Endpoint behavior of the pion distribution amplitude in QCD Sum Rules approach

Alexander Pimikov[1]
in collaboration with S. Mikhailov[1], N. Stefanis[2]

Bogoliubov Lab. Theor. Phys., JINR (Dubna, Russia)[1]
ITP-II, Ruhr-Universität (Bochum, Germany)[2]

Pion distribution amplitude $\varphi_\pi(x,\mu^2)$

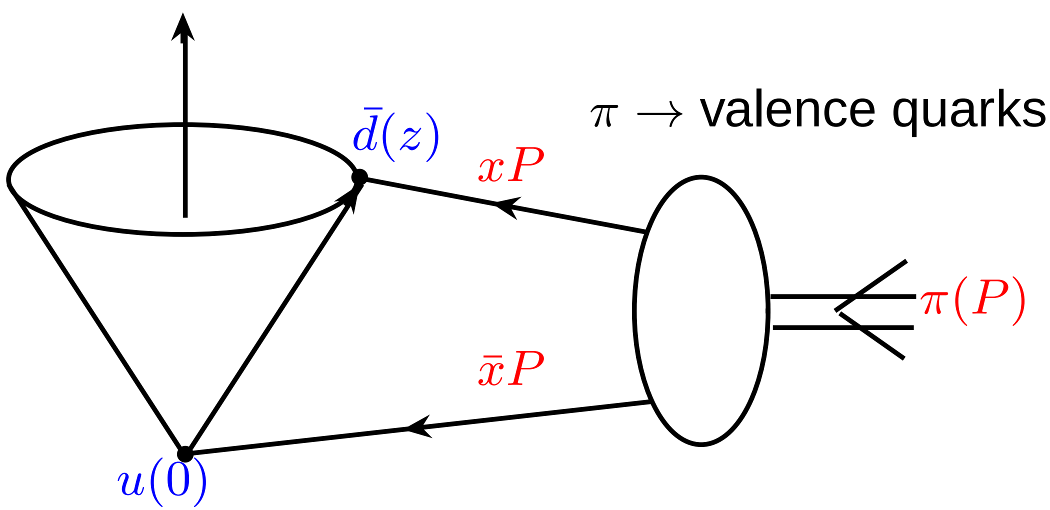
The pion DA parameterizes this matrix element:

$$\langle 0|\,\bar{d}(z)\gamma_\nu\gamma_5\mathcal{C}(z,0)u(0)\,|\,\pi(P)\rangle\Big|_{z^2=0} = if_\pi P_\nu \int_0^1 dx\; e^{ix(zP)}\varphi_\pi(x,\mu^2)$$

Fock–Schwinger string to ensure the gauge-independence:

$$\mathcal{C}(z,0) = \mathcal{P}\,exp\left[ig\int_0^z A_\mu(\tau)d\tau^\mu\right]$$

Pion DA describes the transition of a physical pion into two valence quarks, separated at light cone.



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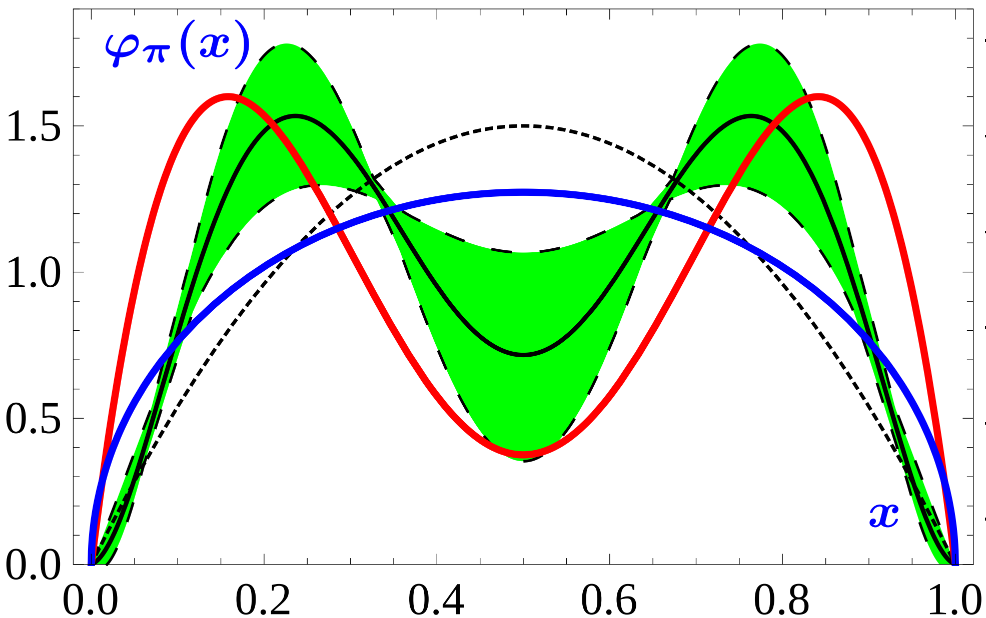
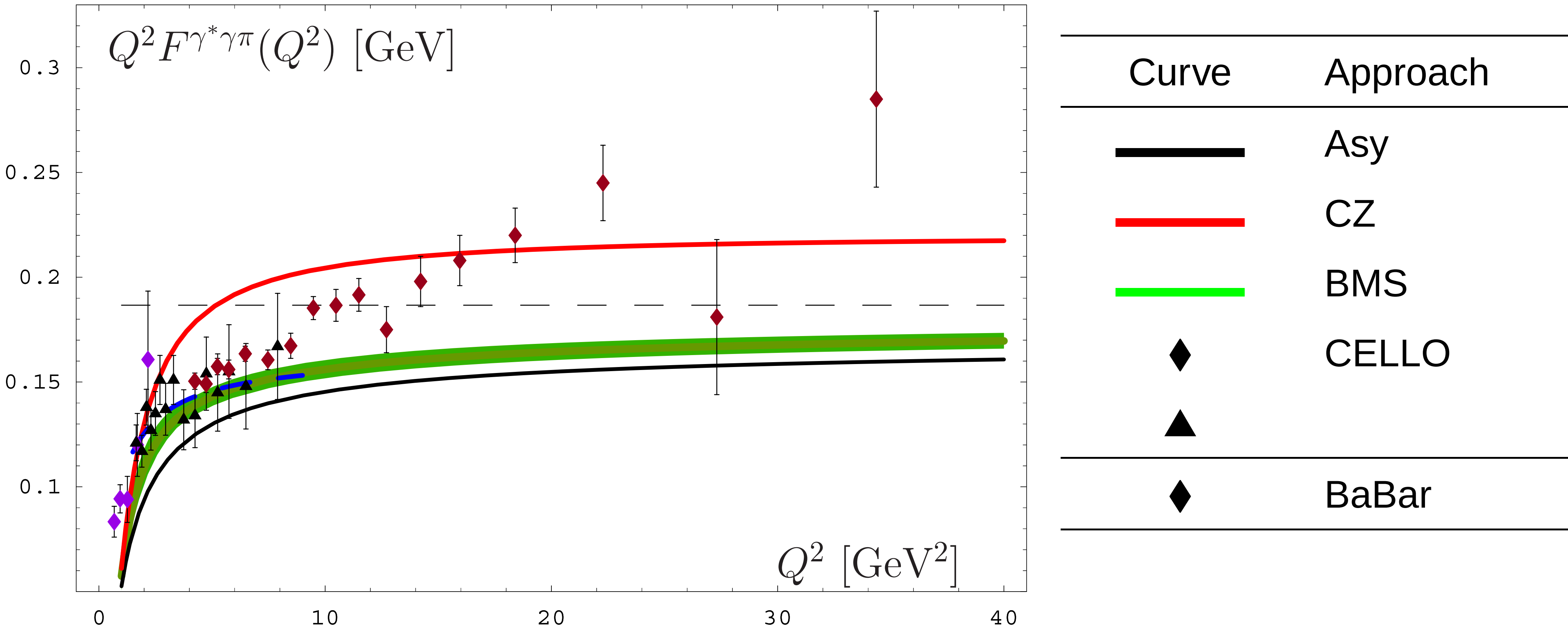


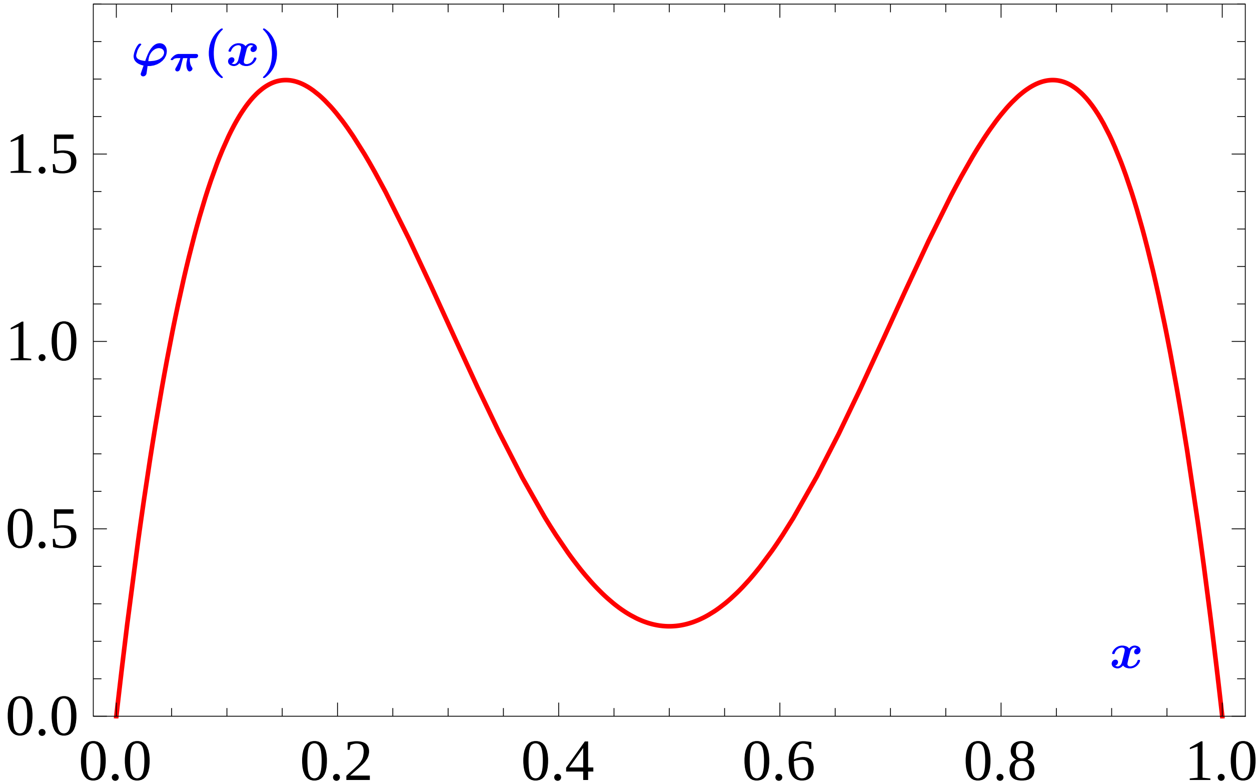
Curve	Approach
	Asymptotic
	BMS from NLC QCD SR
	CZ from QCD SR
	AdS/QCD result

BaBar data on the $\gamma\gamma* \to \pi$ transition FF



- For momentum transfer up to $\mathbf{10}$ GeV$^{\mathbf{2}}$, new BaBar data agree well with the previous CLEO data and prefer the DA with endpoints strongly suppressed.
- The high-$\boldsymbol{Q^2}$ BaBar data show an unexpected growth with $\boldsymbol{Q^2}$ which cannot be understood on the basis of collinear factorization and calls for pion DAs that have their endpoints strongly enhanced.

Models of Pion DA

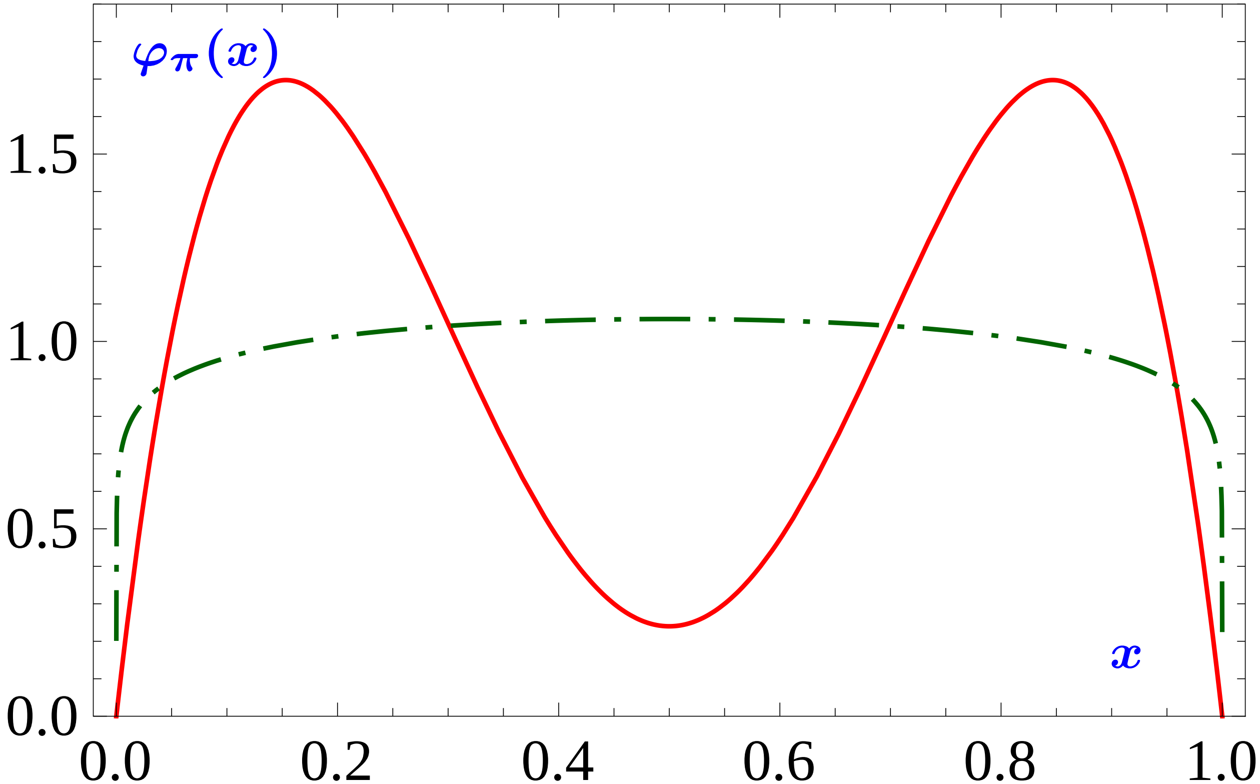


We set $\int_0^1 \varphi_\pi(x)dx = 1\,.$

curve	Model
red line	CZ from QCD SR

All DAs are normalized at the same scale $\mu_0^2 \simeq 1$ GeV^2

Models of Pion DA

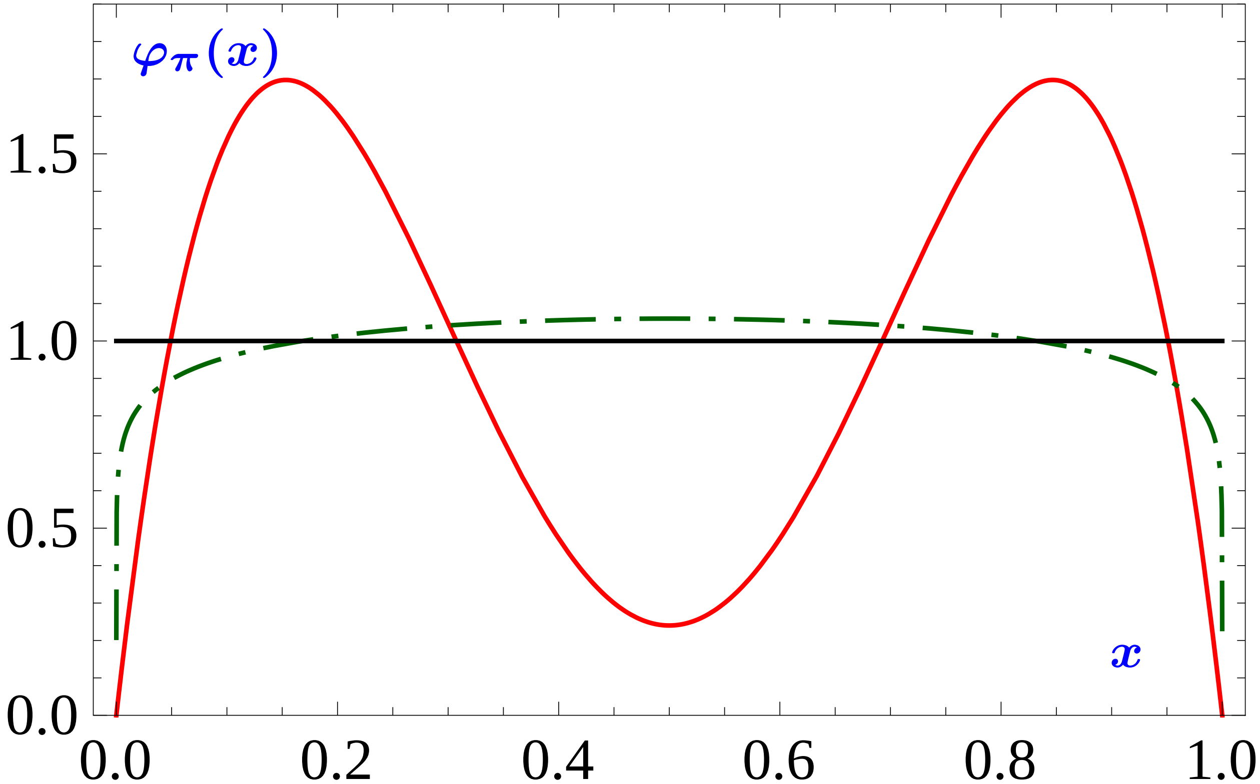


We set $\int_0^1 \varphi_\pi(x)dx = 1\,.$

curve	Model
	CZ from QCD SR
	$\sim x^{0.1}(1-x)^{0.1}$

All DAs are normalized at the same scale $\mu_0^2 \simeq 1$ GeV^2

Models of Pion DA

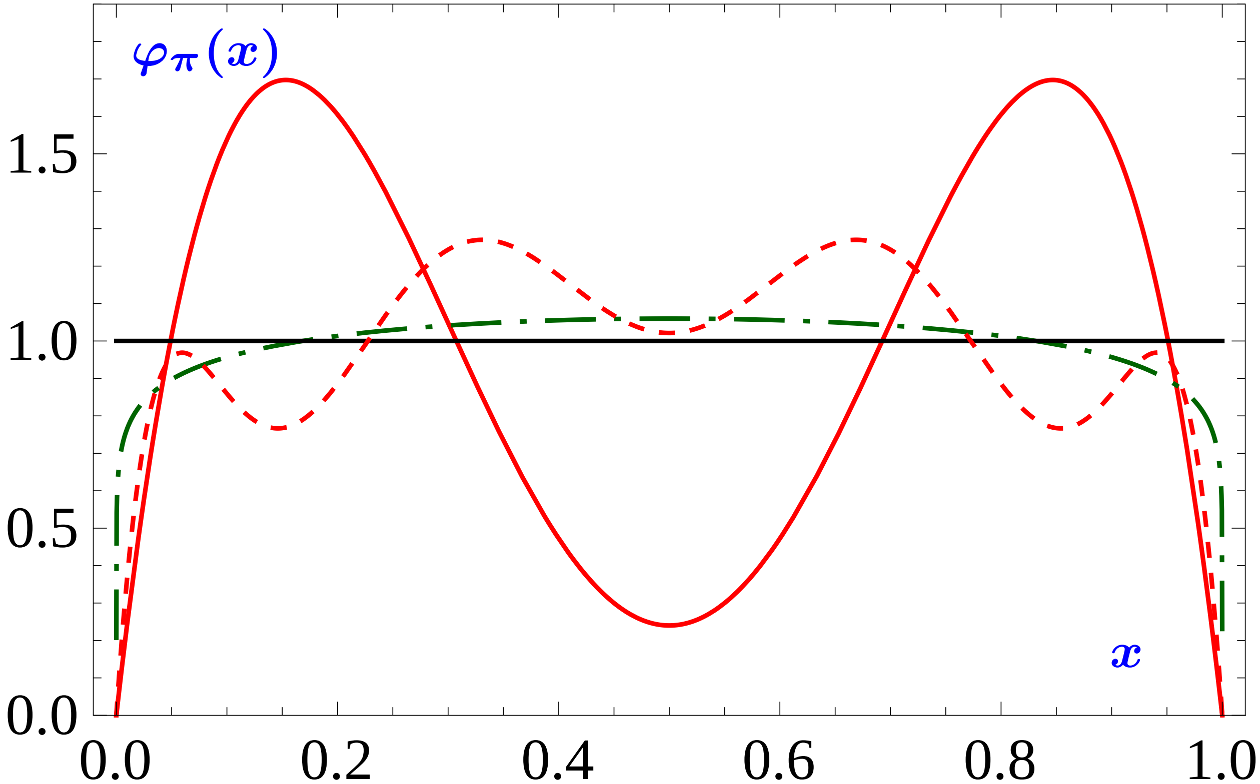


We set $\int_0^1 \varphi_\pi(x)dx = 1$.

curve	Model
red solid line	CZ from QCD SR
green dash-dotted line	$\sim x^{0.1}(1-x)^{0.1}$
black solid line	flat DA $\varphi_\pi(x) = 1$

All DAs are normalized at the same scale $\mu_0^2 \simeq 1$ GeV^2

Models of Pion DA



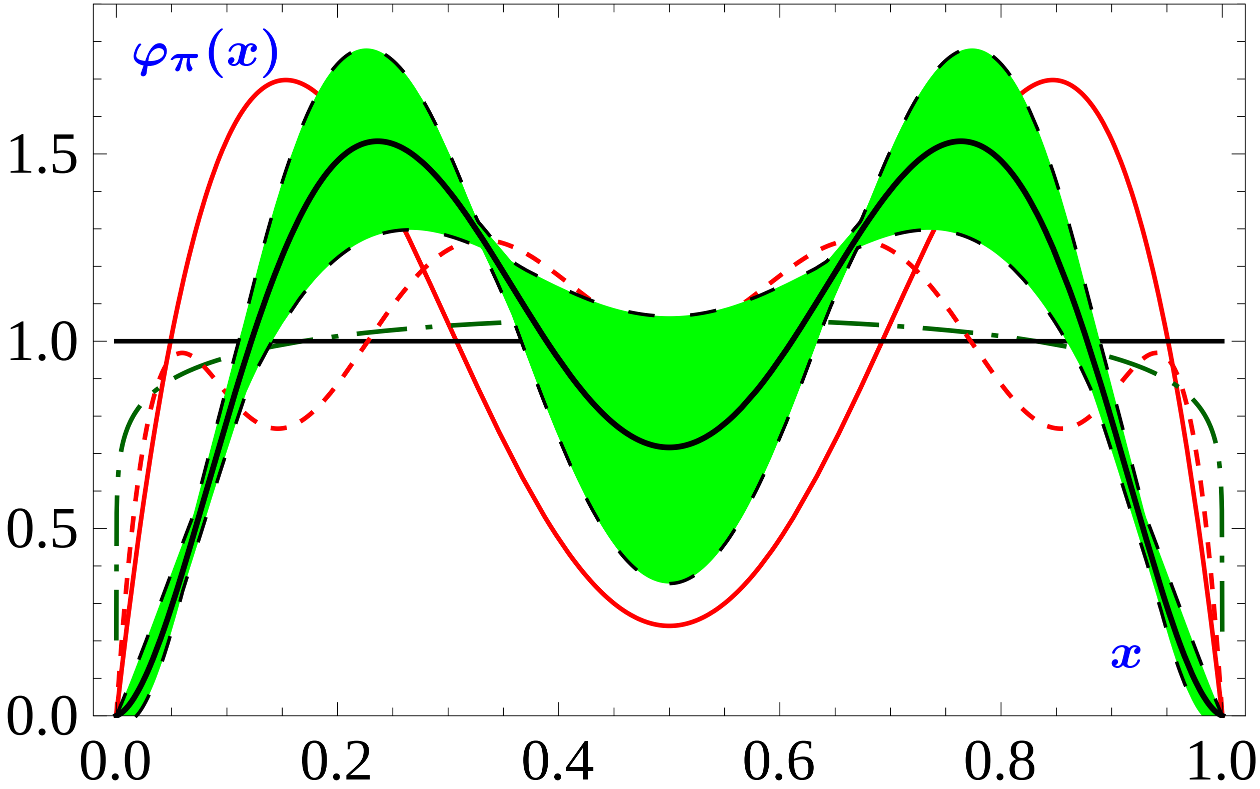
We set $\int_0^1 \varphi_\pi(x)dx = 1\,.$

curve	Model
red solid line	CZ from QCD SR
green dash-dotted line	$\sim x^{0.1}(1-x)^{0.1}$
black solid line	flat DA $\varphi_\pi(x) = 1$
red dashed line	flat-like DA $\varphi^{\text{flat,3}}(x)$

All DAs are normalized at the same scale $\mu_0^2 \simeq 1$ GeV^2

$$\varphi^{\text{flat,N}}(x) = 6x\bar{x}\sum_{n=0}^{N} C_{2n}^{3/2}(2x-1)\frac{2(4n+3)}{3(2n+1)(2n+2)} \quad \text{with} \quad \varphi^{\text{flat},\infty}(x) = 1\,.$$

Models of Pion DA



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green band with black lines	BMS from NLC QCD S

All DAs are normalized at the same scale $\mu_0^2 \simeq 1$ GeV^2

- Our aim is the reconsidering the QCD SR approach with nonlocal condensate (NLC) by focusing attention on end-point behavior of pion DA.

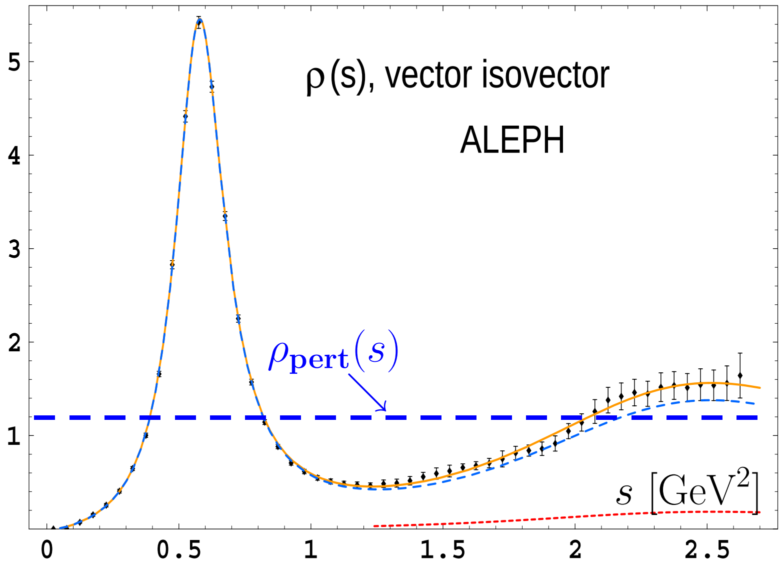
QCD SR Approach

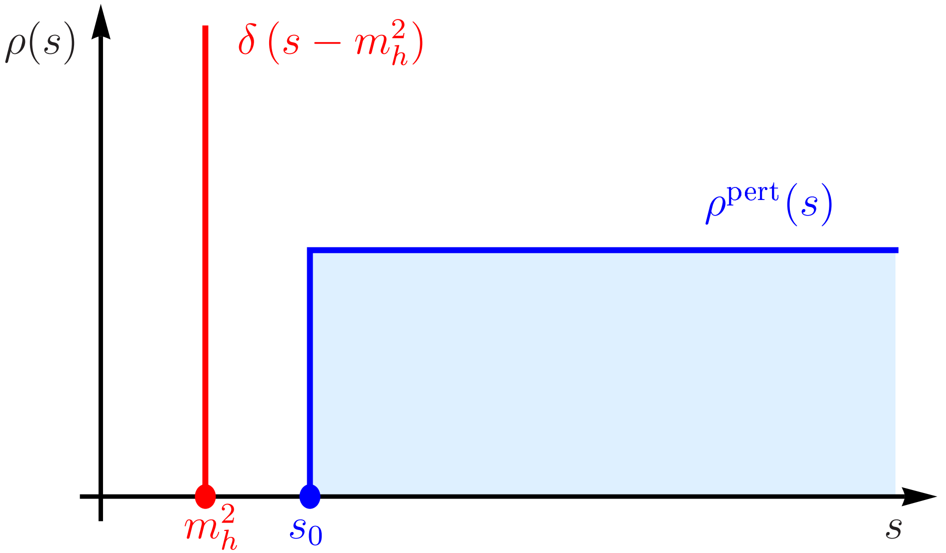
Determination of spectrum parameters from requirement of agreement between two ways for correlator :

- 1th way — Dispersion relation: decay constants f_h and masses m_h,

$$\Pi_{\text{had}}\left(Q^2\right) = \int_0^\infty \frac{\rho_{\text{had}}(s)\ ds}{s+Q^2} + \text{subtractions}\,.$$

  - model spectral density: $\rho_{\text{had}}(s) = f_h^2\,\delta\left(s-m_h^2\right) + \rho_{\text{pert}}(s)\,\theta\left(s-s_0\right)\,.$



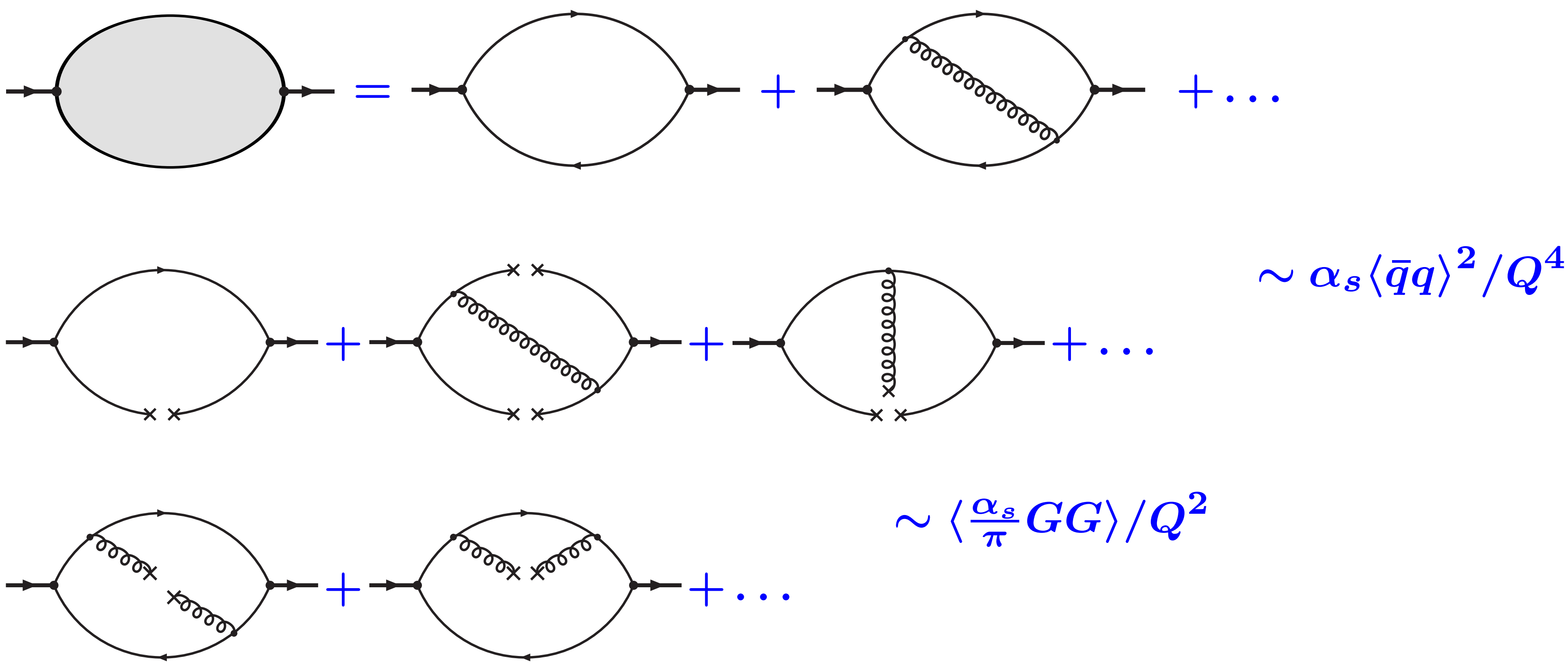


Theoretical part of QCD SR

- 2th way — Operator product expansion:

$$\Pi_{\text{OPE}}\left(Q^2\right) = \Pi_{\text{pert}}\left(Q^2\right) + \sum_n C_n \frac{\langle 0| : O_n : |0\rangle}{Q^{2n}}\,.$$

  - Condensates $\langle 0| : O_n : |0\rangle \equiv \langle O_n \rangle = ?$ (next slides).



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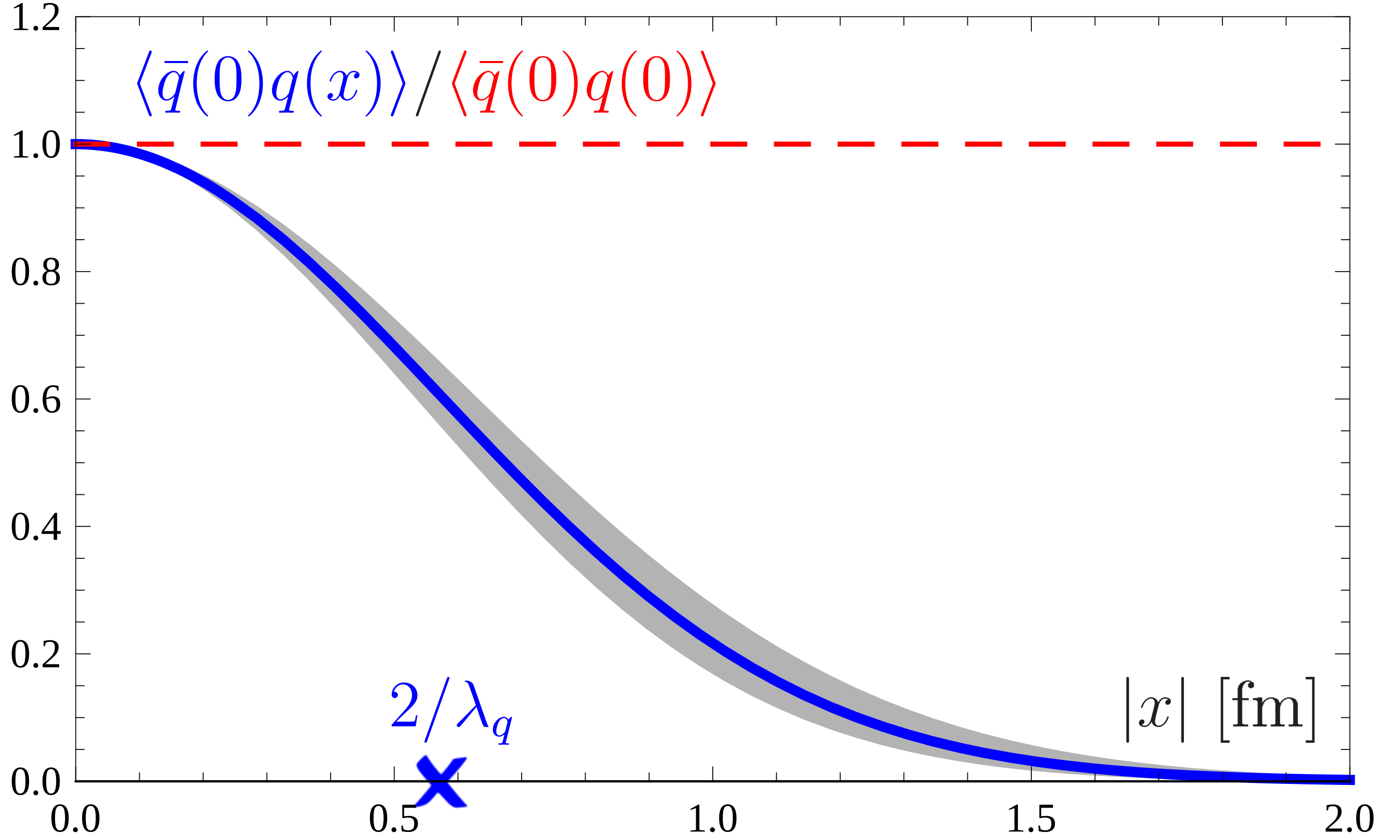
QCD SR reads:

$$\Pi_{\text{had}}\left(Q^2, m_h, f_h\right) = \Pi_{\text{OPE}}\left(Q^2\right)\,.$$

Lattice data of Pisa group



Nonlocality of quark condensates $\lambda_q^2 = 0.42(8)$ GeV^2 from lattice data of Pisa group in comparison with **local limit**.

- Even at $|z| \simeq 0.5$ fm nonlocality is quite important!

Coordinate dependence of condensates

Parameterization for scalar condensate was suggested in works of Bakulev, Mikhailov and Radyushkin:

$$\langle : \bar{q}_A(0) q_A(x) : \rangle = \langle \bar{q} q \rangle \int_0^\infty \boxed{f_S(\alpha)} e^{\alpha x^2/4} d\alpha \,, \text{ where } x^2 < 0.$$

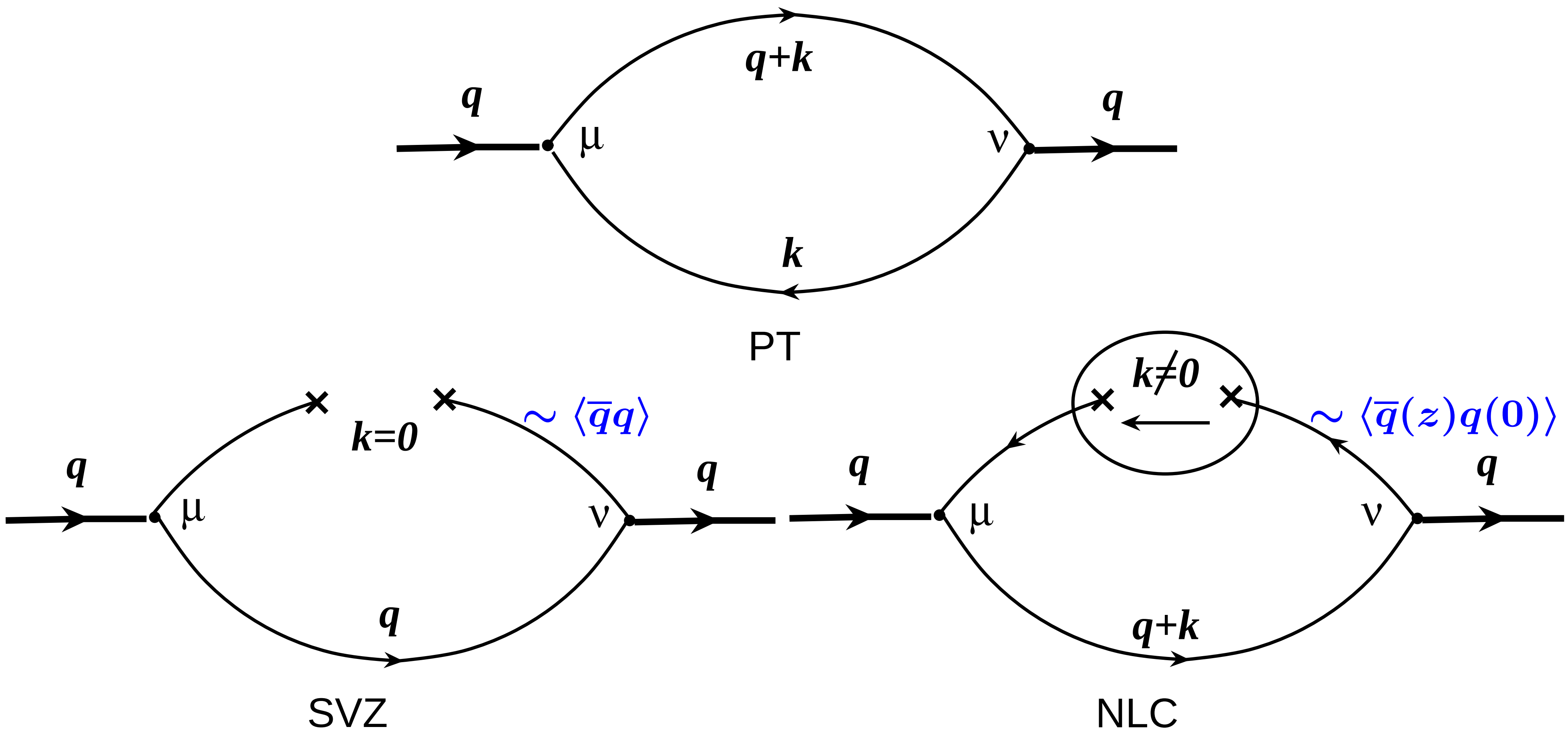
- First approximation which takes into account finite width of

  quark distribution in vacuum: $f_S(\alpha) = \delta\left(\alpha - \frac{\lambda_q^2}{2}\right), \quad \lambda_q^2 = \frac{\langle \bar{q} D^2 q \rangle}{\langle \bar{q} q \rangle}$

- Such representation corresponds to Gaussian form $\sim \exp\left(\lambda_q^2 x^2/8\right)$ of NLC in coordinate representation.

- The **smooth model** $f_S(\alpha) \sim \alpha^{n-1} \exp\left(-\Lambda^2/\alpha - \sigma^2 \alpha\right)$ has a sensible asymptotic form $\langle \bar{q}(0) q(x) \rangle \Big|_{x^2 \to \infty} \sim \exp\left(-\Lambda x\right)$ in x-representation.

Diagrams for $\langle T\left(J_\nu(z)J_\mu(0)\right)\rangle$



- Quarks run through vacuum with nonzero momentum $k \neq 0$:

$$\langle k^2\rangle = \frac{\langle\bar{q}D^2q\rangle}{\langle\bar{q}q\rangle} = \lambda_q^2 = 0.4 - 0.5\,\text{GeV}^2$$

QCD SR for pion DA

Applying the QCD SR technics for correlator of two axial current $\int d^4x\, e^{iqx} \langle 0 | T \left[J^N_{5\mu}(0) J^+_{5\nu}(x) \right] | 0 \rangle$, one can obtain the SR for pion DA $\varphi_\pi(x)$:

$$f_\pi^2\, \varphi_\pi(x) + f_{A_1}^2\, \varphi_{A_1}(x)\, e^{-m_{A_1}^2/M^2} = \int_0^{s_0} \rho_{\text{pert}}(x)\, e^{-s/M^2} ds + \Phi_{\text{npert}}(x, M^2)\,,$$

where $\Phi_{\text{npert}} = \Phi_{\text{G}} + \Phi_{\text{4Q}} + \Phi_{\text{V}} + \Phi_{\text{T}}$,

with singular terms:

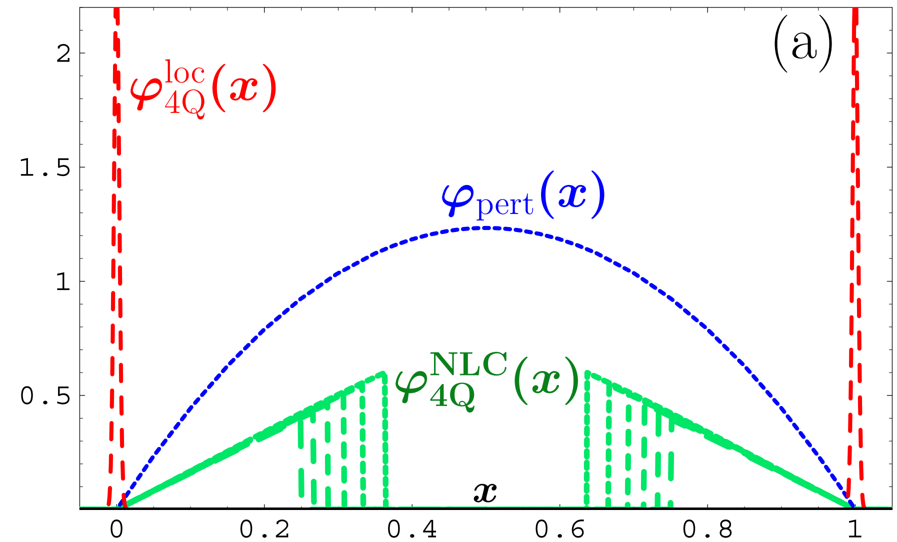
$\Phi_{\text{4Q}} \sim x\theta(\Delta - x)$,

$\Phi_{\text{V}} \sim x\delta'(\Delta - x)$,

$\Phi_{\text{T}} \sim \delta(\Delta - x) + \delta(2\Delta - x) + \theta(\Delta - x)$,

$\Phi_{\text{G}} \sim \delta(\Delta - x)$,

with $\Delta = \lambda_q^2/M^2 \in [0.01, 0.3]$.

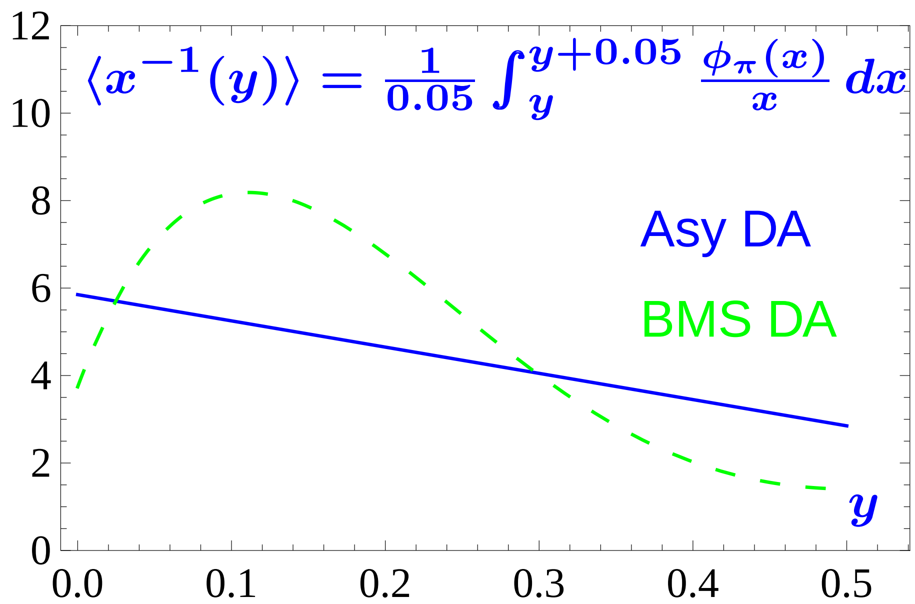


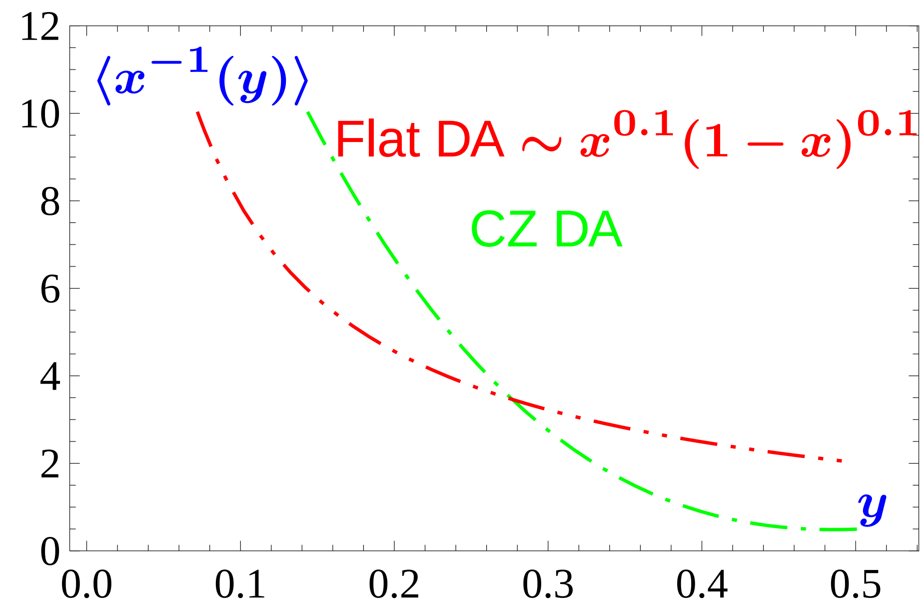
- Since the nonperturbative contribution has singularities, this SR is mainly proposed to study the integral characteristic of π-DA. The exception is end-point region where only 4-quark condensate contribute using minimal Gaussian model.

Integral characteristic of pion DA

Moments: $\langle\xi^{2N}\rangle \equiv \int_0^1 dx\,\varphi_\pi(x)(2x-1)^{2N}\,,\quad \langle x^{-1}\rangle \equiv \int_0^1 dx\,\varphi_\pi(x)x^{-1}\,.$

SVZ	$\langle\xi^0\rangle$	LO	local cond.	f_π
CZ	$\langle\xi^{2N}\rangle$ $N=0,1$	NLO	local cond.	$f_\pi,\ a_2$
BMS	$\langle\xi^{2N}\rangle$, $N=0,1,\ldots,5$ and $\langle x^{-1}\rangle$	NLO	nonlocal cond.	$f_\pi,\ a_2,\ a_4$





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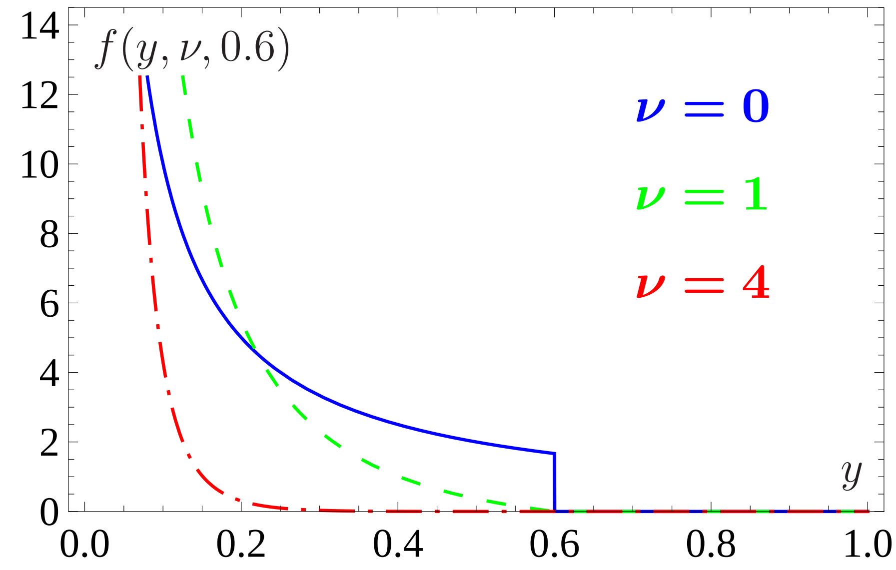
BMS	$\langle \xi^{2N} \rangle$, $N = 0, 1, \ldots, 5$ and $\langle x^{-1} \rangle$	NLO	nonlocal cond.	$f_\pi,\ a_2,\ a_4$
Here	$[D^{(\nu)}\varphi_\pi](x)$, $\nu \in [2, 4]$, $x \in [0.4, 1]$	NLO	nonlocal cond.	$\varphi'_\pi(0)$

The definition of the "integral" derivative:

$$[D^{(\nu+2)}\varphi](x) = \frac{1}{x} \int_0^x dy\, \varphi(y) f(y, \nu, x)\,,$$

where $f(y, \nu, x) = \frac{\theta(x-y)}{\Gamma(\nu+1)\, y} \left(\ln \frac{x}{y} \right)^\nu\,.$



“Integral” derivatives $[D^{(\nu)}\varphi_\pi](x)$ of pion DA

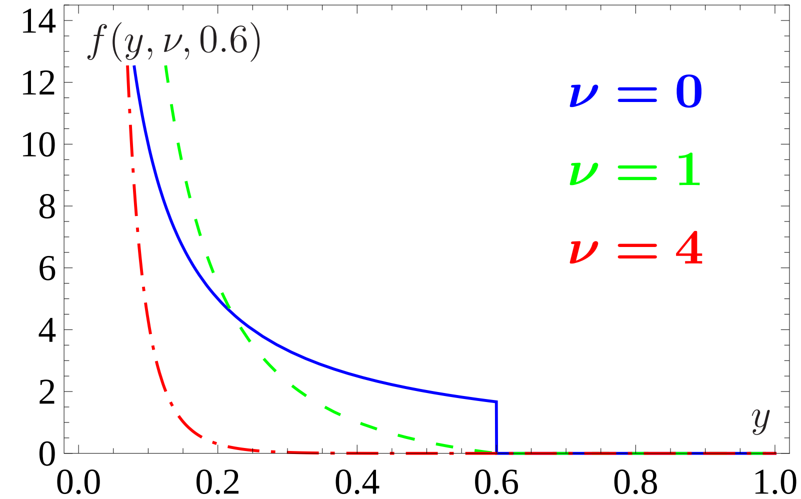
$$[D^{(\nu+2)}\varphi](x) = \frac{1}{x}\int_0^x dy\,\varphi(y) f(y,\nu,x)\,,$$ where $$f(y,\nu,x) = \frac{\theta(x-y)}{\Gamma(\nu+1)\,y}\left(\ln\frac{x}{y}\right)^{\nu}.$$

Properties of the “integral” derivative:

$$[D^{(2)}\varphi](x) = \frac{1}{x}\int_0^x \frac{\varphi(y)}{y}dy \xrightarrow{x\to 1} \langle x^{-1}\rangle\,,$$

$$[D^{(n+1)}\varphi](x) = \frac{1}{x}\int_0^x dy\,[D^{(n)}\varphi](y)\,,$$

$$[D^{(\nu+2)}\varphi](x) = \varphi'(0) + \varphi''(0)\frac{x}{2!2^{\nu+1}} + \ldots\,.$$

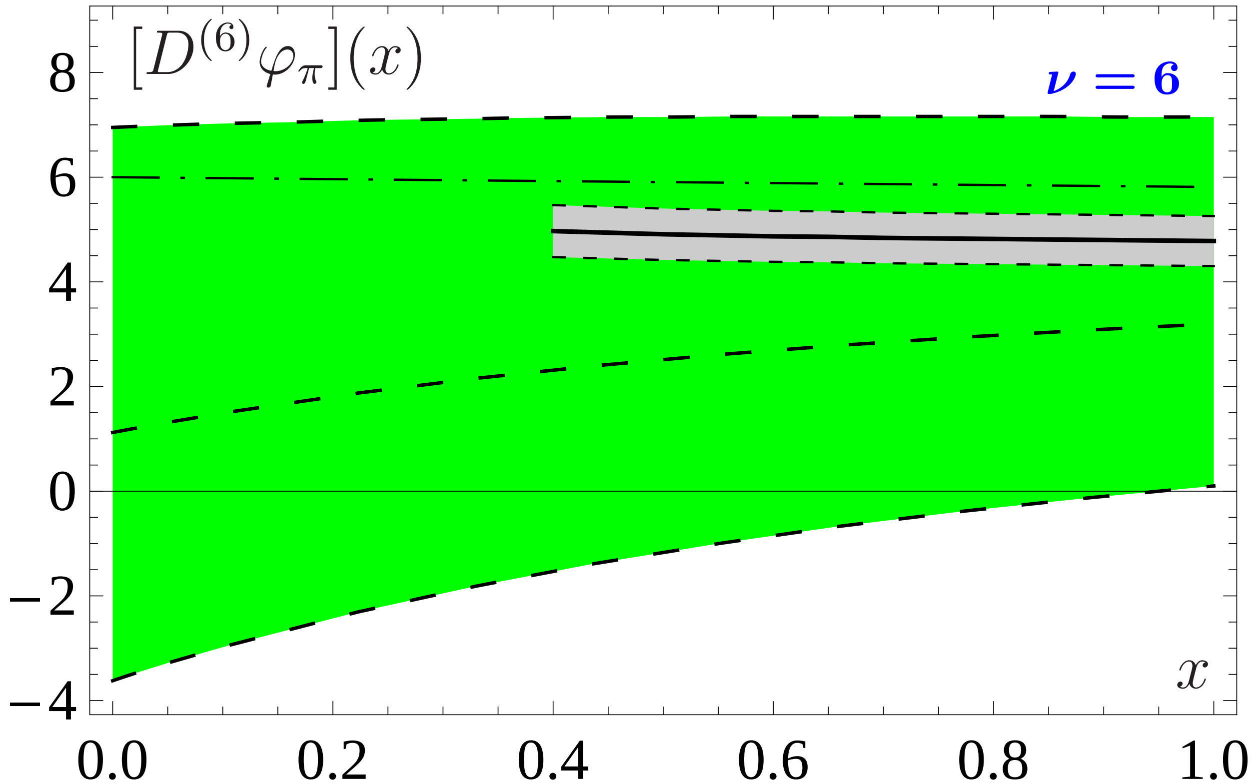


The defined operator $D^{(\nu)}$ reproduces at small x and/or large ν the derivative of $\varphi(x)$ at the origin $x = 0$.

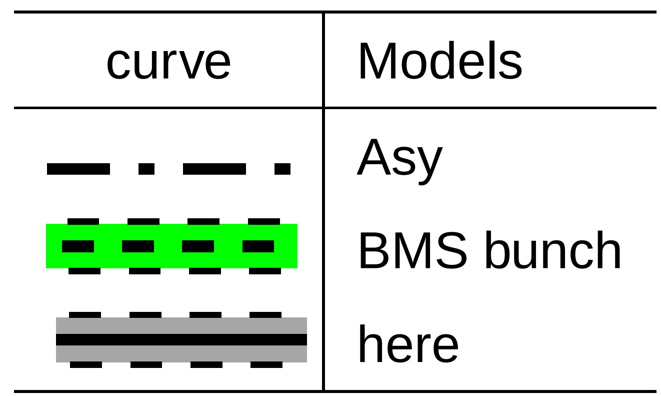
It acts on linear function as a derivative $D^{(\nu)}(ax) = a$ for any ν, x.

QCD SR result on $[D^{(\nu)}\varphi_\pi](x)$

The result of QCD SR analysis is shown by gray strip for $\nu = 2, 3, 6$ and $x \in [0.4, 1]$



in comparison with BMS bunch result and asymptotic model.



- The image of the operator $D^{(\nu+2)}$ for $\nu \geq 4$ is numerically very close to the result obtained with the differentiation method (next slides).

QCD SR result on $[D^{(\nu)}\varphi_\pi](x)$

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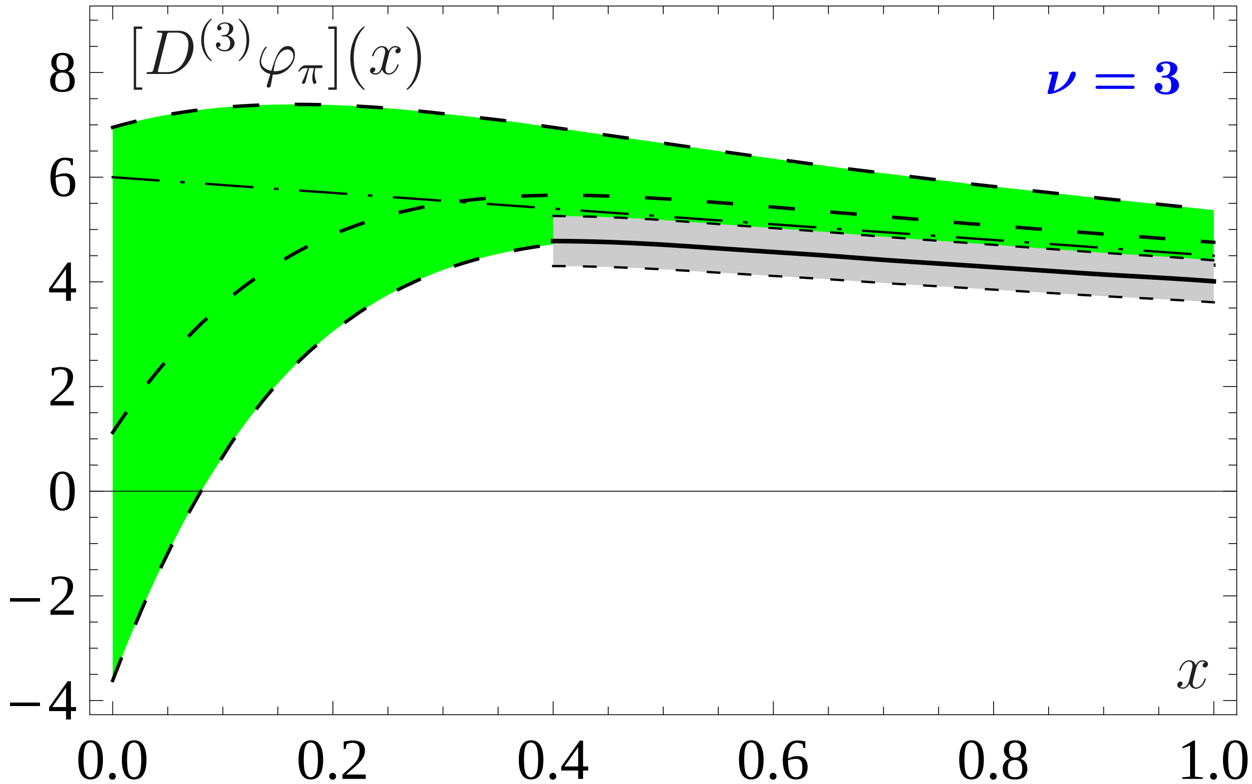

in comparison with BMS bunch result and asymptotic model.

curve	Models
dash-dotted line	Asy
green band with dashed lines	BMS bunch
gray band with solid line	here

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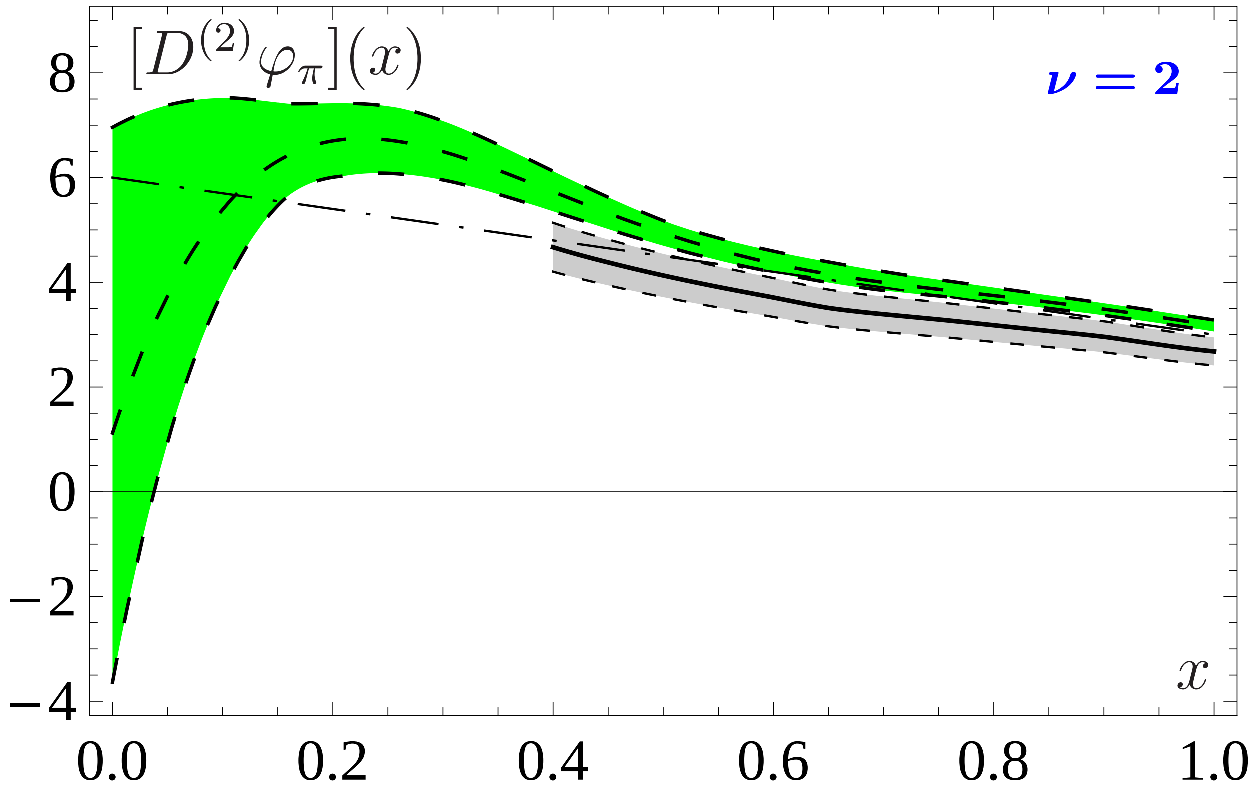

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	Asy
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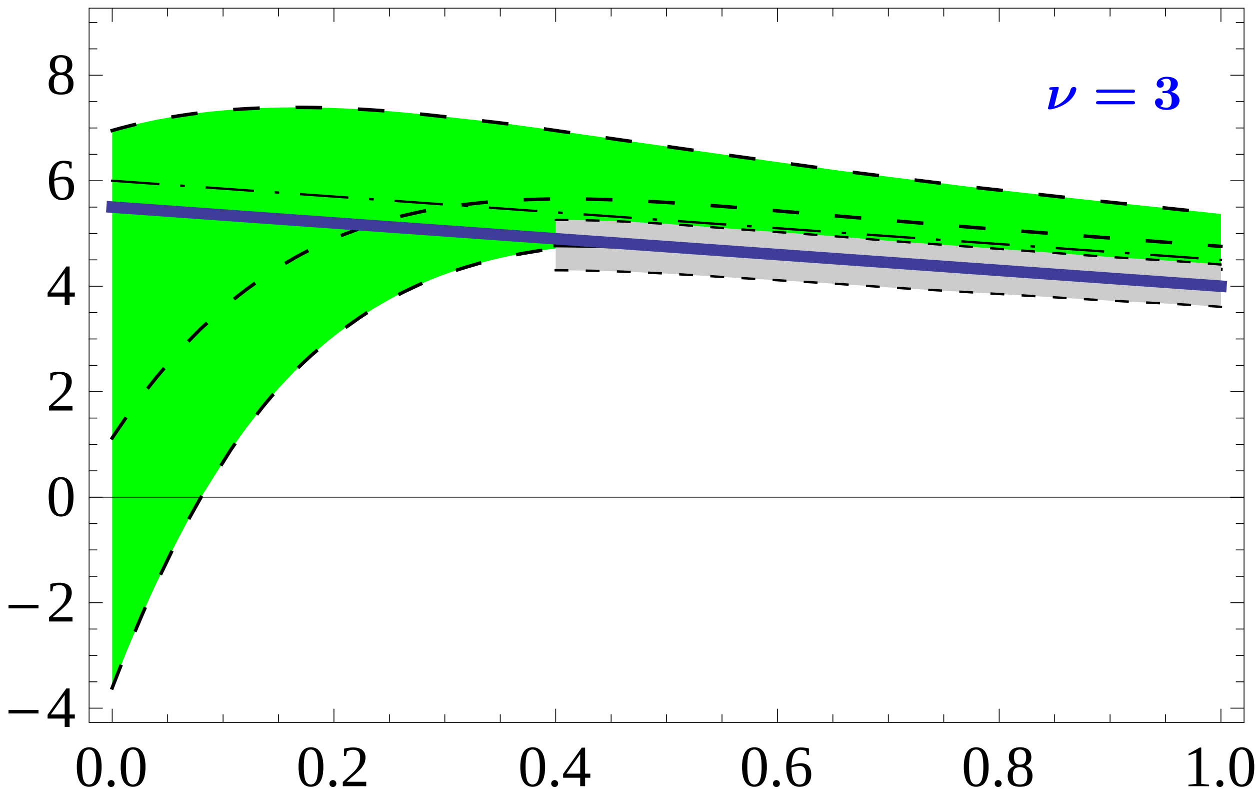


in comparison with BMS bunch result and asymptotic model.

curve	Models
	Asy
	BMS bunch
	here

- Using the expansion for $[D^{(\nu+2)}\varphi_\pi](x)$ and linear dependence of the SR result we estimated $\varphi'_\pi(0)$:

$$[D^{(\nu+2)}\varphi](x) = \varphi'(0) + \varphi''(0)\frac{x}{2!2^{\nu+1}} + \ldots \quad \Longrightarrow \quad \varphi'_\pi(0) = 5.5 \pm 1.5\,.$$

QCD SR for $\varphi'_\pi(0)$

By the differentiating the QCD SR for pion DA at $x = 0$ we arrive to SR for $\varphi'_\pi(0)$

$$f_\pi^2\,\varphi'_\pi(0) = \frac{3}{2\pi^2}M^2\left(1 - e^{-s_0/M^2}\right) - f_{A_1}^2\,\varphi'_{A_1}(0)\,e^{-m_{A_1}^2/M^2} + \frac{144\pi\alpha_S}{81}\langle\bar{q}q\rangle^2\Phi'\,,$$

where only 4-quark condensate contribution is survived.

Then nonperturbative term is mainly defined by the scalar-quark condensate at large and moderate distances

$$\Phi' = \int_0^\infty d\alpha\,\frac{f_S(\alpha)}{\alpha^2} = \langle\bar{q}q\rangle^{-1}\int_0^\infty z^2\langle\bar{q}(0)q(z)\rangle dz^2\,.$$

$$f_S(\alpha) = \begin{cases} \delta(\alpha - \lambda_q^2/2) & \delta\text{-ansatz model} \to \text{gaussian behavior} \\ \alpha^{n-1}e^{-\Lambda^2/\alpha - \alpha\,\sigma^2} & \text{smooth model} \to \text{exponential decay} \end{cases}$$

- The Gaussian model $\sim \mathbf{exp}\left(\lambda_q^2 x^2/8\right)$ of scalar condensate leads to a simple expression for the nonperturbative contribution to SR: $\Phi'_{\text{gaus}} = 4/\lambda_q^4$. Then the QCD SR result is $\varphi'_\pi(0) = 5.3(5)$, where nonlocality parameter $\lambda_q^2 = 0.4\,\text{GeV}^2$ was applied.

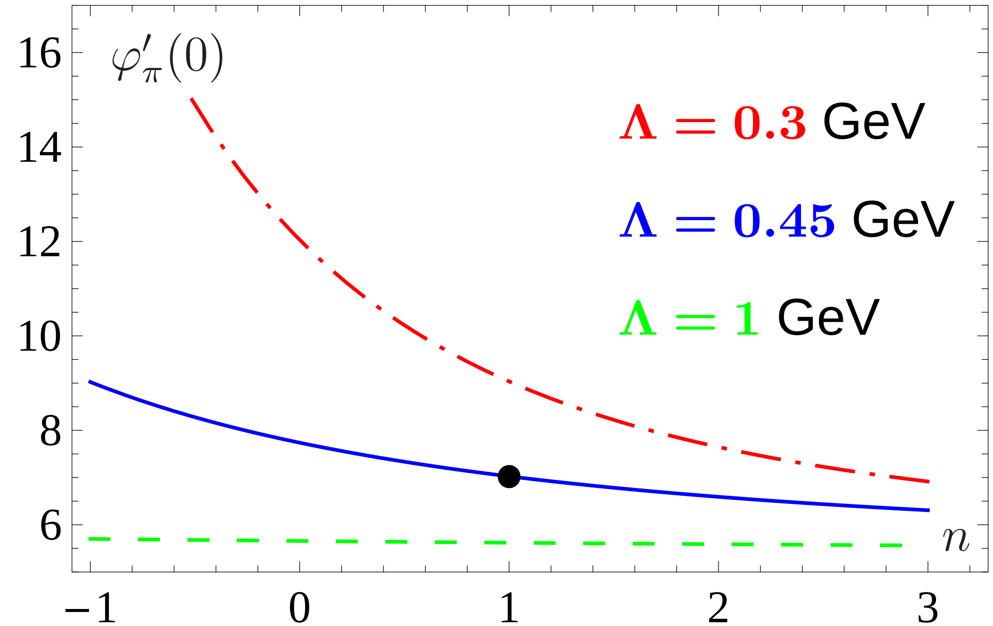
QCD SR for $\varphi'_\pi(0)$ with smooth NLC

- There is an indication from the heavy-quark effective theory **[Radyushkin 91]** that in reality the quark-virtuality distribution f_S should be parameterized in a different way as to ensure that the scalar condensate decreases exponentially at large distances. $\langle \bar{q}(0)q(z)\rangle \sim |z|^{-(2n+1)/2} e^{-\Lambda|z|}$.

It could be realized by this model for f_S:

$$f_S(\alpha; \Lambda, n, \sigma) \sim \alpha^{n-1} e^{-\Lambda^2/\alpha - \alpha\,\sigma^2}\,,$$

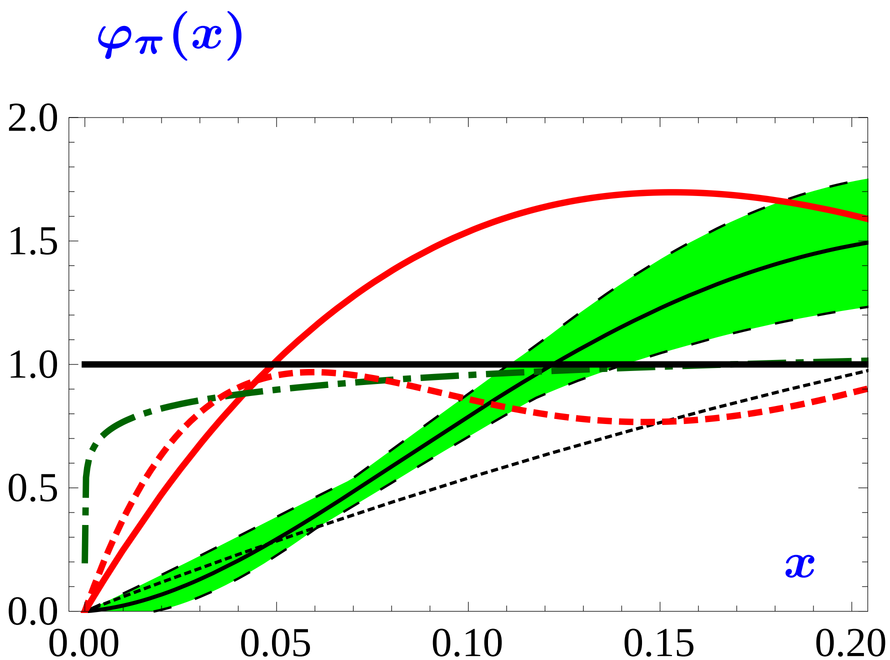
- The analysis of SR for the heavy-light meson obtained in heavy quark effective theory leads to values $\Lambda = 0.45$ GeV and $n = 1$. For this parameters we get $\varphi'_\pi(0) = 7.0(7)$ (black point on Fig.).



- Exponentially decreased models of NLC leads to the larger nonperturbative contribution and as a result the larger value of $\varphi'_\pi(0)$ in comparison with Gaussian model.

Comparisons of results with pion DA models

Approach	$[D^{(3)}\varphi_\pi](0.5)$	$\varphi'_\pi(0)$
Integral LO QCD SR	4.7 ± 0.5	5.5 ± 1.5
Differential LO QCD SR, gaussian decay of NLC	—	5.3 ± 0.5
Differential LO QCD SR, exponential decay of NLC	—	7.0 ± 0.7



Curve	Model	$[D^{(3)}\varphi_\pi](0.5)$	$\varphi'_\pi(0)$
	BMS DA	5.7 ± 1.0	1.7 ± 5.3
	Asy DA	5.25	6
	CZ DA	15.1	26.2
	$\varphi^{\text{flat},3}(x)$	22.5	72
	$\sim x^{0.1}$	227	$\gg 6$

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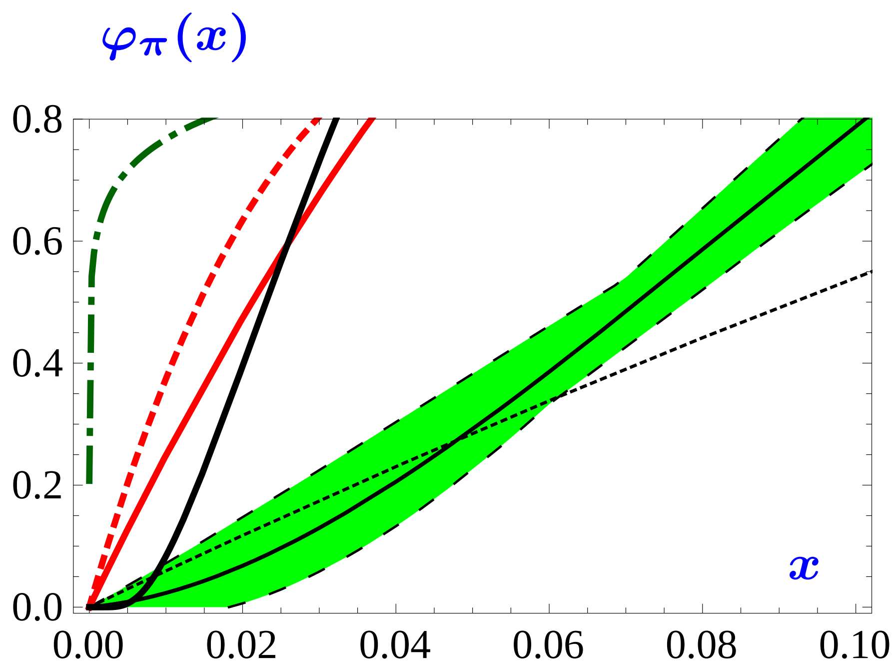

Curve	Model	$[D^{(3)}\varphi_\pi](0.5)$	$\varphi'_\pi(0)$
green band with black solid and dashed lines	BMS DA	5.7 ± 1.0	1.7 ± 5.3
black dashed line	Asy DA	5.25	6
red solid line	CZ DA	15.1	26.2
red dashed line	$\varphi^{\text{flat},3}(x)$	22.5	72
green dash-dotted line	$\sim x^{0.1}$	227	$\gg 6$
black solid line	[WH10]	14	0

Conclusion

- The integral derivative of the pion DA, based on the derived SR, remains smaller ($\mathbf{4.7 \pm 0.5}$) than the asymptotic value ($\mathbf{5.25}$) and overlaps with the range of values determined with the BMS bunch of pion DAs ($\mathbf{5.7 \pm 1.0}$), while there is no agreement with the CZ DA ($\mathbf{15.1}$) and the flat-like models considered ($\mathbf{227}$).
- The same conclusions can be drawn also for the usual derivative of the pion DA, which follows from the differential SR.
- It is worth mentioning that employing the integral and the differential sum rules, we found that the leading-order QCD sum rules, which employ the minimal Gaussian model for the nonlocal condensates, cannot be satisfied by flat-type pion distribution amplitudes.
- We find that slop of pion DA is defined mainly by behavior of scalar quark condensate at large and moderate distances. Choosing a model for the scalar quark condensate having a slower decay at large distances, causes an increase of the nonperturbative contribution to the SR and entail also an increase of the value $\varphi'_{\pi}(\mathbf{0})$.