Nonlocal Cosmological Models and Exact Solutions

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based on
I.Ya. Aref'eva, L.V. Joukovskaya, S.V.,
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To specify different types of cosmic fluids one uses a phenomenological relation between the pressure p and the energy density ϱ

$$p = w\varrho, \qquad p = E_k - V, \qquad \varrho = E_k + V$$

where w is the state parameter.

$$w(t) = -1 - \frac{2}{3}\frac{\dot{H}}{H^2} = -1 + \frac{2E_k}{\varrho}. \tag{1}$$

Contemporary experiments give strong support that

$$w_{DE} = -1 \pm 0.2. \tag{2}$$

We consider the case $w_{DE} < -1$. Null energy condition (NEC) is violated and there are problems of instability. A possible way to evade the instability problem for models with $w_{DE} < -1$ is to yield a phantom model as an effective one, arising from a more fundamental theory.

In the SFT inspired cosmological models the non-local "kinetic" term

$$\phi e^{-\Box_g}\phi, \tag{3}$$

arise as a key ingredient of the action:

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Non-local action in the general form

Let us consider the following nonlocal action:

$$S=\int d^4x\sqrt{-g}\alpha'\left(\frac{R}{16\pi G_N}+\frac{1}{2g_o^2}\left(\phi\mathcal{F}(\Box_g)\phi-V(\phi)\right)-\Lambda\right) \qquad (4)$$

Here G_N is the Newtonian constant: $8\pi G_N=1/M_P^2$, α' and g_o are constants.

Function $\mathcal{F}$ is assumed to be an analytic function

$$\mathcal{F}(\Box_g)=\sum_{n=0}^{\infty}f_n\Box_g^n,\qquad \Box_g\equiv\frac{1}{\sqrt{-g}}\partial_\mu\sqrt{-g}g^{\mu\nu}\partial_\nu. \qquad (5)$$

From the SFT after some approximations we obtained:

$$\mathcal{F}_{SFT}(\Box_g)=(\xi^2\Box_g+1)e^{-2\Box_g}-c, \qquad (6)$$

where c and ξ^2 are constants.

$\mathcal{F}_{SFT}(\Box_g)$ has only simple and (for some values of c and ξ^2) double roots. We consider the case of an arbitrary analytic $\mathcal{F}$, which has only simple and double roots.

In an arbitrary metric the energy-momentum tensor

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}} = \frac{1}{g_o^2}\Big(E_{\mu\nu} + E_{\nu\mu} - g_{\mu\nu}\left(g^{\rho\sigma}E_{\rho\sigma} + W\right)\Big), \qquad (7)$$

where

$$E_{\mu\nu} \equiv \frac{1}{2}\sum_{n=1}^{\infty} f_n \sum_{l=0}^{n-1} \partial_\mu \Box_g^l \phi \partial_\nu \Box_g^{n-1-l}\phi, \qquad (8)$$

$$W \equiv \frac{1}{2}\sum_{n=2}^{\infty} f_n \sum_{l=1}^{n-1} \Box_g^l \phi \Box_g^{n-l}\phi - \frac{f_0}{2}\phi^2 + V(\phi). \qquad (9)$$

In the case of the zero potential $V(\phi) = 0$, using the equation

$$F(\Box_g)\phi = 0, \quad \Longleftrightarrow \quad f_0\phi = -\sum_{n=1}^{\infty} f_n \Box_g^n \phi, \qquad (10)$$

we obtain

$$W = W_0 \equiv \frac{1}{2}\sum_{n=1}^{\infty} f_n \sum_{l=0}^{n-1} \Box_g^l \phi \Box_g^{n-l}\phi. \qquad (11)$$

From action (4) we obtain the following equations

$$G_{\mu\nu} = 8\pi G_N \left(T_{\mu\nu} - \Lambda g_{\mu\nu}\right), \tag{12}$$

$$F(\Box_g)\phi = \frac{dV}{d\phi}, \tag{13}$$

where $G_{\mu\nu}$ is the Einstein tensor.

It is a system of nonlocal nonlinear equations !!!

HOW CAN WE FIND A SOLUTION?

An algorithm of localization in the case of an arbitrary quadratic potential $V(\phi) = C_2\phi^2 + C_1\phi + C_0$.

We can change values of f_0 and Λ such that the potential takes the form $V(\phi) = C_1\phi$. So, we put $C_2 = 0$ and $C_0 = 0$.

There exist 3 cases:

- $C_1 = 0$
- $C_1 \neq 0$ **and** $f_0 \neq 0$
- $C_1 \neq 0$ **and** $f_0 = 0$

Let us start with the case $C_1 = 0$.

Let us consider the characteristic equation $\mathcal{F}(J) = 0$ and denote simple roots as J_i and double roots as $\tilde{J}_k$. A particular solution of equation (13) we seek in the following form

$$\phi_0 = \sum_{i=1}^{N_1} \phi_i + \sum_{k=1}^{N_2} \tilde{\phi}_k, \qquad (14)$$

where

$$(\Box_g - J_i)\phi_i = 0, \qquad (\Box_g - \tilde{J}_k)^2 \tilde{\phi}_k = 0. \qquad (15)$$

The fourth order differential equation $(\Box_g - \tilde{J}_k)(\Box_g - \tilde{J}_k)\tilde{\phi}_k = 0$ is equivalent to the following system:

$$(\Box_g - \tilde{J}_k)\tilde{\phi}_k = \varphi_k, \qquad (\Box_g - \tilde{J}_k)\varphi_k = 0. \tag{16}$$

For any analytical function $\mathcal{F}(J)$, which has simple roots J_i and double roots $\tilde{J}_k$, the energy–momentum tensor

$$T_{\mu\nu}\left(\phi_0\right) = T_{\mu\nu}\left(\sum_{i=1}^{N_1}\phi_i + \sum_{k=1}^{N_2}\tilde{\phi}_k\right) = \sum_{i=1}^{N_1} T_{\mu\nu}(\phi_i) + \sum_{k=1}^{N_2} T_{\mu\nu}(\tilde{\phi}_k), \tag{17}$$

where

$$T_{\mu\nu} = \frac{1}{g_o^2}\Big(E_{\mu\nu} + E_{\nu\mu} - g_{\mu\nu}\left(g^{\rho\sigma}E_{\rho\sigma} + W\right)\Big), \tag{18}$$

$$E_{\mu\nu}(\phi_i) = \frac{\mathcal{F}'(J_i)}{2}\partial_\mu\phi_i\partial_\nu\phi_i, \quad W(\phi_i) = \frac{J_i\mathcal{F}'(J_i)}{2}\phi_i^2, \quad \mathcal{F}' \equiv \frac{d\mathcal{F}}{dJ} \tag{19}$$

$$E_{\mu\nu}(\tilde{\phi}_k) = \frac{\mathcal{F}''(\tilde{J}_k)}{4}\left(\partial_\mu\tilde{\phi}_k\partial_\nu\varphi_k + \partial_\nu\tilde{\phi}_k\partial_\mu\varphi_k\right) + \frac{\mathcal{F}'''(\tilde{J}_k)}{12}\partial_\mu\varphi_k\partial_\nu\varphi_k, \tag{20}$$

$$W(\tilde{\phi}_k) = \frac{\tilde{J}_k\mathcal{F}''(\tilde{J}_k)}{2}\tilde{\phi}_k\varphi_k + \left(\frac{\tilde{J}_k\mathcal{F}'''(\tilde{J}_k)}{12} + \frac{\mathcal{F}''(\tilde{J}_k)}{4}\right)\varphi_k^2. \tag{21}$$

Consider the following local action

$$S_{loc} = \int d^4x\sqrt{-g}\left(\frac{R}{16\pi G_N} - \Lambda\right) + \sum_{i=1}^{N_1} S_i + \sum_{k=1}^{N_2} \tilde{S}_k, \qquad (22)$$

where

$$S_i = -\frac{1}{g_o^2}\int d^4x\sqrt{-g}\frac{\mathcal{F}'(J_i)}{2}\left(g^{\mu\nu}\partial_\mu\phi_i\partial_\nu\phi_i + J_i\phi_i^2\right),$$

$$\tilde{S}_k = -\frac{1}{g_o^2}\int d^4x\sqrt{-g}\left(g^{\mu\nu}\left(\frac{\mathcal{F}''(\tilde{J}_k)}{4}\left(\partial_\mu\tilde{\phi}_k\partial_\nu\varphi_k + \partial_\nu\tilde{\phi}_k\partial_\mu\varphi_k\right) + \right.\right.$$
$$\left.\left. + \frac{\mathcal{F}'''(\tilde{J}_k)}{12}\partial_\mu\varphi_k\partial_\nu\varphi_k\right) + \frac{\tilde{J}_k\mathcal{F}''(\tilde{J}_k)}{2}\tilde{\phi}_k\varphi_k + \left(\frac{\tilde{J}_k\mathcal{F}'''(\tilde{J}_k)}{12} + \frac{\mathcal{F}''(\tilde{J}_k)}{4}\right)\varphi_k^2\right).$$

Remark 1. If $\mathcal{F}(J)$ has an infinity number of roots then one nonlocal model corresponds to infinity number of different local models. In this case the initial nonlocal action (4) generates infinity number of local actions (22).

Remark 2. We should prove that the way of localization is self-consistent. To construct local action (22) we assume that equations (15) are satisfied. Therefore, the method of localization is correct only if these equations can be obtained from the local action S_{loc}. The straightforward calculations show that

$$\frac{\delta S_{loc}}{\delta\phi_i}=0 \;\Leftrightarrow\; \Box_g\phi_i=J_i\phi_i; \qquad \frac{\delta S_{loc}}{\delta\tilde{\phi}_k}=0 \;\Leftrightarrow\; \Box_g\varphi_k=\tilde{J}_k\varphi_k. \tag{23}$$

$$\frac{\delta S_{loc}}{\delta\varphi_k}=0 \qquad \Leftrightarrow \qquad \Box_g\tilde{\phi}_k=\tilde{J}_k\tilde{\phi}_k+\varphi_k. \tag{24}$$

We obtain from S_{loc} the Einstein equations as well:

$$G_{\mu\nu}=8\pi G_N\left(T_{\mu\nu}(\phi_0)-\Lambda g_{\mu\nu}\right), \tag{25}$$

where ϕ_0 is given by (14) and $T_{\mu\nu}(\phi_0)$ can be calculated by (17).

We have obtained such systems of differential equations that any solutions of these systems are particular solutions of the initial nonlocal equations (12) and (13).

Let us consider the model with action (4) in the case $C_1 \neq 0$. If $f_0 \neq 0$, then we indroduce a new scalar field

$$\chi = \phi - \frac{C_1}{f_0} \tag{26}$$

and get the energy–momentum tensor in the form (18) with

$$E_{\mu\nu} = \frac{1}{2}\sum_{n=1}^{\infty} f_n \sum_{l=0}^{n-1} \partial_\mu \Box_g^l \chi \partial_\nu \Box_g^{n-1-l}\chi, \tag{27}$$

$$W = \frac{1}{2}\sum_{n=1}^{\infty} f_n \sum_{l=1}^{n-1} \Box_g^l \chi \Box_g^{n-l}\chi - \frac{f_0}{2}\chi^2 + \frac{C_1^2}{2f_0}. \tag{28}$$

It is easy to see that

$$\mathcal{F}(\Box)\phi = C_1 \qquad \Longleftrightarrow \qquad \mathcal{F}(\Box)\chi = 0. \tag{29}$$

If $f_0 = 0$, then $J = 0$ is a root of the characteristic equation $\mathcal{F}(J) = 0$. It is easy to show, that the function

$$\tilde{\chi} = \phi_0 + \psi, \tag{30}$$

where ϕ_0 and ψ are solutions of the following equations

$$\mathcal{F}(\Box)\phi_0 = 0, \qquad \Box^m\psi = \frac{C_1}{f_m}, \tag{31}$$

m **is the order of the root** $J = 0$**, satisfies**

$$\mathcal{F}(\Box)\tilde{\chi} = C_1. \tag{32}$$

The function ϕ_0 is given by (14), but the sum do not include ϕ_{i_0}, which corresponds to the root $J = 0$, because this function can not be separated from ψ. We consider the case $m = 1$ ($J = 0$ is a simple root).

It is easy to show:

$$T_{\mu\nu}(\tilde{\chi}) = T_{\mu\nu}(\psi) + T_{\mu\nu}(\phi_0), \tag{33}$$

$$W(\psi) = C_1\psi + \frac{f_2 C_1^2}{2f_1^2}, \qquad E_{\mu\nu}(\psi) = \frac{1}{2}f_1\partial_\mu\psi\partial_\nu\psi, \tag{34}$$

For an arbitrary quadratic potential $V(\phi) = C_2\phi^2 + C_1\phi + C_0$ ***there exists the following algorithm of localization*:**

- **Find roots of the function $\mathcal{F}(J)$ and calculate orders of them.**
- **Select an finite number of simple and double roots.**
- **Change values of f_0 and Λ such that the potential takes the form $V(\phi) = C_1\phi$.**
- **Construct the corresponding local action. In the case $C_1 = 0$ one should use formula (22). In the case $C_1 \neq 0$ and $f_0 \neq 0$ one should use (22) with the replacement of the scalar field ϕ by χ. In the case $C_1 \neq 0$ and $f_0 = 0$ the local action is the sum of (22) and (in the case of simple root $J = 0$)**
$$S_\psi = -\frac{1}{2g_o^2}\int d^4x\sqrt{-g}\left(f_1 g^{\mu\nu}\partial_\mu\psi\partial_\nu\psi + 2C_1\psi + \frac{f_2 C_1^2}{f_1^2}\right).$$
- **Vary the obtained local action and get a system of the Einstein equations and equations of motion.**
- **Seek solutions of the obtained local system.**

Exact Solutions in the FRW metric

Let us consider the Friedmann equations, which corresponds to a real simple root J_1:

$$\left\{\begin{aligned} 3H^2 &= \frac{4\pi G\mathcal{F}'(J_1)}{g_0^2}\left(\dot\phi^2 + J_1\phi^2\right) + 8\pi G\Lambda, \\ \dot H &= -\frac{4\pi G\mathcal{F}'(J_1)}{g_0^2}\dot\phi^2, \end{aligned}\right. \tag{35}$$

a dot denotes a time derivative. Exact real solutions of this system are as follows:

At $J_1 > 0$

$$\phi(t) = \pm\frac{\sqrt{3J_1}g_0^2}{6\pi G\mathcal{F}'(J_1)}(t-t_0), \qquad H(t) = -\frac{J_1 g_0^2}{6\pi G\mathcal{F}'(J_1)}(t-t_0), \tag{36}$$

where t_0 is an arbitrary constant. These solutions exist only at

$$\Lambda = -\frac{J_1 g_0^2}{24G^2\pi^2\mathcal{F}'(J_1)}\,. \tag{37}$$

At $J_1 = 0$ the type of solution depends on sign of Λ:

- $\Lambda = 0$

$$H(t) = -\frac{1}{3(t-t_0)}, \qquad \phi(t) = C_1 \pm \frac{\sqrt{3}g_0}{\sqrt{\pi G\mathcal{F}'(0)}}\ln(t-t_0), \tag{38}$$

where t_0 and C_1 are arbitrary constants.

- **If $\Lambda > 0$, then we obtain solutions:**

$$H_1(t) = \frac{2\sqrt{6\pi G\Lambda}}{3}\tanh\left(2\sqrt{6\pi G\Lambda}\,(t-t_0)\right), \tag{39}$$

$$\phi_1(t) = \pm\sqrt{\frac{-g_0^2}{12\pi G\mathcal{F}'(0)}}\arctan\left(\sinh\left(2\sqrt{6\pi G\Lambda}\,(t-t_0)\right)\right) + C_2 \tag{40}$$

Using $\tanh(t + i\pi/2) = \coth(t)$, one gets a new real solution.

- **In the case $\Lambda < 0$**

$$H_2(t) = -\frac{2\sqrt{-6\pi G\Lambda}}{3}\tan\left(2\sqrt{-6\pi G\Lambda}(t-t_0)\right), \tag{41}$$

$$\phi_2(t) = C_2 \pm \sqrt{\frac{g_0^2}{12\pi G\mathcal{F}'(0)}}\operatorname{arctanh}\left(\sin\left(2\sqrt{-6\pi G\Lambda}(t-t_0)\right)\right). \tag{42}$$

SOLUTIONS FOR EQUATIONS OF MOTION

Let us consider nonlocal Klein–Gordon equation in the case of an arbitrary potential:

$$\mathcal{F}(\Box_g)\phi = V'(\phi), \tag{43}$$

where prime is a derivative with respect to ϕ. A particular solution of (43) is a solution of the following system:

$$\sum_{n=0}^{N-1} f_n \Box_g^n \phi = V'(\phi) - C, \qquad f_N \Box_g^N \phi = C, \tag{44}$$

where $N-1$ is a natural number, C is an arbitrary constant.

In the case $f_1 \neq 0$ we can choose $N = 2$. In the spatially flat FRW metric with the interval:

$$ds^2 = -\,dt^2 + a^2(t)\left(dx_1^2 + dx_2^2 + dx_3^2\right), \tag{45}$$

where $a(t)$ is the scale factor, we obtain from (44) the following system:

$$f_1 \Box_g \phi = -\,f_1\left(\ddot{\phi} + 3H\dot{\phi}\right) = V'(\phi) - f_0\phi - C, \qquad f_2 \Box_g^2 \phi = C. \tag{46}$$

The Hubble parameter

$$H = \; -\frac{1}{3\dot{\phi}}\left(\ddot{\phi} + \tilde{V}'(\phi) - \frac{C}{f_1}\right), \tag{47}$$

where

$$\tilde{V}'(\phi) \equiv \frac{1}{f_1}\left(V'(\phi) - f_0\phi\right). \tag{48}$$

Equation

$$(\partial_t^2 + 3H\partial_t)\left(\ddot{\phi} + 3H\dot{\phi}\right) = \frac{C}{f_2}, \tag{49}$$

is as follows

$$(\partial_t^2 + 3H\partial_t)\tilde{V}' = \tilde{V}'''\dot{\phi}^2 + \tilde{V}''(\ddot{\phi} + 3H\dot{\phi}) = \; -\frac{C}{f_2}. \tag{50}$$

We eliminate H and obtain

$$\dot{\phi}^2 = \frac{1}{\tilde{V}'''}\left(\tilde{V}''\tilde{V}' - \frac{C}{f_1}\tilde{V}'' - \frac{C}{f_2}\right). \tag{51}$$

The obtained equation can be solved in quadratures. Its general solution depend on two arbitrary parameters C and t_0, which corresponds to the time shift.

CUBIC POTENTIAL

The case of cubic potential is is connected with the bosonic string field theory. Let us find solutions (43) for

$$V(\phi) = B_3\phi^3 + B_2\phi^2 + B_1\phi + B_0, \tag{52}$$

where B_0**,** B_1**,** B_2**, and** B_3 **are arbitrary constants, but** $B_3 \neq 0$**. For this potential we get (51) in the following form**

$$\dot{\phi}^2 = 4C_3\phi^3 + 6C_2\phi^2 + 4C_1\phi + C_0, \tag{53}$$

where

$$C_0 = \frac{(B_1 - C)(2B_2 - f_0)}{6f_1B_3} - \frac{Cf_1^2}{6f_1f_2B_3}, \qquad C_2 = \frac{2B_2 - f_0}{4f_1}, \tag{54}$$

$$C_1 = \frac{6B_3(B_1 - C) + (2B_2 - f_0)^2}{24f_1B_3}, \qquad C_3 = \frac{3B_3}{4f_1}. \tag{55}$$

Note, that $C_3 \neq 0$ **since** $B_3 \neq 0$**. Using the transformation**

$$\phi = \frac{1}{2C_3}(2\xi - C_2), \tag{56}$$

we get the following equation

$$\dot{\xi}^2 = 4\xi^3 - g_2\xi - g_3, \tag{57}$$

where

$$g_2 = \frac{(2B_2 - f_0)^2 - 12B_3(B_1 - C)}{16f_1^2}, \qquad g_3 = 2C_1C_2C_3 - C_2^3 - C_0C_3^2 = -\frac{3B_3C}{32f_2f_1}\,.$$

A solution of equation (57) is either the Weierstrass elliptic function

$$\xi(t) = \wp(t - t_0, g_2, g_3), \tag{58}$$

or a degenerate elliptic function.

Let us consider degenerated cases. At $g_2 = 0$ **and** $g_3 = 0$

$$\phi_1 = \frac{1}{C_3(t - t_0)^2} - \frac{C_2}{2C_3} = \frac{4f_1}{3B_3(t - t_0)^2} - \frac{2B_2 - f_0}{6B_3}. \tag{59}$$

$$H_1 = \frac{5}{3(t - t_0)}. \tag{60}$$

We are of interest to obtain a bounded solution, which tends to a finite limit at $t \to \infty$**. We have obtained such solutions in**

the following form

$$\phi_2 = D_2 \tanh(\beta(t-t_0))^2 + D_0, \tag{61}$$

$$D_2 = \frac{4}{3B_3} f_1 \beta^2, \qquad D_0 = \frac{1}{18B_3}\left(3(f_0 - 2B_2) - 16 f_1 \beta^2\right), \tag{62}$$

where β **is a root of the following equation**

$$1024 f_2 f_1 \beta^6 + 576 f_1^2 \beta^4 + 324 B_3 B_1 - 27(2B_2 - f_0)^2 = 0. \tag{63}$$

Bounded real solutions for equation (53) correspond to real root of equations (63). Pure image root of (63) correspond to unbounded real solutions for equation (53), because $\tanh(\beta t)^2 = -\tan(\mathrm{i}\beta t)^2$**. The solution** ϕ_2 **exists at**

$$C = \frac{1}{36B_3}\left(64 f_1^2 \beta^4 - 3(2B_2 - f_0)^2 + 36 B_3 B_1\right). \tag{64}$$

$$H_2 = \frac{\beta(2\cosh(\beta t)^2 - 3)}{3\cosh(\beta t)\sinh(\beta t)} -$$

$$- \frac{3B_3(D_2\tanh(\beta t)^2 + D_0)^2 + (2B_2 - f_0)(D_2\tanh(\beta t)^2 + D_0) + B_1}{6 f_1 D_2 \beta \tanh(\beta t)(1 - \tanh(\beta t)^2)}.$$

Conclusions

We have studied the SFT inspired nonlocal models with quadratic potentials and obtained:

Local and non-local Einstein equations have one and the same solutions.

Nonlocality arises in the case of $\mathcal{F}(\Box_g)$ with an infinite number of roots.

One system of non-local Einstein equations $\Leftrightarrow$ Infinity number of systems of local Einstein equations.

In the Friedmann–Robertson–Walker metric the proposed method for the search of exact solutions for field equation allows to get in quadratures solutions, which depend on two arbitrary parameters. Exact solutions have been found for a cubic potential.