Integrable systems and the universal R-matrix[1]

Kh. S. Nirov

Bergische Universität Wuppertal, and INR RAS, Moscow

Quarks 2010, Kolomna, Russia, June 6–12

[1] Based on work in collaboration with H. Boos, F. Göhmann, A. Klümper, and A. V. Razumov

- **Classical integrable systems**

- **Classical integrable systems**
  - Lagrangian and Hamiltonian description

- **Classical integrable systems**
  - Lagrangian and Hamiltonian description
  - Group-theoretic framework

- **Classical integrable systems**
  - Lagrangian and Hamiltonian description
  - Group-theoretic framework
  - Riemann–Hilbert problem

- **Classical integrable systems**
  - Lagrangian and Hamiltonian description
  - Group-theoretic framework
  - Riemann–Hilbert problem
- **Quantum integrable systems**

- **Classical integrable systems**
  - Lagrangian and Hamiltonian description
  - Group-theoretic framework
  - Riemann–Hilbert problem

- **Quantum integrable systems**
  - Bethe Ansatz (H. Bethe, 1931)

- **Classical integrable systems**
  - Lagrangian and Hamiltonian description
  - Group-theoretic framework
  - Riemann–Hilbert problem

- **Quantum integrable systems**
  - Bethe Ansatz (H. Bethe, 1931)
  - Functional relations (R. Baxter, 1971-72)

- **Classical integrable systems**
  - Lagrangian and Hamiltonian description
  - Group-theoretic framework
  - Riemann–Hilbert problem

- **Quantum integrable systems**
  - Bethe Ansatz (H. Bethe, 1931)
  - Functional relations (R. Baxter, 1971-72)
  - Quantum groups (V. G. Drinfeld, 1985-87; M. Jimbo, 1985-87)

- Generalized Toda system

$$q_{tt} = -\nabla_q U, \qquad U = \sum_{\alpha \in \pi} \exp(2\alpha(q)), \quad q(t) = \sum_{\alpha \in \pi} h_\alpha q^\alpha(t)$$

- Generalized Toda system

$$q_{tt} = -\nabla_q U, \qquad U = \sum_{\alpha\in\pi} \exp(2\alpha(q)), \quad q(t) = \sum_{\alpha\in\pi} h_\alpha q^\alpha(t)$$

- Lax pair

$$L_t = [A, L]$$

$$A = -\sum_{\alpha\in\pi} \left(\mathrm{e}^{\operatorname{ad}_q} e_\alpha - \mathrm{e}^{-\operatorname{ad}_q} f_\alpha\right), \quad L = q_t + \sum_{\alpha\in\pi} \left(\mathrm{e}^{\operatorname{ad}_q} e_\alpha + \mathrm{e}^{-\operatorname{ad}_q} f_\alpha\right)$$

- Generalized Toda system

$$q_{tt} = -\nabla_q U, \qquad U = \sum_{\alpha\in\pi} \exp(2\alpha(q)), \quad q(t) = \sum_{\alpha\in\pi} h_\alpha q^\alpha(t)$$

- Lax pair

$$L_t = [A, L]$$

$$A = -\sum_{\alpha\in\pi} \left(\mathrm{e}^{\mathrm{ad}_q} e_\alpha - \mathrm{e}^{-\mathrm{ad}_q} f_\alpha\right), \quad L = q_t + \sum_{\alpha\in\pi} \left(\mathrm{e}^{\mathrm{ad}_q} e_\alpha + \mathrm{e}^{-\mathrm{ad}_q} f_\alpha\right)$$

- The classical Yang–Baxter equation

$$\{L(\zeta) \underset{,}{\otimes} L(\xi)\} = [r(\zeta/\xi), L(\zeta)\otimes I + I\otimes L(\xi)], \qquad r(x) : \mathbb{C}^\times \to \mathfrak{G}\otimes\mathfrak{G}$$

$$[r_{12}(\zeta_{12}), r_{13}(\zeta_{13})] + [r_{12}(\zeta_{12}), r_{23}(\zeta_{23})] + [r_{13}(\zeta_{13}), r_{23}(\zeta_{23})] = 0$$

$$\zeta_{ij} = \zeta_i/\zeta_j, \qquad r_{12} = r \otimes I, \quad r_{23} = I \otimes r, \quad \ldots$$

- Quantization

$$R(\zeta) : \mathbb{C}^\times \to U_\hbar(\mathfrak{G}) \otimes U_\hbar(\mathfrak{G})$$

$$R(\zeta) \equiv R(\zeta, \hbar) = \rho(\zeta, \hbar)\,(I + \hbar\, r(\zeta) + \ldots)$$

- Quantization

$$R(\zeta) : \mathbb{C}^\times \longrightarrow U_\hbar(\mathfrak{G}) \otimes U_\hbar(\mathfrak{G})$$

$$R(\zeta) \equiv R(\zeta, \hbar) = \rho(\zeta, \hbar)\,(I + \hbar\, r(\zeta) + \ldots)$$

- Let $\mathcal{R} \in \mathcal{A} \otimes \mathcal{A}$ such that

$$\Delta^{\mathrm{op}}(a) = \mathcal{R}\Delta(a)\mathcal{R}^{-1}$$

for all $a \in \mathcal{A}$. Suppose $\mathcal{A}$ is *quasitriangular*

$$\Delta \otimes \mathrm{id}(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{23}, \qquad \mathrm{id} \otimes \Delta(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{12}.$$

Then $\mathcal{R}$ is the *universal R-matrix*. It satisfies *the QYBE*

$$\mathcal{R}_{12}\,\mathcal{R}_{13}\,\mathcal{R}_{23} = \mathcal{R}_{23}\,\mathcal{R}_{13}\,\mathcal{R}_{12}$$

- Quantization

$$R(\zeta) : \mathbb{C}^{\times} \to U_\hbar(\mathfrak{G}) \otimes U_\hbar(\mathfrak{G})$$
$$R(\zeta) \equiv R(\zeta, \hbar) = \rho(\zeta, \hbar)\,(I + \hbar\, r(\zeta) + \ldots)$$

- Let $\mathcal{R} \in \mathcal{A} \otimes \mathcal{A}$ such that

$$\Delta^{\mathrm{op}}(a) = \mathcal{R}\Delta(a)\mathcal{R}^{-1}$$

for all $a \in \mathcal{A}$. Suppose $\mathcal{A}$ is *quasitriangular*

$$\Delta \otimes \mathrm{id}(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{23}, \qquad \mathrm{id} \otimes \Delta(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{12}.$$

Then $\mathcal{R}$ is the *universal R-matrix*. It satisfies *the QYBE*

$$\mathcal{R}_{12}\,\mathcal{R}_{13}\,\mathcal{R}_{23} = \mathcal{R}_{23}\,\mathcal{R}_{13}\,\mathcal{R}_{12}$$

- Representation

$$R(\zeta_1, \zeta_2) := \varphi_{\zeta_1} \otimes \varphi_{\zeta_2}(\mathcal{R})$$

$$R_{12}(\zeta_{12})R_{13}(\zeta_{13})R_{23}(\zeta_{23}) = R_{23}(\zeta_{23})R_{13}(\zeta_{13})R_{12}(\zeta_{12})$$

$$R_{12}(\zeta_{12}) \in \mathrm{End}(V_1 \otimes V_2 \otimes \mathrm{id}_3), \quad R_{23}(\zeta_{23}) \in \mathrm{End}(\mathrm{id}_3 \otimes V_2 \otimes V_3), \quad \ldots$$

L-operators

- Defining

$$\hat{L}(\zeta_{12}) = \chi_{\zeta_1} \otimes \varphi_{\zeta_2}(\mathcal{R}), \qquad \chi_\zeta : \mathcal{A} \to \mathcal{C},$$

we obtain an element $\hat{L}(\zeta)$ called an *L-operator* and satisfying

$$\hat{R}(\zeta_{12})(\hat{L}(\zeta_1) \times \hat{L}(\zeta_2)) = (\hat{L}(\zeta_2) \times \hat{L}(\zeta_1))\hat{R}(\zeta_{12})$$

Here $\hat{R} = RP$ with $P(v_1 \otimes v_2) = v_2 \otimes v_1$.

L-operators

- Defining

$$\hat{L}(\zeta_{12}) = \chi_{\zeta_1} \otimes \varphi_{\zeta_2}(\mathcal{R}), \qquad \chi_\zeta : \mathcal{A} \to \mathcal{C},$$

we obtain an element $\hat{L}(\zeta)$ called an *L-operator* and satisfying

$$\hat{R}(\zeta_{12})(\hat{L}(\zeta_1) \times \hat{L}(\zeta_2)) = (\hat{L}(\zeta_2) \times \hat{L}(\zeta_1))\hat{R}(\zeta_{12})$$

Here $\hat{R} = RP$ with $P(v_1 \otimes v_2) = v_2 \otimes v_1$.

- Another type of L-operators

$$\check{L}(\zeta_{12}) = \varphi_{\zeta_1} \otimes \psi_{\zeta_2}(\mathcal{R}), \qquad \psi_\zeta : \mathcal{A} \to \mathcal{D}$$

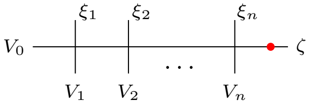
and the corresponding relations

$$\check{R}(\zeta_{12})(\check{L}(\zeta_1) \times \check{L}(\zeta_2)) = (\check{L}(\zeta_2) \times \check{L}(\zeta_1))\check{R}(\zeta_{12})$$

Now $\check{R} = PR$, and note that $\check{R} \neq \hat{R}$.

Traces of monodromy-type matrices

- Transfer matrix



$$T(\zeta;\xi_1,\xi_2,\dots,\xi_n|\alpha) = \mathrm{tr}_{\varphi_0}\left(R_{01}(\zeta/\xi_1)R_{02}(\zeta/\xi_2)\cdots R_{0n}(\zeta/\xi_n)\tau_\alpha\right)$$

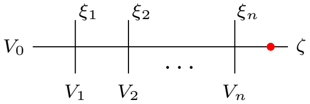
Here $R_{0i}(\zeta/\xi_i)$ acts on $V_0 \otimes V_1 \otimes \dots \otimes V_n$ and the trace is over V_0.

$$[T(\zeta;\xi_1,\dots,\xi_n), T(\zeta';\xi_1,\dots,\xi_n)] = 0$$

$$Z(\zeta_1,\dots,\zeta_m|\xi_1,\dots,\xi_n) = \mathrm{tr}\left(T(\zeta_1;\xi_1,\dots,\xi_n)\cdots T(\zeta_m;\xi_1,\dots,\xi_n)\right)$$

Traces of monodromy-type matrices

- Transfer matrix



$$T(\zeta;\xi_1,\xi_2,\dots,\xi_n|\alpha) = \mathrm{tr}_{\varphi_0}\left(R_{01}(\zeta/\xi_1)R_{02}(\zeta/\xi_2)\cdots R_{0n}(\zeta/\xi_n)\tau_\alpha\right)$$

Here $R_{0i}(\zeta/\xi_i)$ acts on $V_0 \otimes V_1 \otimes \ldots \otimes V_n$ and the trace is over V_0.

$$[T(\zeta;\xi_1,\dots,\xi_n), T(\zeta';\xi_1,\dots,\xi_n)] = 0$$

$$Z(\zeta_1,\dots,\zeta_m|\xi_1,\dots,\xi_n) = \mathrm{tr}\left(T(\zeta_1;\xi_1,\dots,\xi_n)\cdots T(\zeta_m;\xi_1,\dots,\xi_n)\right)$$

- Baxter's Q-matrix

$$Q(\zeta;\xi_1,\dots,\xi_n|\beta) = \mathrm{tr}_{\chi,\psi}\left(L(\zeta/\xi_1)\times\ldots\times L(\zeta/\xi_n)\kappa_\beta\right)$$

$$[T(\zeta;\xi), Q(\zeta';\xi)] = 0, \qquad [Q(\zeta;\xi), Q(\zeta';\xi)] = 0$$

$$T(\zeta;\xi)Q(\zeta;\xi) = A(\zeta/\xi)Q(q^{-2}\zeta;\xi) + B(\zeta/\xi)Q(q^2\zeta;\xi)$$

- An important observation:

$$\mathcal{R} \in \mathcal{B}_+ \otimes \mathcal{B}_- \subset \mathcal{A} \otimes \mathcal{A}$$

with $\mathcal{B}_+$ spanned by e_i, h_i, and $\mathcal{B}_-$ by f_i, h_i. Here $\mathcal{A}$ is $U_\hbar(\mathfrak{g}'(A))$

$$[h_i, h_j] = 0,$$
$$[h_i, e_j] = a_{ij}\, e_j, \qquad [h_i, f_j] = -a_{ij}\, f_j,$$
$$[e_i, f_j] = \delta_{ij}\, \frac{q^{h_i} - q^{-h_i}}{q^{d_i} - q^{-d_i}},$$
$$\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1 - a_{ij} \\ k \end{bmatrix}_{q^{d_i}} (e_i)^{1-a_{ij}-k} e_j (e_i)^k = 0, \qquad \& \{e_i \leftrightarrow f_i\}$$

- We have a Hopf algebra with comultiplication ($q = \mathrm{e}^\hbar$)

$$\Delta(h_i) = h_i \otimes 1 + 1 \otimes h_i,$$
$$\Delta(e_i) = e_i \otimes 1 + q^{-d_i h_i} \otimes e_i, \qquad \Delta(f_i) = f_i \otimes q^{d_i h_i} + 1 \otimes f_i$$

The prescription for untwisted affine Lie algebras

- The system $\Delta_+(A)$ is supplied with a *normal order*
  (i) all multiple roots follow each other in an arbitrary order;
  (ii) each nonsimple root $\alpha + \beta$, where α is not proportional to β, is to be placed between α and β.
  - We also add that for any root $\gamma \in \Delta_+(A)$

$$\gamma + m\delta \prec k\delta \prec (\delta - \gamma) + \ell\delta$$

The prescription for untwisted affine Lie algebras

- The system $\Delta_+(A)$ is supplied with a *normal order*
  (i) all multiple roots follow each other in an arbitrary order;
  (ii) each nonsimple root $\alpha + \beta$, where α is not proportional to β, is to be placed between α and β.
  - We also add that for any root $\gamma \in \Delta_+(A)$

$$\gamma + m\delta \prec k\delta \prec (\delta - \gamma) + \ell\delta$$

- Constructing the root vectors corresponding to the positive roots of $\mathfrak{g}(A^{(1)})$ from the root vectors corresponding to the simple positive roots $e_{\alpha_0} = e_{\delta-\theta}$ and e_{α_i}

The prescription for untwisted affine Lie algebras

- The system $\Delta_+(A)$ is supplied with a *normal order*
  - (i) all multiple roots follow each other in an arbitrary order;
  - (ii) each nonsimple root $\alpha + \beta$, where α is not proportional to β, is to be placed between α and β.
  - We also add that for any root $\gamma \in \Delta_+(A)$

$$\gamma + m\delta \prec k\delta \prec (\delta - \gamma) + \ell\delta$$

- Constructing the root vectors corresponding to the positive roots of $\mathfrak{g}(A^{(1)})$ from the root vectors corresponding to the simple positive roots $e_{\alpha_0} = e_{\delta-\theta}$ and e_{α_i}
- The universal R-matrix according to Khoroshkin and Tolstoy

$$\mathcal{R} = \mathcal{R}_{\prec\delta}\,\mathcal{R}_{\sim\delta}\,\mathcal{R}_{\succ\delta}\,\mathcal{K}$$

with $\mathcal{K}$ given by the expression

$$\mathcal{K} = \exp\left(\hbar \sum_{i,j=1}^{r} (b_{ij}\, h_{\alpha_i} \otimes h_{\alpha_j})\right), \qquad b\, a = I_r$$

- The 1st factor

$$\mathcal{R}_{\prec\delta} = \prod_{m\geq 0}^{\curvearrowright} \exp_{q^{-(\gamma,\gamma)}} \left((q - q^{-1})\, s_{m,\,\gamma}^{-1}\, e_{\gamma+m\delta} \otimes f_{\gamma+m\delta}\right),$$

where

$$[e_{\gamma+m\delta},\, f_{\gamma+m\delta}] = s_{m,\,\gamma}\, \frac{q^{h_{\gamma+m\delta}} - q^{-h_{\gamma+m\delta}}}{q - q^{-1}}$$

and $h_{\gamma+m\delta} = \sum_i k_i h_i$ if $\gamma + m\delta = \sum_i k_i \alpha_i$.

- The 1st factor

$$\mathcal{R}_{\prec\delta} = \prod_{m\geq 0}^{\curvearrowright} \exp_{q^{-(\gamma,\gamma)}} \left((q - q^{-1})\, s_{m,\gamma}^{-1}\, e_{\gamma+m\delta} \otimes f_{\gamma+m\delta} \right),$$

where

$$[e_{\gamma+m\delta},\, f_{\gamma+m\delta}] = s_{m,\gamma}\, \frac{q^{h_{\gamma+m\delta}} - q^{-h_{\gamma+m\delta}}}{q - q^{-1}}$$

and $h_{\gamma+m\delta} = \sum_i k_i h_i$ if $\gamma + m\delta = \sum_i k_i \alpha_i$.

- The 2nd factor

$$\mathcal{R}_{\sim\delta} = \exp\left((q - q^{-1}) \sum_{m>0} \sum_{i,j=1}^{r} u_{m,\,ij}\, e_{m\delta,\,\alpha_i} \otimes f_{m\delta,\,\alpha_j} \right),$$

where

$$[e_{\alpha_i+m\delta},\, e_{n\delta,\alpha_j}] = t_{n,ij}\, e_{\alpha_i+(m+n)\delta}, \qquad u_m\, t_m = I_r$$

- The 1st factor

$$\mathcal{R}_{\prec\delta} = \prod_{m\geq 0}^{\curvearrowright} \exp_{q^{-(\gamma,\gamma)}} \left((q - q^{-1})\, s_{m,\,\gamma}^{-1}\, e_{\gamma+m\delta} \otimes f_{\gamma+m\delta} \right),$$

where

$$[e_{\gamma+m\delta},\, f_{\gamma+m\delta}] = s_{m,\,\gamma}\, \frac{q^{h_{\gamma+m\delta}} - q^{-h_{\gamma+m\delta}}}{q - q^{-1}}$$

and $h_{\gamma+m\delta} = \sum_i k_i h_i$ if $\gamma + m\delta = \sum_i k_i \alpha_i$.

- The 2nd factor

$$\mathcal{R}_{\sim\delta} = \exp\left((q - q^{-1}) \sum_{m>0} \sum_{i,j=1}^{r} u_{m,\,ij}\, e_{m\delta,\,\alpha_i} \otimes f_{m\delta,\,\alpha_j} \right),$$

where

$$[e_{\alpha_i+m\delta},\, e_{n\delta,\alpha_j}] = t_{n,ij}\, e_{\alpha_i+(m+n)\delta}, \qquad u_m\, t_m = I_r$$

- The 3rd factor

$$\mathcal{R}_{\succ\delta} = \prod_{m\geq 0}^{\curvearrowright} \exp_{q^{-(\gamma,\gamma)}} \left((q - q^{-1})\, s_{m,\,\delta-\gamma}^{-1}\, e_{(\delta-\gamma)+m\delta} \otimes f_{(\delta-\gamma)+m\delta} \right)$$

- The universal R-matrix belongs to $U_\hbar(\mathfrak{b}'_+(A^{(1)})) \otimes U_\hbar(\mathfrak{b}'_-(A^{(1)}))$. We define φ_ζ, $\zeta \in \mathbb{C}^\times$, by the relations

$$\varphi_\zeta(h_{\alpha_i}) = \varphi(h_{\alpha_i}),$$
$$\varphi_\zeta(e_{\alpha_i}) = \zeta^{s_i}\varphi(e_{\alpha_i}), \qquad \varphi_\zeta(f_{\alpha_i}) = \zeta^{-s_i}\varphi(f_{\alpha_i}),$$

where $s_i \in \mathbb{Z}$ and $\varphi = \pi \circ \varepsilon$, with ε being a homomorphism of $U_\hbar(\mathfrak{g}'(A^{(1)}))$ to $U_\hbar(\mathfrak{g}(A))$, and π – a representation of $U_\hbar(\mathfrak{g}(A))$.

- The universal R-matrix belongs to $U_\hbar(\mathfrak{b}'_+(A^{(1)})) \otimes U_\hbar(\mathfrak{b}'_-(A^{(1)}))$. We define φ_ζ, $\zeta \in \mathbb{C}^\times$, by the relations

$$\varphi_\zeta(h_{\alpha_i}) = \varphi(h_{\alpha_i}),$$
$$\varphi_\zeta(e_{\alpha_i}) = \zeta^{s_i}\varphi(e_{\alpha_i}), \qquad \varphi_\zeta(f_{\alpha_i}) = \zeta^{-s_i}\varphi(f_{\alpha_i}),$$

  where $s_i \in \mathbb{Z}$ and $\varphi = \pi \circ \varepsilon$, with ε being a homomorphism of $U_\hbar(\mathfrak{g}'(A^{(1)}))$ to $U_\hbar(\mathfrak{g}(A))$, and π – a representation of $U_\hbar(\mathfrak{g}(A))$.
- For L-operators of type $\hat{L}$ we take $\chi_\zeta : U_\hbar(\mathfrak{b}'_+(A^{(1)})) \to \mathcal{C}$, and for L-operators of type $\check{L}$ we take $\psi_\zeta : U_\hbar(\mathfrak{b}'_-(A^{(1)})) \to \mathcal{D}$.

- The universal R-matrix belongs to $U_\hbar(\mathfrak{b}'_+(A^{(1)})) \otimes U_\hbar(\mathfrak{b}'_-(A^{(1)}))$. We define φ_ζ, $\zeta \in \mathbb{C}^\times$, by the relations

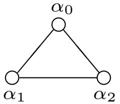
$$\varphi_\zeta(h_{\alpha_i}) = \varphi(h_{\alpha_i}),$$
$$\varphi_\zeta(e_{\alpha_i}) = \zeta^{s_i}\varphi(e_{\alpha_i}), \qquad \varphi_\zeta(f_{\alpha_i}) = \zeta^{-s_i}\varphi(f_{\alpha_i}),$$

  where $s_i \in \mathbb{Z}$ and $\varphi = \pi \circ \varepsilon$, with ε being a homomorphism of $U_\hbar(\mathfrak{g}'(A^{(1)}))$ to $U_\hbar(\mathfrak{g}(A))$, and π – a representation of $U_\hbar(\mathfrak{g}(A))$.
- For L-operators of type $\hat{L}$ we take $\chi_\zeta : U_\hbar(\mathfrak{b}'_+(A^{(1)})) \to \mathcal{C}$, and for L-operators of type $\check{L}$ we take $\psi_\zeta : U_\hbar(\mathfrak{b}'_-(A^{(1)})) \to \mathcal{D}$.
- Here, for the unital associative algebras $\mathcal{C}$ and $\mathcal{D}$ we take the tensor product of q-deformed oscillator algebras, each defined by the generators a_i, $a_i^\dagger$ and D_i, $i = 1, \ldots, r$, with the relations

$$[D_i, a_i] = -a_i, \qquad [D_i, a_i^\dagger] = a_i^\dagger,$$
$$a_i a_i^\dagger = 1 - q^2 q^{2D_i}, \qquad a_i^\dagger a_i = 1 - q^{2D_i}$$

- Of special interest

$$A_2^{(1)} = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$$



$\delta = \alpha_0 + \alpha_1 + \alpha_2$

Taking $\pi = \pi^{(1,0)}$ we come to the R-matrix

$$R(\zeta) = q^{2/3} \mathrm{e}^{\lambda_3(q^2\zeta^s) - \lambda_3(q^{-2}\zeta^s)} \Big[E_{11} \otimes E_{11} + E_{22} \otimes E_{22} + E_{33} \otimes E_{33}$$
$$+ \frac{q^{-1}(1-\zeta^s)}{1 - q^{-2}\zeta^s} (E_{11} \otimes E_{22} + E_{11} \otimes E_{33} + E_{22} \otimes E_{11} + E_{22} \otimes E_{33} + E_{33} \otimes E_{11}$$
$$+ E_{33} \otimes E_{22}) + \frac{1 - q^{-2}}{1 - q^{-2}\zeta^s} (\zeta^{s_1} E_{12} \otimes E_{21} + \zeta^{s_1+s_2} E_{13} \otimes E_{31} + \zeta^{s_2} E_{23} \otimes E_{32}$$
$$+ \zeta^{s-s_1} E_{21} \otimes E_{12} + \zeta^{s-s_1-s_2} E_{31} \otimes E_{13} + \zeta^{s-s_2} E_{32} \otimes E_{23}) \Big],$$

where

$$\lambda_3(\zeta) = \sum_{m \in \mathbb{Z}_+} \frac{1}{q^{2m} + 1 + q^{-2m}} \frac{\zeta^m}{m} = \sum_{m \in \mathbb{Z}_+} \frac{1}{[3]_{q^m}} \frac{\zeta^m}{m},$$

$$\lambda_3(q^2\zeta) + \lambda_3(\zeta) + \lambda_3(q^{-2}\zeta) = -\log(1-\zeta)$$

- Gauge invariance:

$$R^{(s,s_1,s_2)}(\zeta_{12}) = [G(\zeta_1) \otimes G(\zeta_2)] R^{(1,0,0)}(\zeta_{12}^s) [G(\zeta_1) \otimes G(\zeta_2)]^{-1},$$

where

$$G(\zeta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta^{-s_1} & 0 \\ 0 & 0 & \zeta^{-s_1-s_2} \end{pmatrix}$$

- Gauge invariance:

$$R^{(s,s_1,s_2)}(\zeta_{12}) = [G(\zeta_1) \otimes G(\zeta_2)] R^{(1,0,0)}(\zeta_{12}^s) [G(\zeta_1) \otimes G(\zeta_2)]^{-1},$$

where

$$G(\zeta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta^{-s_1} & 0 \\ 0 & 0 & \zeta^{-s_1 - s_2} \end{pmatrix}$$

- The *transfer matrices* of inhomogeneous vertex models related to R-matrices with different choices of s, s_1 and s_2 are connected by a similarity transformation and a change of the spectral parameters. The corresponding *partition functions* in the case of the toroidal boundary conditions are connected by a change of the spectral parameters.
Analogously, the Q-*operators* obtained from L-operators corresponding to different values of s, s_1 and s_1 are connected by a change of the spectral parameters and a similarity transformation.

L-operators of type $\hat{L}$

- Choosing the evaluation map

$$\chi_\zeta(h_{\delta-\alpha-\beta}) = -D_1 - D_2, \quad \chi_\zeta(h_\alpha) = 2D_1 - D_2, \quad \chi_\zeta(h_\beta) = -D_1 + 2D_2,$$

$$\chi_\zeta(e_{\delta-\alpha-\beta}) = \frac{1}{q-q^{-1}}\, a_1\, a_2\, q^{-D_1-2D_2}\, \zeta^{s-s_1-s_2},$$

$$\chi_\zeta(e_\alpha) = \frac{1}{q-q^{-1}}\, a_1^\dagger\, \zeta^{s_1}, \qquad \chi_\zeta(e_\beta) = \frac{1}{q-q^{-1}}\, q^{D_1} a_2^\dagger\, \zeta^{s_2},$$

we come to the L-operator ($s = 1$, $s_1 = s_2 = 0$ for brevity)

$$\hat{L}(\zeta) = \mathrm{e}^{\lambda_3(q^{-2}\zeta)} \begin{pmatrix} q^{D_1} & q^{-2} a_1\, q^{-D_1-D_2}\, \zeta & a_1 a_2\, q^{-D_1-3D_2}\, \zeta \\ a_1^\dagger\, q^{D_1} & q^{-D_1+D_2} - q^{-2}\, q^{D_1-D_2} \zeta & -a_2\, q^{D_1-3D_2}\, \zeta \\ 0 & a_2^\dagger\, q^{D_2} & q^{-D_2} \end{pmatrix}$$

- The evaluation map

$$\chi_\zeta(h_{\delta-\alpha-\beta}) = -D_1 - D_2, \quad \chi_\zeta(h_\alpha) = 2D_1 - D_2, \quad \chi_\zeta(h_\beta) = -D_1 + 2D_2,$$

$$\chi_\zeta(e_{\delta-\alpha-\beta}) = -\frac{1}{q-q^{-1}}\, a_1 a_2\, q^{-D_1} \zeta^{s-s_1-s_2},$$

$$\chi_\zeta(e_\alpha) = \frac{1}{q-q^{-1}}\, a_1^\dagger\, \zeta^{s_1}, \qquad \chi_\zeta(e_\beta) = \frac{1}{q-q^{-1}}\, q^{-D_1} a_2^\dagger\, \zeta^{s_2}$$

leads to the L-operator ($s = 1$, $s_1 = s_2 = 0$ for brevity)

$$\hat{L}(\zeta) = \frac{\mathrm{e}^{-\lambda_3(q^2\zeta)}}{1-\zeta} \begin{pmatrix} q^{D_1} - q^{-2} q^{-D_1}\zeta & -a_1\, q^{-3D_1+D_2}\,\zeta & -a_1 a_2\, q^{-D_1-D_2}\,\zeta \\ a_1^\dagger\, q^{D_1} & q^{-D_1+D_2} & a_2\, q^{D_1-D_2}\,\zeta \\ q^{-1} a_1^\dagger a_2^\dagger & a_2^\dagger\, q^{-2D_1+D_2} & q^{-D_2} - q^{D_2}\zeta \end{pmatrix}$$

✓ Gauge invariance

$$\hat{L}^{(s,s_1,s_2)}(\zeta_{12}) = \gamma_{\zeta_1}(G(\zeta_2)\hat{L}^{(1,0,0)}(\zeta_{12}^s)G^{-1}(\zeta_2)),$$

where γ_ζ, $\zeta \in \mathbb{C}^\times$, is given by

$$\gamma_\zeta(a_i) = a_i \zeta^{-s_i}, \qquad \gamma_\zeta(a_i^\dagger) = a_i^\dagger \zeta^{s_i}, \qquad \gamma_\zeta(D_i) = D_i, \qquad i = 1, 2.$$

L-operators of type $\check{L}$

- Here we work with the mapping ψ_ζ given by

$$\psi_\zeta(h_{\delta-\alpha-\beta}) = -D_1 - D_2, \quad \psi_\zeta(h_\alpha) = 2D_1 - D_2, \quad \psi_\zeta(h_\beta) = -D_1 + 2D_2,$$

$$\psi_\zeta(f_{\delta-\alpha-\beta}) = \frac{1}{q-q^{-1}}\, q^{-D_1-2D_2} a_1^\dagger a_2^\dagger\, \zeta^{-s+s_1+s_2},$$

$$\psi_\zeta(f_\alpha) = \frac{1}{q-q^{-1}}\, a_1 \zeta^{-s_1}, \qquad \psi_\zeta(f_\beta) = \frac{1}{q-q^{-1}}\, a_2\, q^{D_1} \zeta^{-s_2}.$$

This leads to the L-operator (again $s=1$, $s_1=s_2=0$)

$$\check{L}(\zeta) = \mathrm{e}^{\lambda_3(q^{-2}\zeta)} \begin{pmatrix} q^{D_1} & a_1\, q^{-D_1+D_2} & 0 \\ q^{-2}\, a_1^\dagger\, q^{D_1-2D_2}\, \zeta & q^{-D_1+D_2} - q^{-2} q^{D_1-D_2} \zeta & a_2\, q^{D_1-D_2} \\ q^{-3}\, a_1^\dagger a_2^\dagger\, q^{-2D_2}\, \zeta & -q^{-2}\, a_2^\dagger\, q^{-D_2}\, \zeta & q^{-D_2} \end{pmatrix}$$

- While choosing the evaluation map

$$\psi_\zeta(h_{\delta-\alpha-\beta}) = -D_1 - D_2, \quad \psi_\zeta(h_\alpha) = 2D_1 - D_2, \quad \psi_\zeta(h_\beta) = -D_1 + 2D_2,$$
$$\psi_\zeta(f_{\delta-\alpha-\beta}) = -\frac{1}{q-q^{-1}}\, q^{-D_1}\, a_1^\dagger\, a_2^\dagger\, \zeta^{-s+s_1+s_2},$$
$$\psi_\zeta(f_\alpha) = \frac{1}{q-q^{-1}}\, a_1\, \zeta^{-s_1}, \qquad \psi_\zeta(f_\beta) = \frac{1}{q-q^{-1}}\, a_2\, q^{-D_1} \zeta^{-s_2},$$

we obtain one more L-operator (here $s=1$, $s_1 = s_2 = 0$)

$$\check{L}(\zeta) = \frac{\mathrm{e}^{-\lambda_3(q^2\zeta)}}{1-\zeta} \begin{pmatrix} q^{D_1} - q^{-2} q^{-D_1}\zeta & a_1\, q^{-D_1+D_2} & a_1 a_2\, q^{-D_1-D_2} \\ -q^{-2}\, a_1^\dagger\, q^{-D_1}\,\zeta & q^{-D_1+D_2} & a_2\, q^{-D_1-D_2} \\ -q^{-1}\, a_1^\dagger a_2^\dagger\,\zeta & a_2^\dagger\, q^{D_2}\,\zeta & q^{-D_2} - q^{D_2}\zeta \end{pmatrix}$$

✓ Gauge invariance

$$\check{L}^{(s,s_1,s_2)}(\zeta_{12}) = G(\zeta_1)\gamma_{\zeta_2}(\check{L}^{(1,0,0)}(\zeta_{12}^s))G^{-1}(\zeta_1)$$