\!R+\big(\frac{17}{6}m^2-\frac23\mu^2\big)\,{}^B\!R_{nn}\\
&\qquad+\big(\frac92 m^2-\frac13\mu^2\big)\,k^2_{\alpha\beta}+\big(-2m^2+\frac13\mu^2\big)\,k^2\\
&\qquad-4m^2P-\frac32 m\,k\,p-p^2\Big)\;+\;O\big(\mathfrak{M}^3\big). \qquad (33)
\end{aligned}$$

This result confirms the general properties of one-loop ultraviolet divergences in any dimension d proven in [2]. These divergences are contained in both series of the expansion (31). For an even d they are analytic and polynomial in both M and m, for an odd d they have a structure $\sqrt{Mm}$ times a finite polynomial in M and m. Finally, their background dimensionality is always bounded by $O(\mathfrak{M}^d)$.

5 Conclusions

Thus we have constructed the covariant curvature expansion in massive brane induced gravity models, found its peculiar structure (15) nonanalytic in the DGP scale and derived a nontrivial cutoff (16) of this general expansion. We discussed several lowest orders of this expansion for a quantum scalar field in a curved bulk spacetime with a kinetic term on the brane, presented its effective potential and ultraviolet divergences for the case of a 4-dimensional brane.

These results might find important applications. Although a comparison of our massive model with the massless DGP model of [16] is not straightforward, we can observe a common feature in their cutoff properties. In both theories their cutoff (16) is different from the bulk one M and is modified by the DGP scale m. For the tree-level DGP model with the Planck mass $M=M_P$, playing the role of the bulk cutoff, this cutoff equals $M_{\rm cutoff}=(m^2M)^{1/3}$ [6]. With m identified with the cosmological horizon scale, this is about $(1000{\rm km})^{-1}$ which is much below the submillimeter scale capable of featuring the infrared modifications of the Einstein theory [17]. As we see, the situation with the local expansion for the *quantum* action is much better — the cutoff (16) is a geometric average of M and m, which is much higher,

$$(mM)^{1/2}\gg(m^2M)^{1/3}, \qquad (34)$$

and comprises $(0.1\mathrm{mm})^{-1}$. This supports the conjecture [18] that the replacement of the weak field perturbation theory by a derivative expansion, as is the case of the local Schwinger-DeWitt series (probably with the non-perturbative resummation of powers of a local potential term of the operator (1) [19]), might improve the range of validity of the calculational scheme.

Obviously, the Schwinger-DeWitt technique in brane induced gravity models turns out to be much more complicated than in models without spacetime boundaries or in case of boundaries with local Dirichlet and Neumann boundary conditions. It does not reduce to a simple bookkeeping of local surface terms like the one reviewed in [11]. Nevertheless it looks complete and self-contained, because it provides in a systematic way a manifestly covariant calculational procedure for a wide class of boundary conditions including tangential derivatives (in fact of any order).

To summarize, we developed a new scheme of calculating quantum effective action for the braneworld DGP-type system in curved spacetime. This scheme gives a systematic curvature expansion by means of a manifestly covariant technique. Combined with the method of fixing the background covariant gauge for diffeomorphism invariance developed in [12, 20] this gives the universal background field method of the Schwinger-DeWitt type in gravitational brane systems.

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