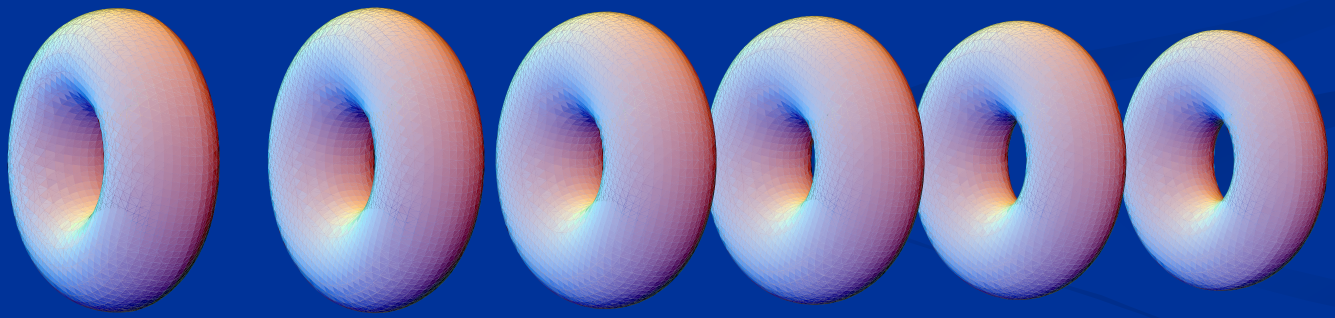
Durham University

Monopoles and Skyrmions

Ya. Shnir



Quarks-2010

Коломна, Россия, 8 Июня 2010

Outline

- Abelian and non-Abelian magnetic monopoles
  - BPS monopoles
  - Rational maps and construction of multimonopoles
  - SU(2) axially symmetric multimonopoles
  - Monopole-antimonopole chains
  - Multimonopole moduli space
- Skyrmions
  - Skyrme model
  - Construction of multiskyrmions
  - Axially symmetric multiskyrmions
  - Rational maps and multiskyrmions
  - Skyrmion-antiSkyrmion chains
  - Multiskyrmions moduli space
- Summary and outlook

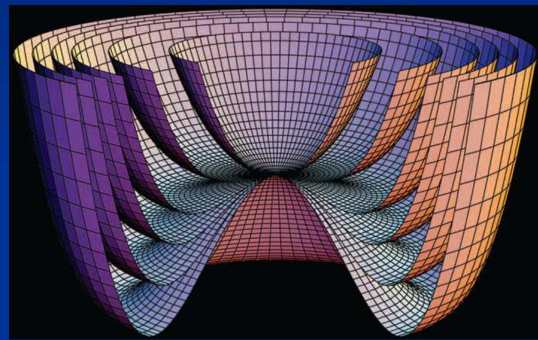
Yang-Mills-Higgs Theory

$$S = \frac{1}{2}\int d^4x \, \{ \, F_{\mu\nu}F^{\mu\nu} + \ (D_\mu\Phi)(D^\mu\Phi) - V(\Phi)\}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ie[A_\mu, A_\nu]$$

$$D_\mu\Phi = \partial_\mu\Phi + ie[A_\mu, \Phi]$$

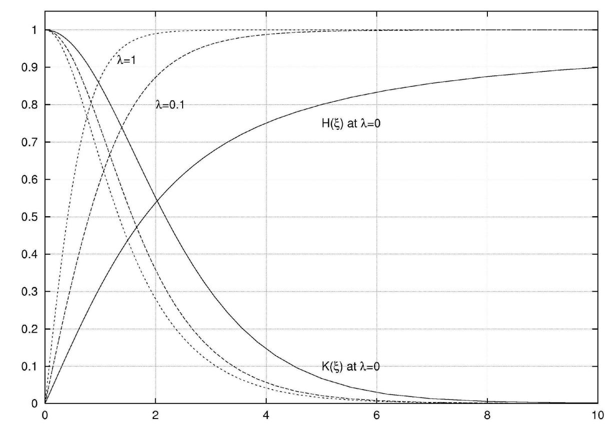
$$V(\Phi) = \lambda \, (\Phi^2 - \eta^2)^2$$



´t Hooft-Polyakov static spherically symmetric solution

$$\phi^a = \frac{r^a}{er^2}H(e\eta r); \qquad A_n^a = \varepsilon_{amn}\frac{r^m}{er^2}(1 - K(e\eta r))$$

$R_c \sim m_v^{-1}$



BPS monopoles

Bogomolny equations: $\lambda = 0$, $B_k = D_k \Phi$

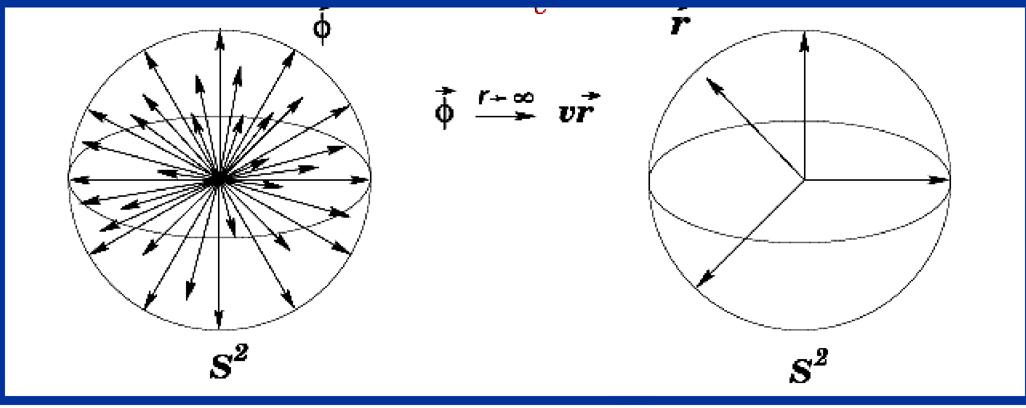
BPS monopole mass: $M = 4\pi\eta/e$ ***Homotopy group*** $\pi_2(S^2)$

- **Long-range scalar field** $\Phi \sim 1/r$
- **No net interaction between the BPS monopoles**
- **Analytical solution of the BPS equations:**

$$K = \frac{\xi}{\sinh \xi}; \quad H = \xi \coth \xi - 1$$

Sir M. Atiyah, R. Ward (1977),
P. Forgacs *et al* (1981),
W. Nahm (1982),
P. Sutcliffe (1996) *and other*

Magnetic charge of a monopole is a topological number $\Phi : S^2 \rightarrow S^2$



Self-dual monopoles vs non-self dual monopoles

BPS monopoles:

- In the limit η=0 the energy becomes:

$$E = \mathrm{Tr} \int d^3x \, \left\{ \frac{1}{4}(\varepsilon_{ijk}F_{ij} \pm D_i\Phi)^2 \mp \frac{1}{2}\varepsilon_{ijk}F_{ij}D_k\Phi \right\}$$

- The first order Bogomol'nyi equations $B_k = \pm D_k\Phi$ yield absolute minimum:

$$M = 4\pi g$$

- No net interaction between the BPS monopoles: the electromagnetic repulsion is compensated by the long-range scalar interaction.

Non self-dual monopoles:

- They are solutions of the second order Yang-Mills equations: $\partial_\mu F_{\mu\nu} = 0$
- $E > M_{BPS}$ even if g=0 (deformations of the topologically trivial sector)
- The constituents are non BPS monopoles and/or vortices in a static equilibrium; separation is relatively small, there are no long-range forces

Rational map monopoles

There is a transformation of a monopole into a rational map from the Riemanian sphere to inself: *R: $S^2 \mapsto S^2$* (P. Sutcliffe, N.Manton et al)

$$R(z) = \frac{a(z)}{b(z)} = \frac{a_1 z^{n-1} + \cdots + a_n}{z^n + b_1 z^{n-1} + \cdots + b_n}, \qquad z = x_1 + i x_2$$

Construction of the rational maps monopoles:

- Represent BPS equation in spherical coordinates $r, z, \bar{z}$
- Impose a complex gauge $\Phi = -iA_r = \frac{i}{2} U^{-1}\partial_r U, \quad A_z = U^{-1}\partial_z U, \quad A_{\bar{z}} = 0$
- Construct the monopoles using

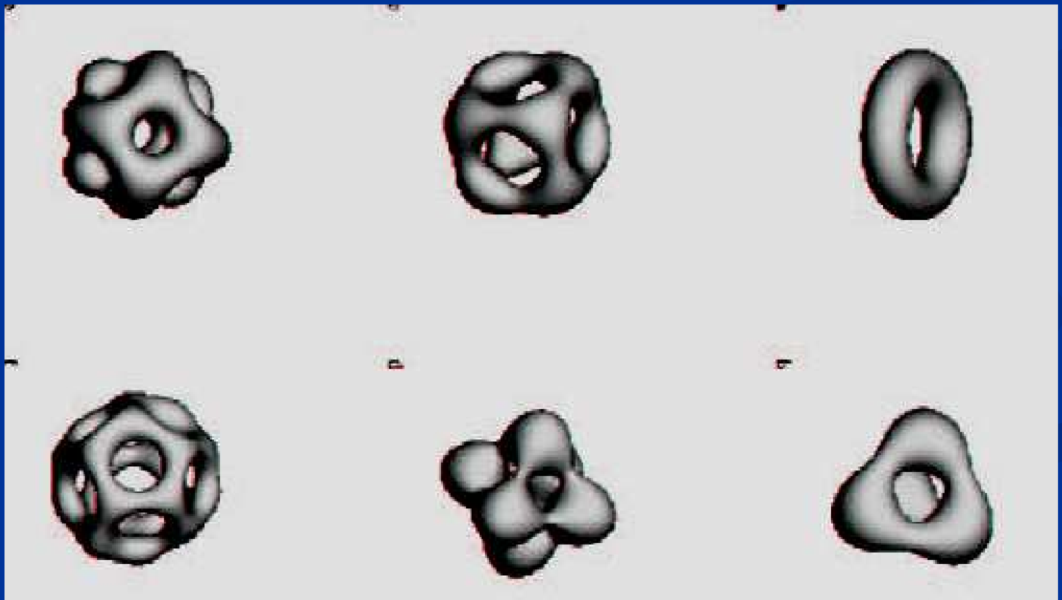
$$U \sim \exp\left\{ \frac{2r}{1+|R|^2} \begin{pmatrix} |R|^2 - 1 & 2\bar{R} \\ 2R & 1 - |R|^2 \end{pmatrix} \right\}$$

R = 1/z: **one spherically symmetric monopole centered at the origin;**

$$R(z) = \frac{a_1 z + a_2}{z^2 + b_1 z + b_2}$$: **two monopoles**

$$R(z) = \frac{i\sqrt{3}z^2 - 1}{z(z^2 - i\sqrt{3})}$$: **Tetrahedral monopoles (degree 3 map)**

$$R(z) = \frac{z^4 + 2i\sqrt{3}z^2 + 1}{z^4 - 2iz^2\sqrt{3} + 1}$$: **Octahedral monopoles (degree 4 map)**



Numerical Technique

- Energy minimization approach (static solutions in 2d and in 3d) – supercomputers and grid systems

- Solution of system of PDE's obtained by imposing some symmetry conditions, boundary problem in in 2d and in 3d: Newton-Raphson iterative procedure (FIDISOL/CADSOL package); boundary problem in d1 (COLSYS package)

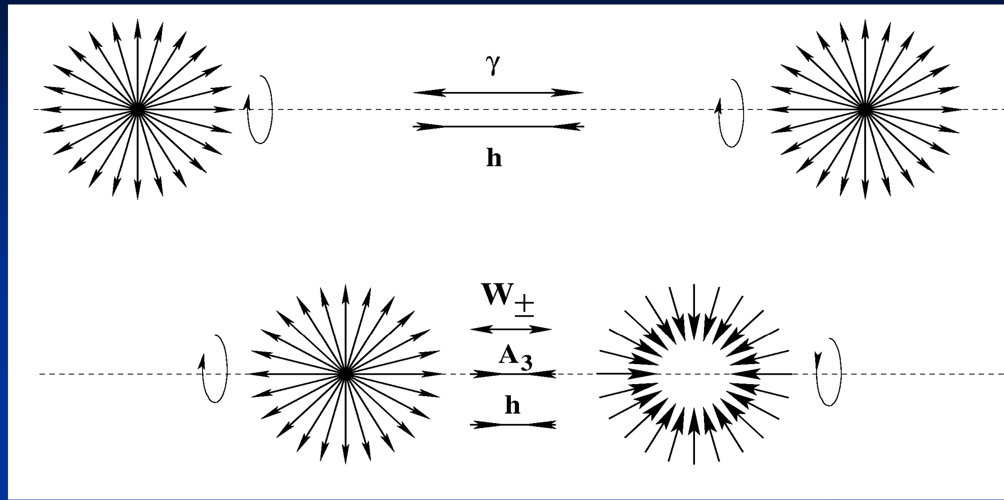
- Gradient flow method

- Dynamics of the solitons: (pseudo) spectral methods, symplectic methods.

Non-BPS axially symmetric monopoles

MA pair: magnetic dipole

(Taubes, Nahm, Rüber, Kleihaus,Kunz & Shnir)



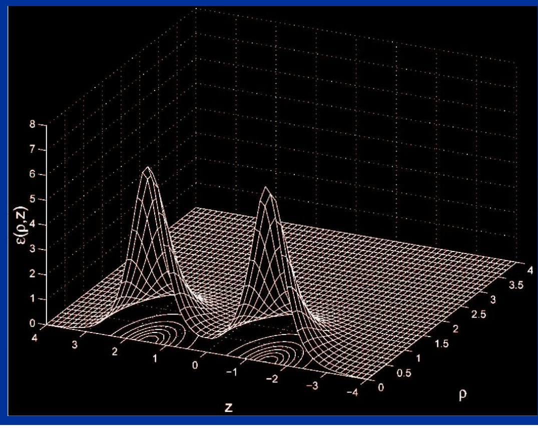
$$A_\mu dx^\mu = \left(\frac{K_1}{r}dr + (1-K_2)d\theta\right)\frac{\tau_\varphi^{(n)}}{2e} - n\sin\theta\left(K_3\frac{\tau_r^{(n,m)}}{2e} + (1-K_4)\frac{\tau_\theta^{(n,m)}}{2e}\right)d\varphi;$$

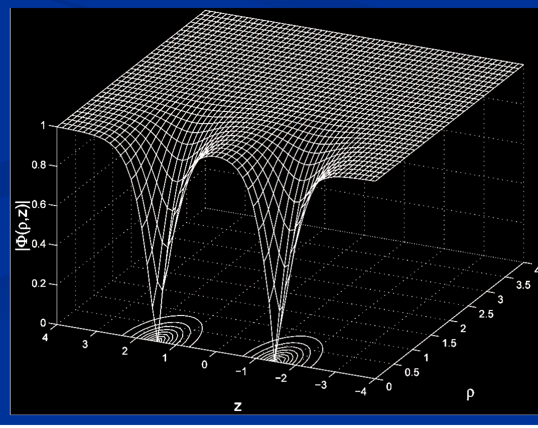
$$\Phi = \Phi^a\frac{\tau^a}{2} = ´\left(H_1\frac{\tau_r^{(n,m)}}{2} + H_2\frac{\tau_\theta^{(n,m)}}{2}\right) .$$

Magnetic charge:

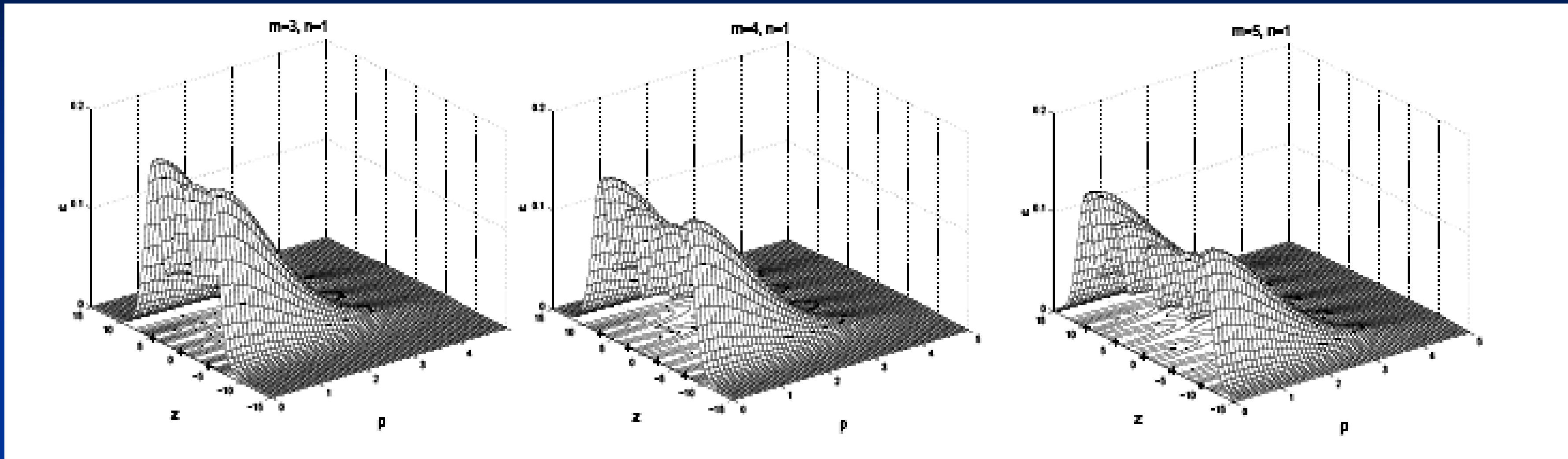
$$Q = \frac{1}{2\pi}\int_{S_2}(\Phi d\Phi \wedge d\Phi)$$

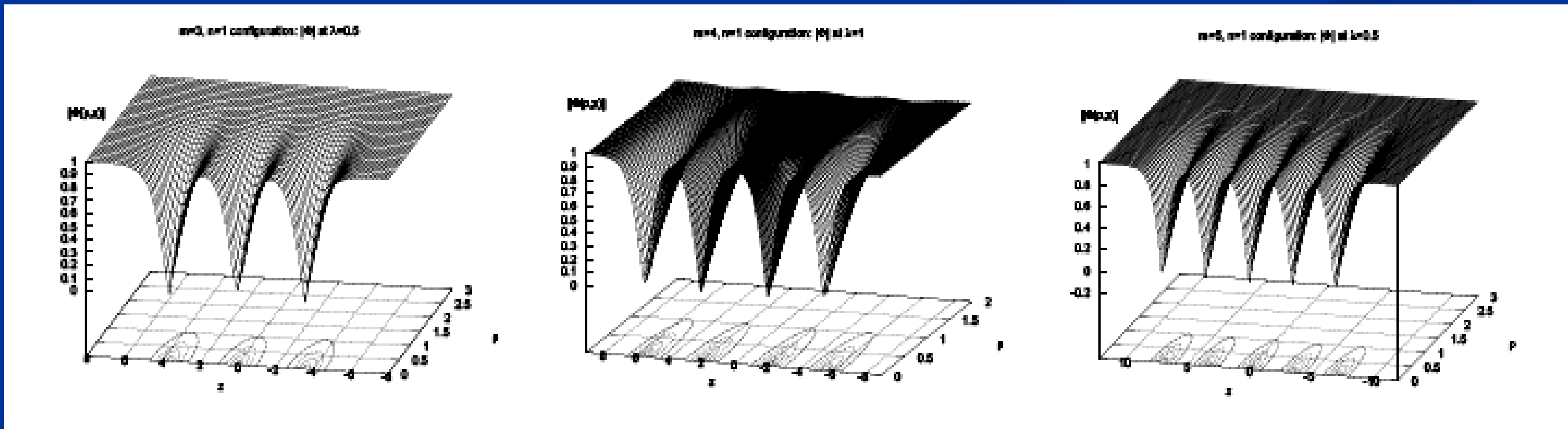
$$= \frac{1}{2}n\left[1-(-1)^m\right]$$





Monopole-antimonopole chains





Effective electromagnetic interaction

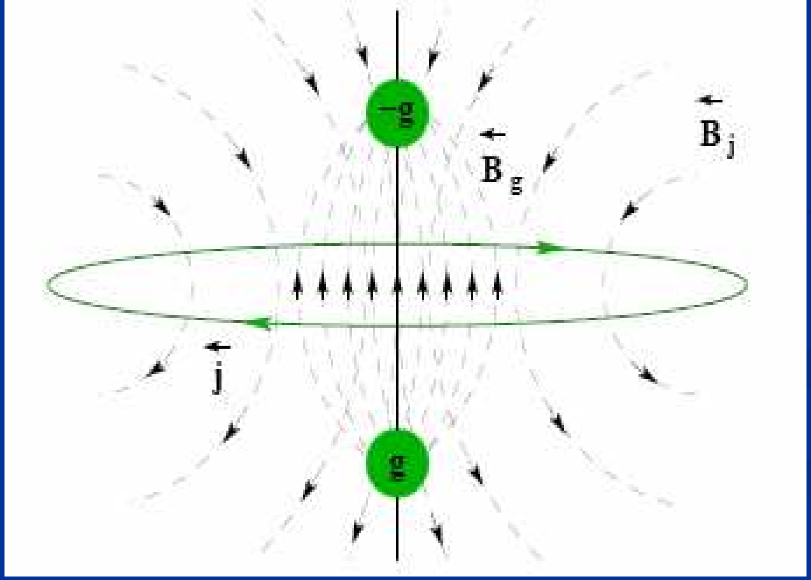
`t Hooft tensor: $$\mathcal{F}_{\mu\nu} = \left\{ \hat{\Phi} F_{\mu\nu} - \frac{i}{2e} \hat{\Phi} D_\mu \hat{\Phi} D_\nu \hat{\Phi} \right\}$$

The electric and magnetic (topological) currents:

$$\partial^\mu \mathcal{F}_{\mu\nu} = 4\pi j^{el}_\nu; \qquad \partial^{\mu *} \mathcal{F}_{\mu\nu} = 4\pi j^{mag}_\nu$$

The magnetic field if generated by the magnetic charges and electric currents

$$j^{el}_r = j^{el}_\theta = 0; \qquad 4\pi j^{el}_\varphi = \partial^2_r \mathcal{A}_\varphi + \sin\theta \ \partial_\theta \frac{1}{r^2 \sin\theta} \partial_\theta \mathcal{A}_\varphi$$



Effective electromagnetic interaction

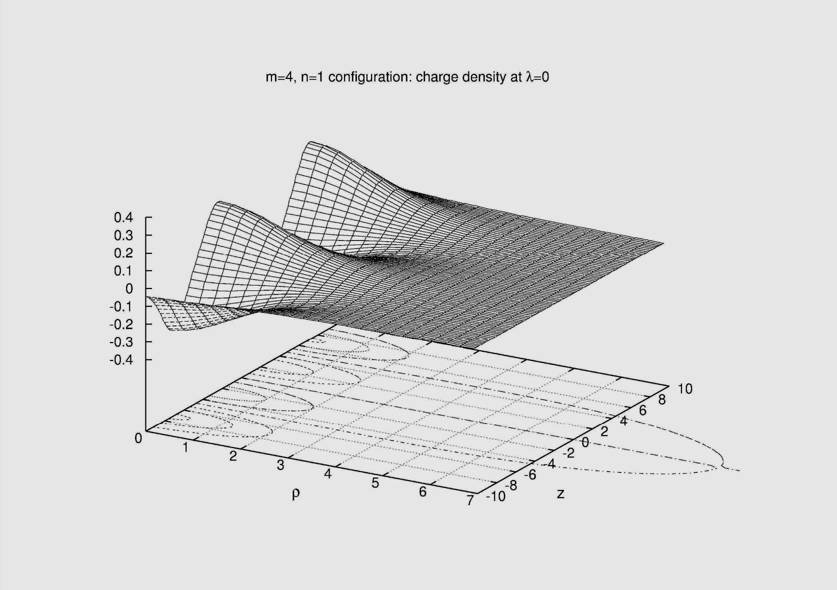
`t Hooft tensor: $$\mathcal{F}_{\mu\nu} = \left\{ \hat{\Phi} F_{\mu\nu} - \frac{i}{2e} \hat{\Phi} D_\mu \hat{\Phi} D_\nu \hat{\Phi} \right\}$$

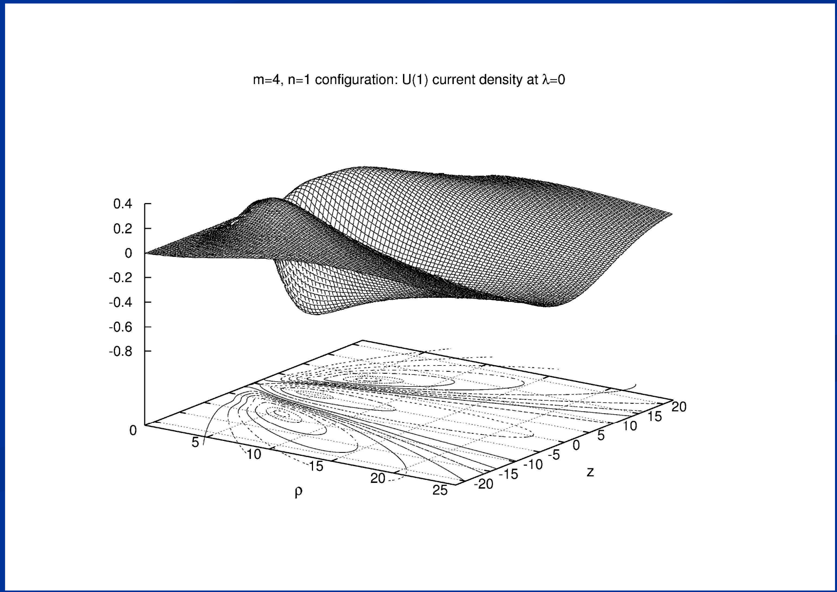
The electric and magnetic (topological) currents:

$$\partial^\mu \mathcal{F}_{\mu\nu} = 4\pi j_\nu^{el}; \qquad \partial^{\mu *} \mathcal{F}_{\mu\nu} = 4\pi j_\nu^{mag}$$

The magnetic field if generated by the magnetic charges and electric currents

$$j_r^{el} = j_\theta^{el} = 0; \qquad 4\pi j_\varphi^{el} = \partial_r^2 \mathcal{A}_\varphi + \sin\theta \ \partial_\theta \frac{1}{r^2 \sin\theta} \partial_\theta \mathcal{A}_\varphi$$





Skyrmions

Skyrme field $U \in \mathbf{SU(2)}; \quad R_\mu = (\partial_\mu U)U^{-1}$

$$L = -\int d^3x \left\{ \frac{F_\pi^2}{16}\,\mathrm{Tr}(R_\mu R^\mu) - \frac{1}{32e^2}\,\mathrm{Tr}([R_\mu, R_\nu][R^\mu, R^\nu]) + \frac{m_\pi^2 F_\pi^2}{8}\,\mathrm{Tr}(1-U) \right\}$$

Nonlinear pion theory: $U = \phi_0 + i\sigma_k \cdot \phi_k, \quad \phi_0^2 + \phi_k^2 = 1$

Baryon charge is identified with the topological number $U: \; S^3 \to S^3$

$$B = \frac{1}{24\pi^2}\int d^3x \varepsilon_{ijk}\,\mathrm{Tr}(R_i R_j R_k)$$

Homotopy group $\pi_3(S^3)$

B= 1: **Spherically symmetric skyrmion** $U = \exp\{if(r)\hat{\mathbf{n}}\cdot\sigma\}$

$$\hat{\mathbf{n}} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$$

Boundary conditions:

$f(r)$, with $f(0) = \pi k$, and $f(\infty) = 0$.

The approximation : $f(r) = 4\arctan[\exp(r)]$

Skyrmion's interaction

There is no self-dual skyrmions: E ⩾ *B*

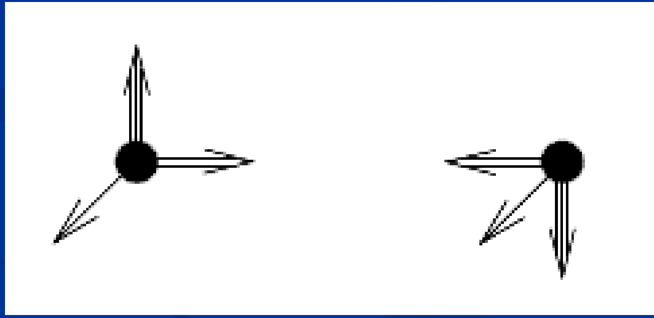
A single Skyrmion is approximated by a triplet of orthogonal dipoles

Field equations: $\partial_\mu \left(R^\mu + \frac{1}{4}[R^\nu,[R_\nu R^\mu]]\right) = 0$

Asymptotically $\phi_k \to \frac{\mathbf{d}_k \cdot \mathbf{r}_k}{4\pi r^3} + O(r^{-3}); \quad \phi_0 \to 1$

The dipole-dipole interaction energy $E_{int} = \frac{2d^2}{3\pi R^3}(\cos\alpha - 1)[1 - 3(\vec{\mathbf{R}}\cdot\vec{\mathbf{n}})]$

Attractive channel: $\tilde{\mathbf{R}} \perp \tilde{\mathbf{n}}$



There are 6 zero modes of the B=1 Skyrmion:
3 translations + 3 rotations

Translations: $U \to iU^{-\frac{1}{2}}(\partial_k U)U^\dagger U^{\frac{1}{2}}\left(\int d^3x\ \mathbf{Tr}(\partial_i U \partial_i U^\dagger)\right)^{-\frac{1}{2}}$

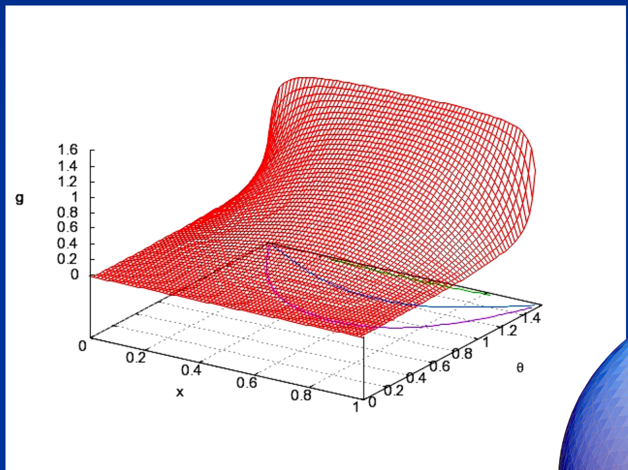
Rotations (Adkins, Nappi & Witten (1983)); $U \to A(t)UA(t)^\dagger$
rigid body approximation

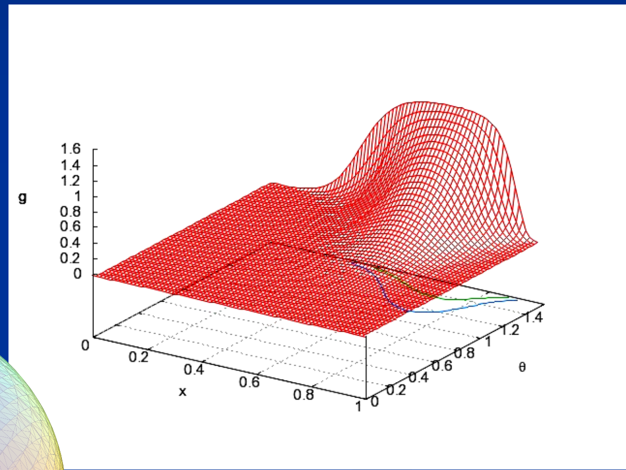
Axially symmetric multiSkyrmions

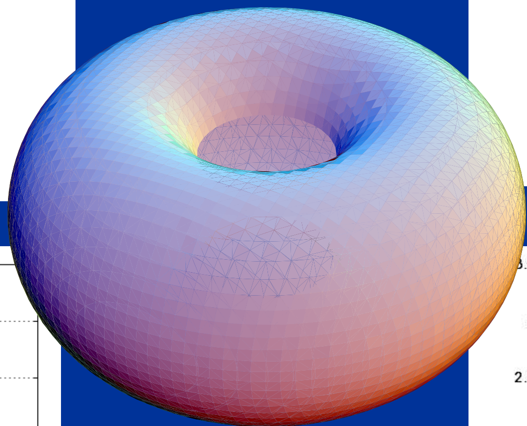
Trigonometric parametrisation:

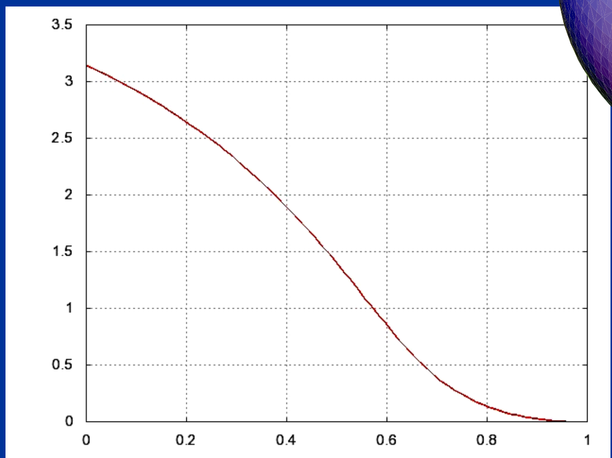
$$\phi^\alpha = \sin f \sin g \, n^\alpha; \quad \phi^3 = \sin f \cos g; \quad \phi^4 = \cos f; \quad n^\alpha = (\cos n\varphi, \sin n\varphi)$$

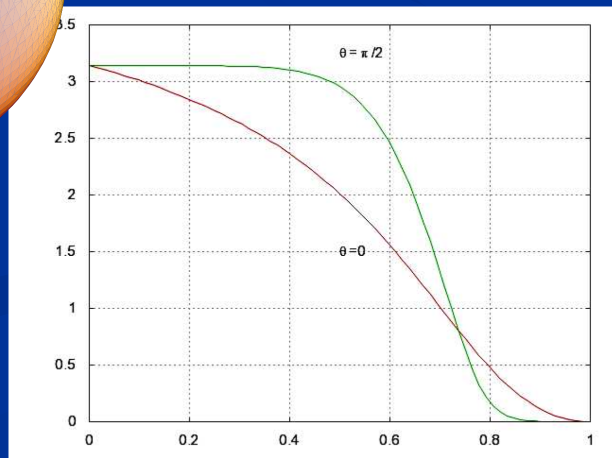
B= 2:











Rational map Skyrmions

(N. S. Manton, C. Houghton and P. Sutcliffe)

Stereographic projection:

$z = \tan(\theta/2)e^{i\varphi}$

$\hat{\mathbf{n}}_z = \frac{1}{1+|z|^2}(z + z^*,\ i(z^* - z),\ 1 - |z|^2)$

$\hat{\mathbf{n}}_R :\ S^2 \to S^2$

$U = \exp\{if(r)\hat{\mathbf{n}}_R \cdot \sigma\}$

$\hat{\mathbf{n}}_R = \frac{1}{1+|R|^2}(R + R^*,\ i(R^* - R),\ 1 - |R|^2)$

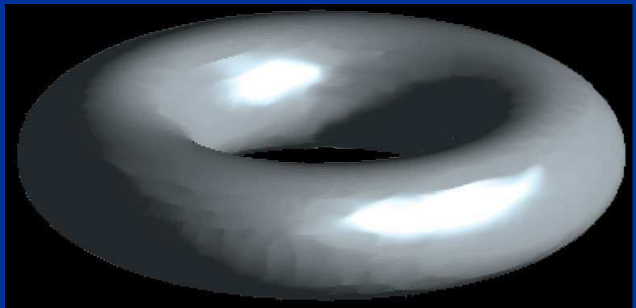
The holomorphic map of degree B: $R = a(z)/b(z)$

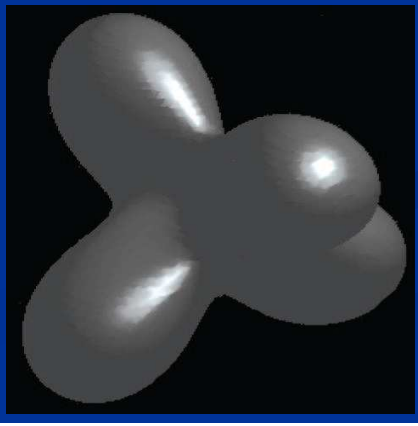
B= 4: $R(z) = \frac{z^4 + 2i\sqrt{3}z^2 + 1}{z^4 - 2i\sqrt{3}z^2 + 1}$

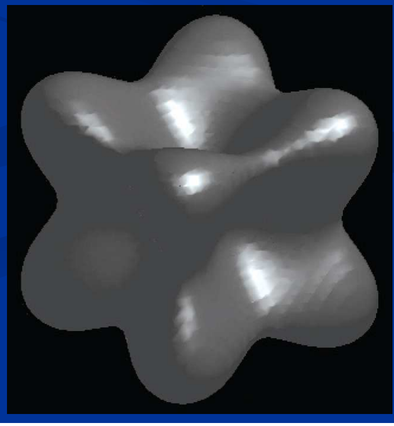
B= 7: $R(z) = \frac{z^7 - 7z^5 - 7z^2 - 1}{z^7 + 7z^5 - 7z^2 + 1}$

(Octahedral Skyrmions)

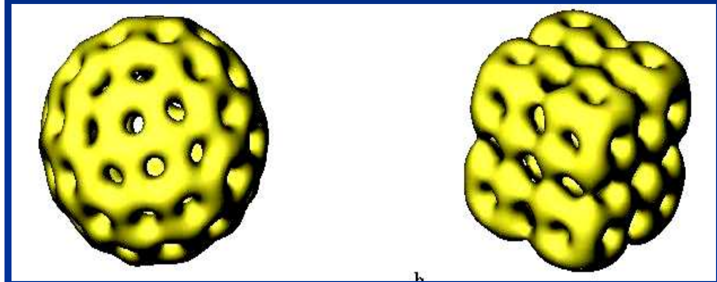
(Icosahedral Skyrmions)



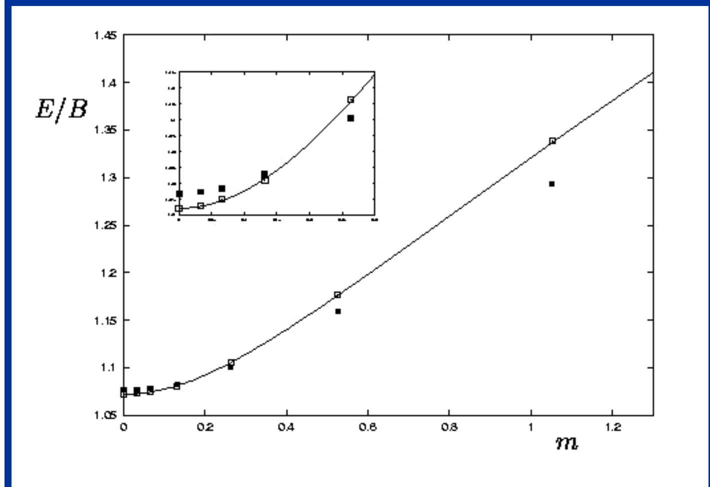




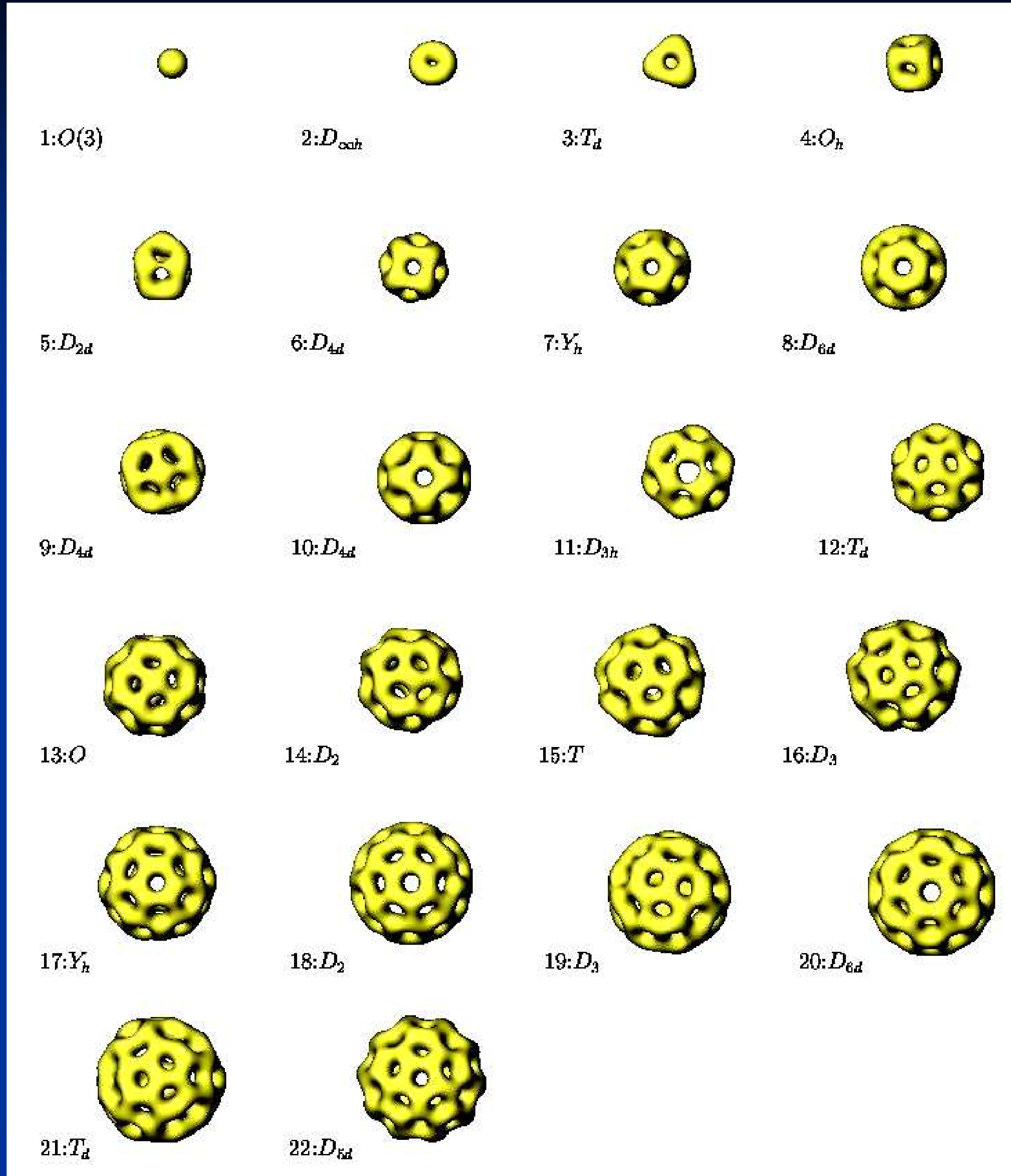
R. A. Battye, N. S. Manton, C.Houghton and P. Sutcliffe (1996,2004)



Shell vs. Crystal



Shell wins for m $\geq$ 0.16



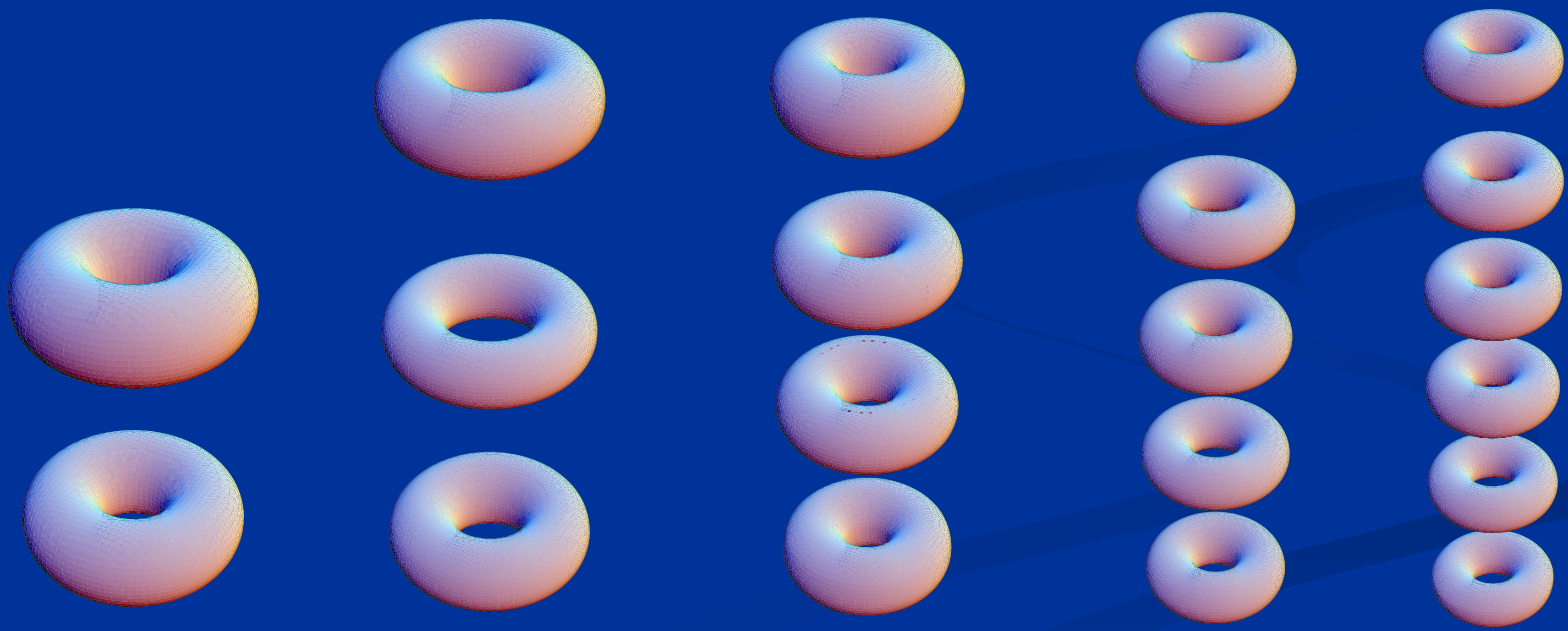
Skyrmion-antiSkyrmion chains

(P. Sutcliffe, S.Krusch, Y.Shnir, T.Tchrakian)

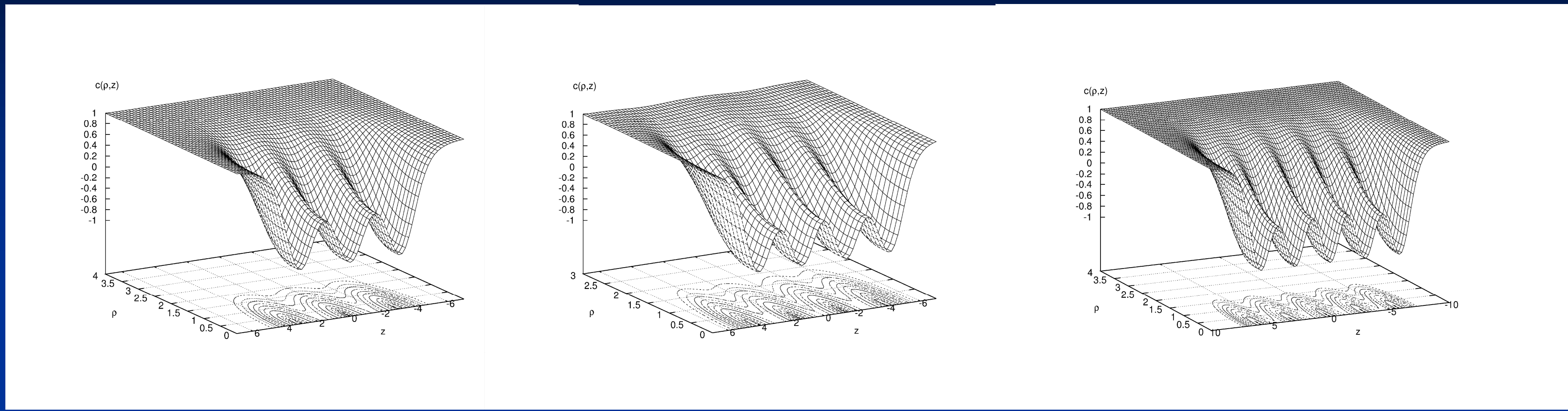
$$\phi^\alpha = \sin f \sin g\, n^\alpha; \quad \phi^3 = \sin f \cos g; \quad \phi^4 = \cos f;$$

$$Q = -\tfrac{1}{2} n \cos(m\theta)\Big|_{\theta=0}^{\theta=\pi} = \tfrac{1}{2} n\, [1 - (-1)^m]$$

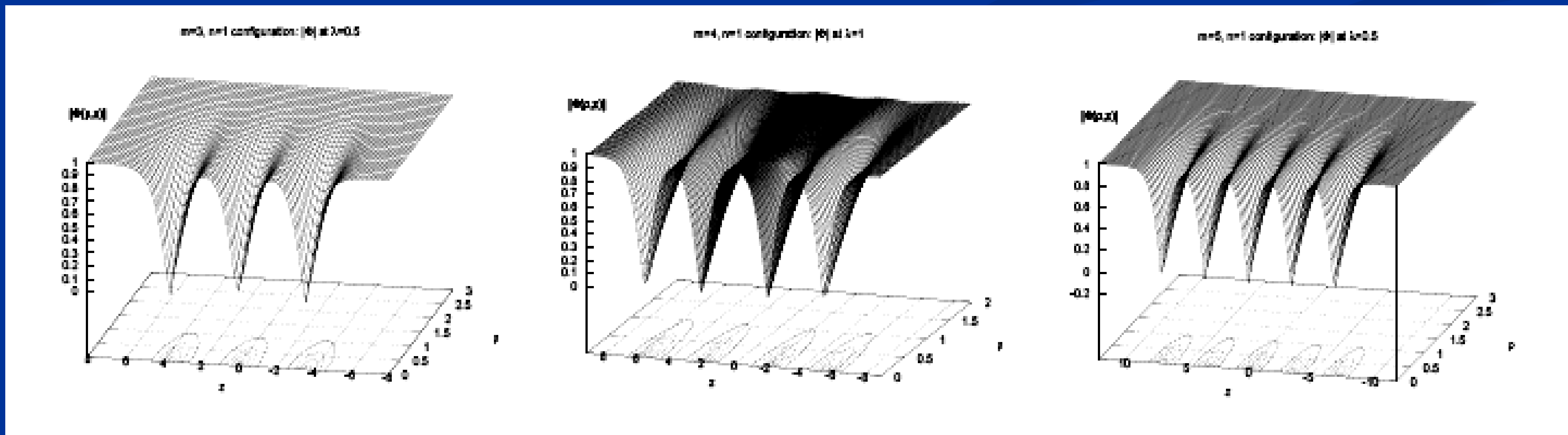
$$f(0) = \pi, \quad f(\infty) = 0$$
$$\partial_r g(0) = 0, \quad g(\infty) = m\pi$$



$\phi^4(r,\theta) \equiv \cos f(r,\theta)$ (Skyrmion - antiSkyrmion chains)



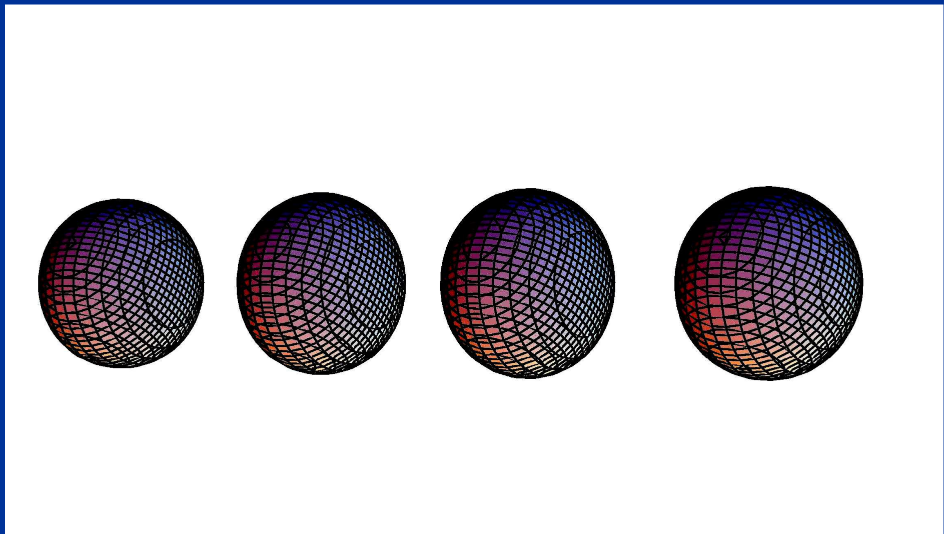
$|\Phi(r,\theta)|^2$ (MAM chains)

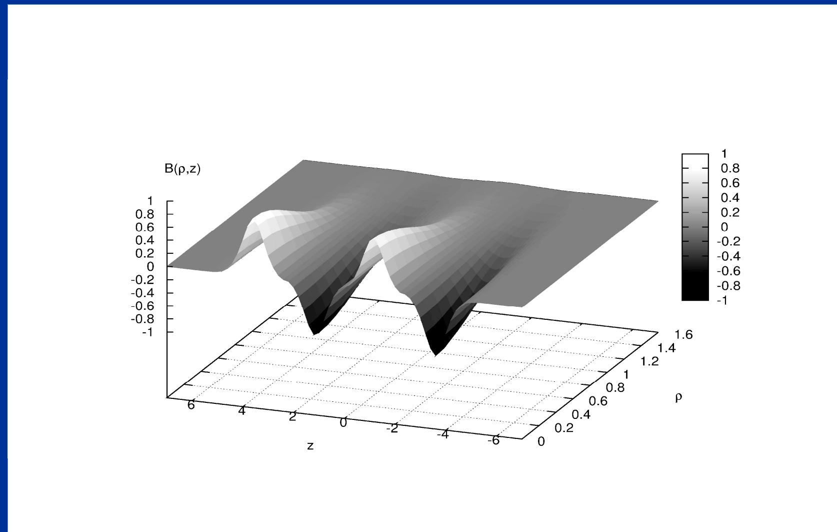


Skyrmion-antiSkyrmion chains: mass term

$$L = -\int d^3x \left\{ \frac{F_\pi^2}{16}\ \mathrm{Tr}(R_\mu R^\mu) - \frac{1}{32e^2}\ \mathrm{Tr}([R_\mu, R_\nu][R^\mu, R^\nu]) + \frac{m_\pi^2 F_\pi^2}{8}\ \mathrm{Tr}(1-U) \right\}$$

n=1 chains exist for m $\geq$ 0.10





Skyrmions and instantons

Skyrme field can be generated from the holonomy of a Yang-Mills instanton (M. Atiyah & N.Manton (1989), P. Sutcliffe (2010))

$$U(\mathbf{x}) = P \exp \left\{ \int_{-\infty}^{\infty} A_4(\mathbf{x}, x_4) dx_4 \right\}$$

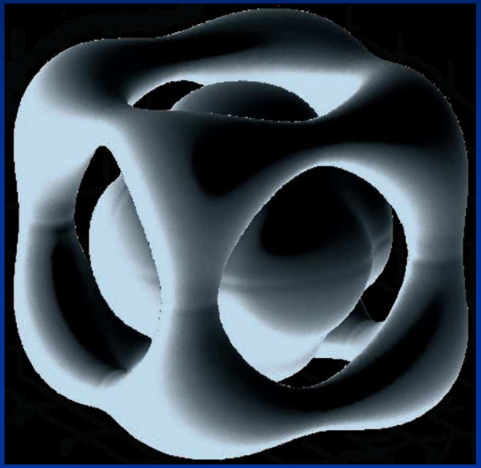
$$U(\mathbf{x}) : \mathbb{R}^3 \mapsto SU(2)$$

Baryon charge is equal to the Pontryagin charge:

$$B = \frac{1}{24\pi^2} \int d^3x \varepsilon_{ijk} \, \mathrm{Tr}(R_i R_j R_k) \Longleftrightarrow N = -\frac{1}{16\pi^2} \, \mathrm{Tr} \int d^4x F_{\mu\nu} \widetilde{F}^{\mu\nu}$$

- Remarkable approximation to the exact solution of the Skyrme model (about 1%)
- Sakai-Sugimoto conjecture about a correspondence between Yang-Mills-Chern-Simons instantons on a curved four-manifold and an extended Skyrme model (Skyrme field and an infinite tower of massive vector mesons)

Summary and Outlook



- There is certain similarity between the monopoles and the Skyrmions.
- Rational maps approach works both for BPS monopoles and for Skyrmions.
- Axially-symmetric sphaleron solutions representing chains of solitons in alternating order exist in both models.
- Non-trival holonomy and Skyrmion-anti-Skyrmion chains?
- Skyrmed monopoles and rational maps?
- Gauged Skyrmions?