# Grassmanians and $\mathcal{N} = 4$ SYM

## Dual theory for the S-matrix

Jaroslav Trnka[†]

[†]Department of Physics, Princeton University

*Based on collaboration with*

**Nima Arkani-Hamed, Freddy Cachazo and Jacob Bourjaily**

N. Arkani-Hamed, F. Cachazo, C. Cheung, J. Kaplan 0903.2110, 0907.5418, 0909.0483

N. Arkani-Hamed, J. Bourjaily, F. Cachazo, J. T. 0912.4912, 0912.5289

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Also other people working in the same field:

A. Hodges, L. Mason, D. Skinner, L. Dolan, P. Goddard, M. Spradlin, A. Volovich, C. Wen, D. Nandan, J. Drummond, J. Henn, L. Ferro, G. Korchemsky, E. Sokatchev,...

Shortcuts for pictures taken from talks by: FC - F. Cachazo, JB - J. Bourjaily, CC - C. Cheung

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- no reference to underlying space-time
- exchange manifest locality for simplicity of the S-matrix
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Why we choose $\mathcal{N}=4$ SYM as a playground?

- "simplest" quantum field theory
- best choice for testing new ideas
- tree level in $\mathcal{N}=4$ SYM is identical to tree level in QCD

# Preliminaries

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Spinor-helicity formalism

- for massless particles we can rewrite $p_\alpha$ in terms of spinors $\lambda_a$, $\tilde{\lambda}_{\dot{a}}$

$$p_\alpha = \sigma_\alpha^{a\dot{a}} \lambda_a \tilde{\lambda}_{\dot{a}}$$

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- scalar products

$$(p_1 - p_2)^2 = \langle 12\rangle[12] \qquad \langle 12\rangle = \epsilon^{ab}\lambda_{1a}\lambda_{2b}, \qquad [12] = \epsilon^{\dot{a}\dot{b}}\tilde{\lambda}_{1\dot{a}}\tilde{\lambda}_{2\dot{b}}$$

- mixed product:

$$\langle i|P|j] = \sum_{k\in P}\langle ik\rangle[kj] \qquad e.g. \quad \langle 1|2+3|4] = \langle 12\rangle[24] + \langle 13\rangle[34]$$

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- they are covariant under the action of little group

$$\lambda \to t\lambda, \qquad \tilde{\lambda} \to t^{-1}\tilde{\lambda}$$

  leaving $p_\alpha$ invariant.

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Color ordering Berends, Giele, Mangano, Parke, Xu

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Maximally-Helicity-Violating amplitudes, $k = 2$ Parke, Taylor [1985]

- closed simple form for tree level amplitude: Parke-Taylor formula

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Let us denote the number of negative helicities $k$.

- $\mathcal{M}(n, k)$ and $\mathcal{M}(n, n-k)$ are related by parity.
- we denote $N^{k-2}$MHV amplitude $\mathcal{M}(n, k)$

# Localization in twistor space

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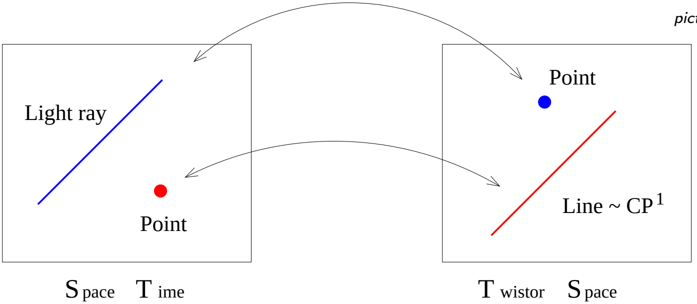
Twistor space: Penrose [1960s]

*picture from FC*



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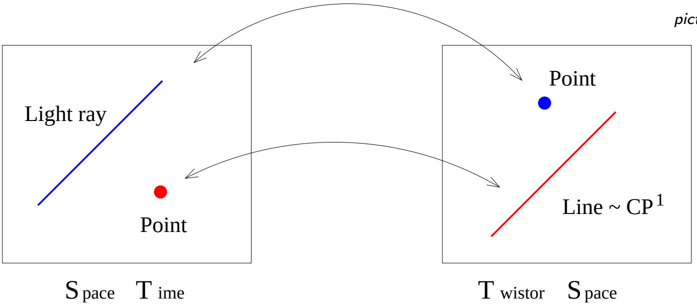


Twistor variable $W$ lives in $\mathbb{C}P^3$, supersymmetric analogue $\mathcal{W}$ in $\mathbb{C}P^{3|4}$.

$$W = \begin{pmatrix} \mu \\ \lambda \end{pmatrix} \qquad \mathcal{W} = \begin{pmatrix} \tilde{\mu} \\ \tilde{\lambda} \\ \tilde{\eta} \end{pmatrix}$$

# Localization in twistor space

Connection to usual momentum space

$$\mathcal{M}(\mathcal{W}_a) = \int d^2\lambda_a e^{\tilde{\mu}_a\lambda_a}\mathcal{M}(\lambda_a, \tilde{\lambda}_a, \tilde{\eta}_a)$$

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Tree level amplitude in twistor space

Witten [2003], Roiban, Spradlin, Volovich [2004]

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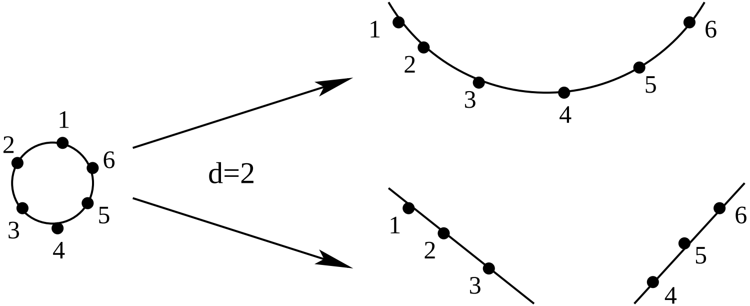

# CSW expansion

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- amplitude is a sum over curves:

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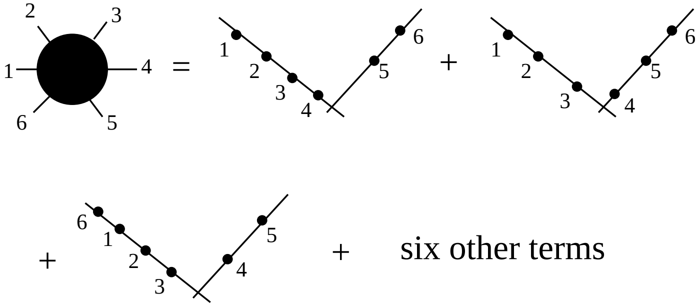


Degree of the map is determined by $k$.

# BCFW recursion relations

Britto, Cachazo, Feng [2004], Britto, Cachazo, Feng, Witten [2005]

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BCFW shift:

$$\lambda_i \to \lambda_i + z\lambda_j, \qquad \tilde{\lambda}_j \to \lambda_j - z\lambda_i$$

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- then the amplitude $\mathcal{M}(z)$ becomes complex
- if the amplitude vanishes for $z \to \infty$ we can use Cauchy's theorem

$$\mathcal{M}(0) = \oint \frac{dz}{z}\mathcal{M}(z) = \sum_{z_P} \frac{\mathcal{M}(z_P)}{z_P}$$

where $z_P$ are value of $z$ at poles, $p^2(z_P) = 0$

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Adjacent shift: $j = i + 1 \quad \to \quad$ minimize the number of diagrams

Non-adjacent shift: connection to gravity?

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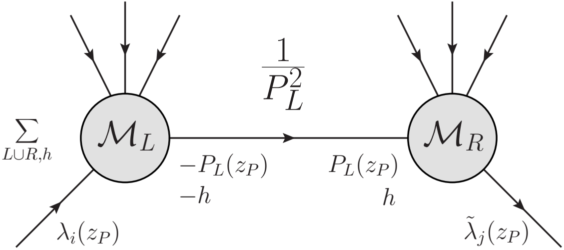

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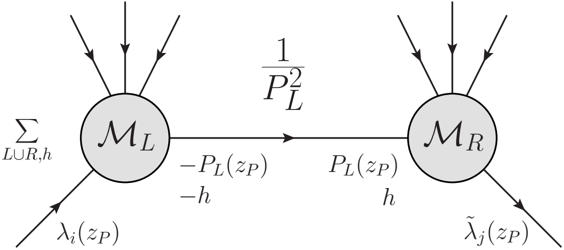


The amplitude is a sum over factorization channels

$$\mathcal{M} = \sum_{L,h} \mathcal{M}_L(z_P, h) \frac{1}{P_L^2} \mathcal{M}_R(z_P, -h)$$

where the sub-amplitudes $\mathcal{M}_L$ and $\mathcal{M}_R$ are evaluated at $z = z_P$ while the denominator is at $z = 0$

# BCFW recursion relations

Britto, Cachazo, Feng [2004], Britto, Cachazo, Feng, Witten [2005]

Example for 6pt NMHV $(k=3)$:

$$\mathcal{M}(+-+-+-) = \frac{[13]^4\langle 46\rangle^4}{[12][23]\langle 45\rangle\langle 56\rangle\langle 6|1+2|3]\,\langle 4|2+3|1]\,s_{123}}$$

$+$ 2 other cyclically related terms

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$+$ 2 other cyclically related terms

Spurious poles $\langle 6|1+2|3]$ and $\langle 4|2+3|1]$ cancel.

Much more compact form than in terms of Feynman diagrams and computationally much faster!

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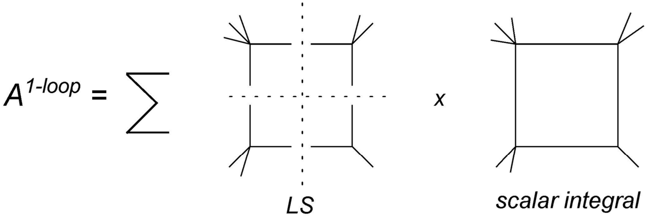

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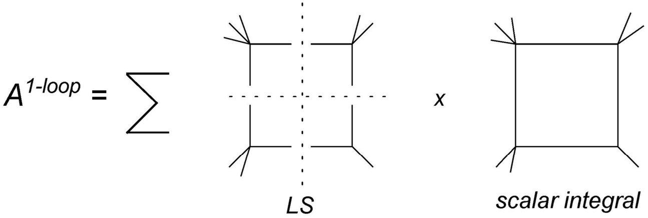
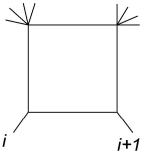



Relation to BCFW expansion: sum over leading singularities of one-loop graphs. For adjacent shifts (particles $i$, $i+1$), sum over following graphs:



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Grassmanian $G(k,n)$: space ok $k$-planes containing origin in $\mathbb{C}^n$

$$\begin{pmatrix} c_{11} & c_{12} & \ldots & c_{1k} & c_{1k+1} & \ldots & c_{1n-1} & c_{1n} \\ c_{21} & c_{22} & \ldots & c_{2k} & c_{2k+1} & \ldots & c_{2n-1} & c_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ c_{k1} & c_{k2} & \ldots & c_{kk} & c_{kk+1} & \ldots & c_{kn-1} & c_{kn} \end{pmatrix}$$

GL(k) invariant

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Momentum conservation revisited:

$$\delta^4\left(\sum_{a=1}^{n} p_a^{\alpha}\right) \to \delta^4\left(\sum_{a=1}^{n}\lambda_a^{\alpha}\lambda_a^{\dot\alpha}\right) \to \delta^4(\lambda\cdot\tilde\lambda)$$

where we can think about $\lambda$, $\tilde\lambda$ as 2-planes in $n$-dimensions

$$\lambda = \begin{pmatrix} \lambda_1^1 & \lambda_2^1 & \ldots & \lambda_n^1 \\ \lambda_1^2 & \lambda_2^2 & \ldots & \lambda_n^2 \end{pmatrix} \qquad \tilde\lambda = \begin{pmatrix} \lambda_1^{\dot 1} & \lambda_2^{\dot 1} & \ldots & \lambda_n^{\dot 1} \\ \lambda_1^{\dot 2} & \lambda_2^{\dot 2} & \ldots & \lambda_n^{\dot 2} \end{pmatrix}$$

that are orthogonal.

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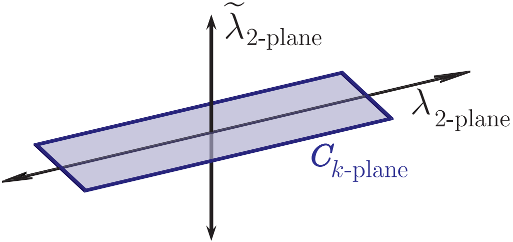
We consider a $C$-plane that is orthogonal to $\tilde{\lambda}$-plane and contains $\lambda$-plane.

*picture from JB*



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The delta function exactly does this job:

$$\prod_{\alpha=1}^{k} \delta^{4|4}(C_{\alpha a}\mathcal{W}_a) \rightarrow \prod_{\alpha=1}^{k} \delta^{0|4}(C_{\alpha a}\tilde{\eta}_a)\cdot\delta^2(C_{\alpha a}\tilde{\lambda}_a)\cdot\int d^2\rho_\alpha \delta^2(\lambda_a - \rho_\beta C_{\beta a})$$

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In fact, $C$ plane is orthogonal to full $2|4$-plane.

Now we integrate over all these $C$ planes with a natural cyclic measure of minors

$$(12\dots k) = \begin{vmatrix} c_{11} & \dots & c_{1k} \\ \vdots & \vdots & \vdots \\ c_{k1} & \dots & c_{kk} \end{vmatrix}$$

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picking some particular helicity configuration of external gluons

$$\prod_{\alpha=1}^{k}\delta^4(C_{\alpha a}W_a) \to \prod_{\alpha=1}^{k}\delta^2(C_{\alpha a}\tilde{\lambda}_a)\cdot\int d^2\rho_\alpha\delta^2(\lambda_a-\rho_\beta C_{\beta a})$$

we get the result for QCD!

$C$-plane is orthogonal just to 2-plane $\tilde{\lambda}$.

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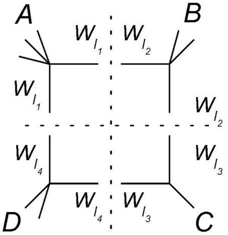
Leading singularities in twistor space Mason, Skinner [2009], Kaplan [2009] :



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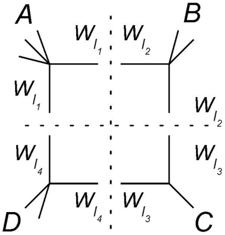


just corresponds to

$$\int \mathcal{D}^{3|4}\mathcal{W}_{I_1}\ldots\mathcal{D}^{3|4}\mathcal{W}_{I_4}\,\mathcal{M}_A(\mathcal{W}_{I_1},\mathcal{W}_{I_2},\ldots)\ldots\mathcal{M}_D(\mathcal{W}_{I_4},\mathcal{W}_{I_1},\ldots)$$

$\rightarrow$ leading singularities are manifestly superconformal invariant!

# Grassmanian proposal

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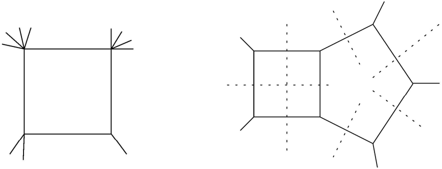
The conjecture is that

- all leading singularities of all loop graphs are residues of this integrals



- there exists a contour which gives a tree level amplitude
- locality is guaranteed by the residue theorem

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Counting:

- we start with $n \cdot k$ variables in $C_{\alpha a}$
- $GL(k)$ removes $k^2$ degrees of freedom
- delta functions remove $2n-4$ variables
- we are left with $(k-2)(n-k-2)$ variables and $\mathcal{L}_{n,k}$ can be interpreted as multi-dimensional contour integral in $\mathbb{C}^{(k-2)(n-k-2)}$

We also see that that there is no solution for constraints for $k = 0, 1, n-1, n$ which makes perfect sense!

# Grassmanian proposal

Arkani-Hamed, Cachazo, Cheung, Kaplan [2009]

$$\mathcal{L}_{n,k} = \frac{1}{\mathrm{vol[GL(k)]}} \int \frac{d^{k\times n}C_{\alpha a} \prod_{\alpha=1}^{k} \delta^{4|4}(C_{\alpha a}\mathcal{W}_a)}{(12\dots k)(23\dots k-1)\dots(n\dots k-1)}$$

Counting:

- we start with $n \cdot k$ variables in $C_{\alpha a}$
- $GL(k)$ removes $k^2$ degrees of freedom
- delta functions remove $2n-4$ variables
- we are left with $(k-2)(n-k-2)$ variables and $\mathcal{L}_{n,k}$ can be interpreted as multi-dimensional contour integral in $\mathbb{C}^{(k-2)(n-k-2)}$

We also see that that there is no solution for constraints for $k = 0, 1, n-1, n$ which makes perfect sense!

MHV amplitude: no integration, the result is determined just by Jacobian

# Example: 6pt NMHV

For $n = 6$, $k = 3$ we have just one integral left. Therefore, $\mathcal{L}_{n,k}$ is a 1-dimensional contour integral.

$$\mathcal{L}_{6,3} = \int \frac{(i_1 i_2 i_3)^4 \, d\tau}{(123)(\tau)(234)(\tau)(345)(\tau)(456)(\tau)(561)(\tau)(612)(\tau)}$$

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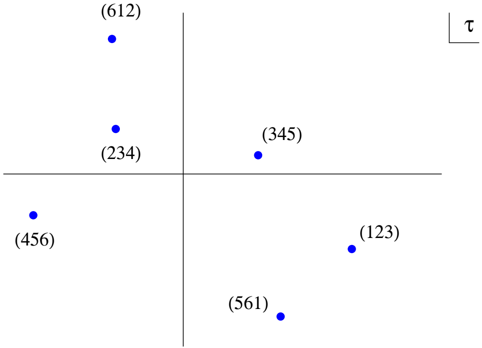
There are six poles in the complex plane

*picture from FC*



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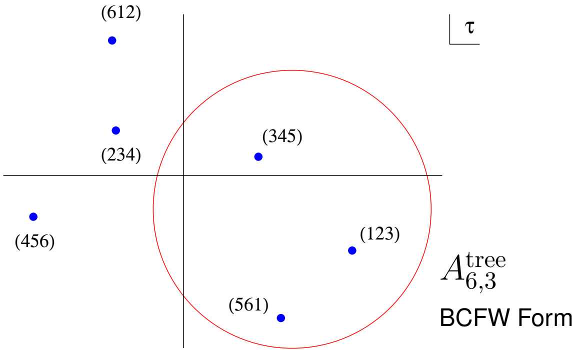
Tree level contour (BCFW form)

*picture from FC*



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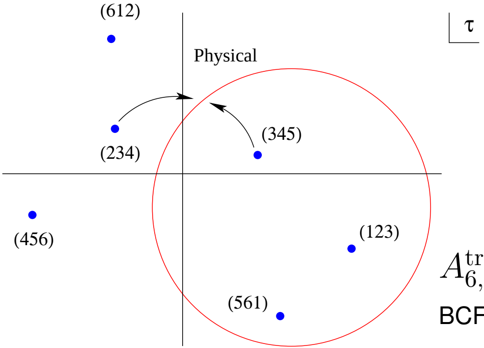


$A_{6,3}^{\text{tree}}$

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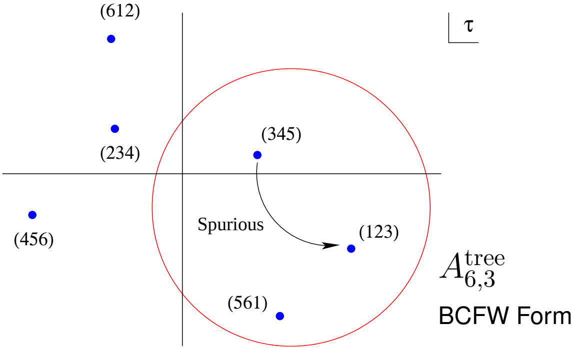
Spurious poles

*picture from FC*



# Dual Super-Conformal invariance

We can introduce $\mathcal{Z}_i$ momentum twistors associated with momenta $x_i = p_i - p_{i-1}$ in dual space. Hodges [2009]

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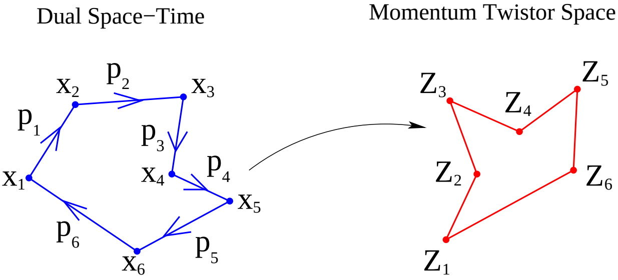


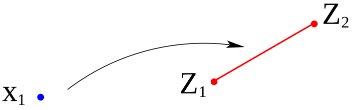

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$$\mathcal{L}_{n,k} \sim \frac{1}{\mathrm{vol}[\mathrm{GL(k-2)}]} \int \frac{d^{(k-2)\times n} D_{\alpha a} \prod_{\alpha=1}^{k-2} \delta^{4|4}(D_{\alpha a}\mathcal{Z}_a)}{(12\ldots k-2)(23\ldots k-1)\ldots(n\ldots k-3)}$$

is Grassmanian $G(k-2,n)$ which is invariant under second (dual) conformal symmetry (acting on $x_i = p_i - p_{i-1}$).

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is Grassmanian $G(k-2,n)$ which is invariant under second (dual) conformal symmetry (acting on $x_i = p_i - p_{i-1}$).

In fact, all residues are then invariant under both conformal and dual conformal symmetries → Yangian invariant!

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It was proven that

$$\int d^{k\times n}C\, f(C)\, \delta^{4|4}(C\cdot\mathcal{W})$$

under the action of Yangian generators, transforms into total derivative only if

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There is also a claim that every super-conformal invariant is of this form. If this is true, then $\mathcal{L}(n,k)$ is the unique way how to write Yangian invariants!

# Relation to twistor strings

Arkani-Hamed, Bourjaily, Cachazo, JT [2009]

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We want to write the integral in the form (for NMHV)

$$A = \int d^{(n-5)}\tau \frac{h(\tau)}{f_1(\tau)\dots f_{n-5}(\tau)}$$

where

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Construction of this map for NMHV, (also Nandan, Volovich, Wen [2009])
e.g. for 7pt we have

$$A_7 = \int_{f_6=f_7=0} d\tau_1 d\tau_2 \frac{(612)(235)}{(671)(123)(345)} \cdot \frac{1}{[(234)(456)(612)]} \frac{1}{[(235)(567)(712)]}$$

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This is also a connection with Witten's twistor string theory.

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**Unification**

# Local physics from Grassmanian

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- Finally, all residues cancel in pairs except nine which are exactly CSW diagrams!
- CSW diagrams are generated from Lagrangian in the light-cone gauge $\rightarrow$ space-time is reintroduced!

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In fact, we rewrote each (non-local) residue as a sum of

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**Emergent space-time**