

# Integrability of high energy scattering amplitudes in $N = 4$ SUSY

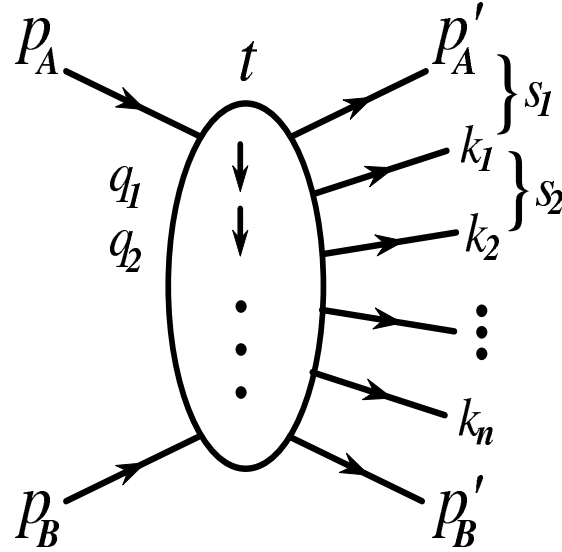
L. N. Lipatov

Petersburg Nuclear Physics Institute



1. Multi-Regge processes
2. Integrability of the BFKL dynamics
3. Pomeron in  $N = 4$  SUSY
4. BDS amplitudes at large energies
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6. Integrable open Heisenberg spin chain
8. Discussion

# 1 Amplitudes in multi-Regge kinematics



$$M_{2 \rightarrow 2+n}^{BFKL} \sim \frac{s_1^{\omega_1}}{|q_1|^2} g T_{c_2 c_1}^{d_1} C(q_2, q_1) \frac{s_2^{\omega_2}}{|q_2|^2} \dots g T_{c_{n+1} c_n}^{d_n} C(q_{n+1}, q_n) \frac{s_{n+1}^{\omega_{n+1}}}{|q_{n+1}|^2},$$

Reggeon-Reggeon-gluon vertex

$$C(q_2, q_1) = \frac{q_2 q_1^*}{q_2^* - q_1^*}, \quad \omega_r = -\frac{\alpha_s N_c}{2\pi} \left( \ln \frac{|q_r^2|}{\mu^2} - \frac{1}{\epsilon} \right), \quad \sigma_t = \sum_n \int d\Gamma_n |M_{2 \rightarrow 2+n}|^2$$

## 2 Analyticity, unitarity and bootstrap

Steinmann relations for overlapping channels

$$\Delta_{s_r} \Delta_{s_{r+1}} M_{2 \rightarrow 2+n} = 0$$

Dispersion representation for  $M_{2 \rightarrow 3}$  in the Regge ansatz

$$M_{2 \rightarrow 3} = c_1 (-s)^{j(t_2)} (-s_1)^{j(t_1) - j(t_2)} + c_2 (-s)^{j(t_1)} (-s_2)^{j(t_2) - j(t_1)}$$

Dispersion representation for  $M_{2 \rightarrow 4}$  in the Regge ansatz

$$\begin{aligned} M_{2 \rightarrow 4} = & d_1 (-s)^{j_3} (-s_{012})^{j_2 - j_3} (-s_1)^{j_1 - j_2} + d_2 (-s)^{j_1} (-s_{123})^{j_2 - j_1} (-s_3)^{j_3 - j_2} \\ & + d_3 (-s)^{j_3} (-s_{012})^{j_1 - j_3} (-s_2)^{j_2 - j_1} + d_4 (-s)^{j_1} (-s_{123})^{j_3 - j_1} (-s_2)^{j_2 - j_3} \\ & + d_5 (-s)^{j_2} (-s_1)^{j_1 - j_2} (-s_3)^{j_3 - j_2}, \quad j_r = j(t_r) \end{aligned}$$

Bootstrap relation in LLA (BFKL (1975-1978))

$$\pi \omega(t_1) M_{2 \rightarrow 2+n} = \sum_r \mathfrak{S}_{s_{0r}} M_{2 \rightarrow 2+n} = \sum_t M_{2 \rightarrow 2+t} M_{2+t \rightarrow 2+n}$$

### 3 BFKL and BKP equations

Balitsky-Fadin-Kuraev-Lipatov equation

$$E \Psi(\vec{\rho}_1, \vec{\rho}_2) = H_{12} \Psi(\vec{\rho}_1, \vec{\rho}_2), \quad \sigma_t \sim s^\Delta, \quad \Delta = -\frac{\alpha_s N_c}{2\pi} E,$$

Holomorphic separability

$$H_{12} = h_{12} + h_{12}^*, \quad h_{12} = \ln(p_1 p_2) + \frac{1}{p_1} \ln(\rho_{12}) p_1 + \frac{1}{p_2} \ln(\rho_{12}) p_2 - 2\gamma$$

Bartels-Kwiecinski-Praszalowicz equation at  $N_c \rightarrow \infty$

$$E \Psi(\vec{\rho}_1, \dots, \vec{\rho}_n) = H \Psi(\vec{\rho}_1, \dots, \vec{\rho}_n), \quad H = \frac{1}{2} \sum_{k=1}^n H_{k,k+1} = h + h^*$$

Holomorphic factorization and Möbius invariance (L.L.)

$$\Psi(\vec{\rho}_1, \vec{\rho}_2, \dots, \vec{\rho}_n) = \sum_{r,s} a_{r,s} \Psi_r(\rho_1, \dots, \rho_n) \Psi_s(\rho_1^*, \dots, \rho_n^*),$$

$$M_a^2 \Psi_r = m(m-1) \Psi_r, \quad M_a^{*2} \Psi_s = \tilde{m}(\tilde{m}-1) \Psi_s, \quad m = \frac{1}{2} + i\nu + \frac{n}{2}$$

## 4 Integrability at $N_c \rightarrow \infty$

Monodromy and transfer matrices (L. (1993))

$$t(u) = L_1 \dots L_n = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}, \quad L_k = \begin{pmatrix} u + \rho_k p_k & p_k \\ -\rho_k^2 p_k & u - \rho_k p_k \end{pmatrix},$$

$$T(u) = A(u) + D(u), \quad [T(u), h] = 0$$

Yang-Baxter equation (L. (1993))

$$t_{r'_1}^{s_1}(u) t_{r'_2}^{s_2}(v) l_{r_1 r_2}^{r'_1 r'_2}(v - u) = l_{s'_1 s'_2}^{s_1 s_2}(v - u) t_{r_2}^{s'_2}(v) t_{r_1}^{s'_1}(u), \quad \hat{l} = u \hat{1} + i \hat{P}$$

Duality symmetry (L. (1999))

$$p_r \rightarrow \rho_{r+1, r} \rightarrow p_{r+1}$$

Heisenberg spin model (L. (1994); F., K.(1995))

$$\vec{S}_k = (\rho_k \partial_k, \partial_k, -\rho_k^2 \partial_k)$$

# 5 Pomeron in $N = 4$ SUSY

BFKL kernel in two loops in  $N = 4$  SUSY (K.,L. (2000))

$$\omega = 4 \hat{a} (2\Psi(1) - \Psi(M) - \Psi(M^*)) + 4 \hat{a}^2 \Delta(n, \gamma), \quad \hat{a} = \frac{g^2 N_c}{16\pi^2},$$

$$\Delta(n, \gamma) = \phi(M) + \phi(M^*) - \frac{\rho(M) + \rho(M^*)}{2\hat{a}/\omega}, \quad M = \gamma + \frac{|n|}{2},$$

$$\rho(M) = \beta'(M) + \frac{1}{2}\zeta(2), \quad \beta'(z) = \frac{1}{4} \left[ \Psi'\left(\frac{z+1}{2}\right) - \Psi'\left(\frac{z}{2}\right) \right]$$

Maximal transcendentality (2002) and integrability for  $\gamma$  (L (1997))

$$\phi(M) = 3\zeta(3) + \Psi''(M) - 2\Phi(M) + 2\beta'(M) \left( \Psi(1) - \Psi(M) \right),$$

$$\Phi(M) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k+M} \left( \Psi'(k+1) - \frac{\Psi(k+1) - \Psi(1)}{k+M} \right)$$

Large coupling asymptotics for the intercept (KLOV, BPST)

$$j = 2 - \frac{1}{2\pi} \hat{a}^{-1/2}$$

## 6 Elastic BDS amplitude at $N = 4$ SUSY

Regge asymptotics at  $s/t \rightarrow \infty$

$$M_{2 \rightarrow 2}^{BDS} = \Gamma(t) \left( \frac{-s}{\mu^2} \right)^{\omega(t)} \Gamma(t)$$

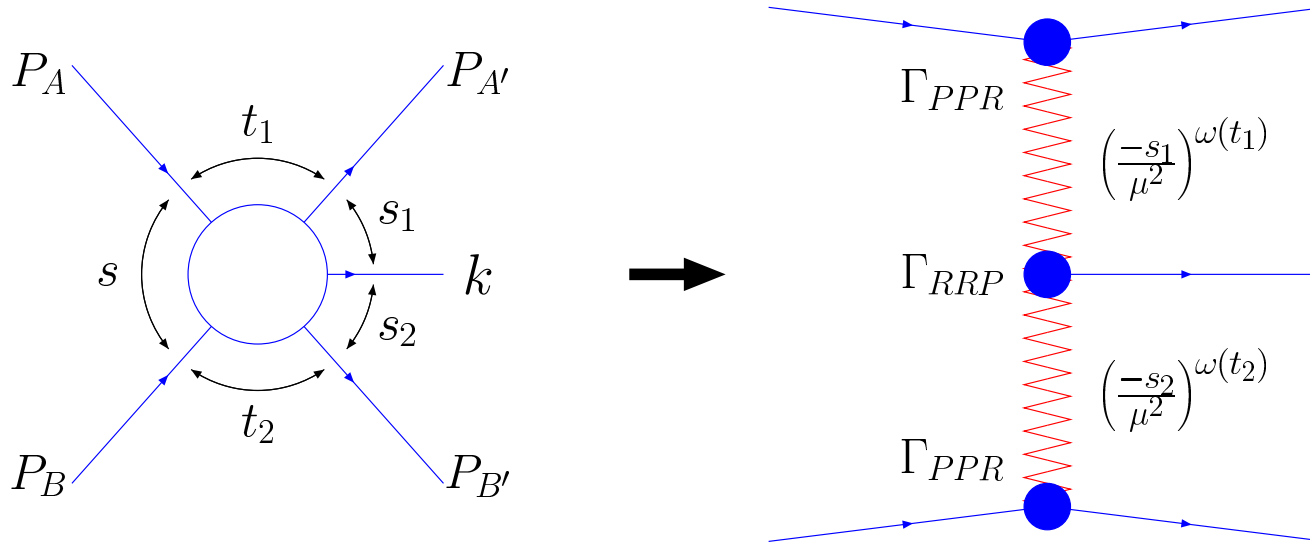
Reggeized gluon trajectory

$$\omega(t) = -\frac{\gamma_K(a)}{4} \ln \frac{-t}{\mu^2} + \int_0^a \frac{da'}{a'} \left( \frac{\gamma_K(a')}{4\epsilon} + \beta(a') \right)$$

Reggeon residues

$$\begin{aligned} \ln \Gamma(t) = & \ln \frac{-t}{\mu^2} \int_0^a \frac{da'}{a'} \left( \frac{\gamma_K(a')}{8\epsilon} + \frac{\beta(a')}{2} \right) + \frac{C(a)}{2} + \frac{\gamma_K(a)}{2} \zeta_2 \\ & - \int_0^a \frac{da'}{a'} \ln \frac{a}{a'} \left( \frac{\gamma_K(a')}{4\epsilon^2} + \frac{\beta(a')}{\epsilon} + \delta(a') \right) \end{aligned}$$

# 7 One particle production (BLS)



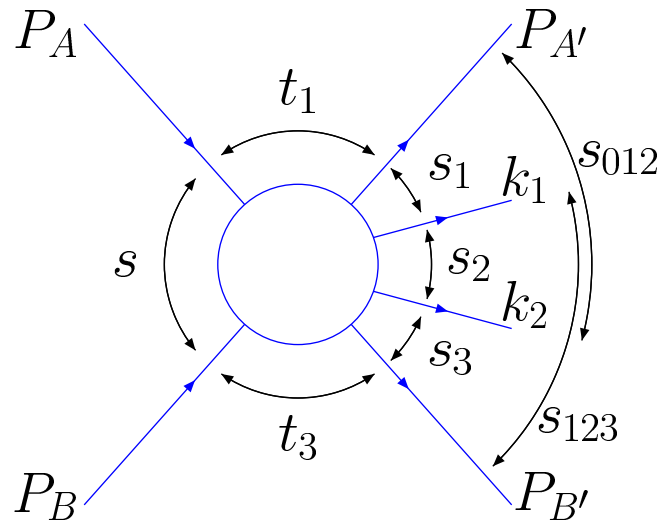
Reggeon-Reggeon-gluon vertex

$$\ln \Gamma_{21} = -\frac{1}{2} \left( \omega(t_1) + \omega(t_2) - \int_0^a \frac{da'}{a'} \left( \frac{\gamma_K(a')}{4\epsilon} + \beta(a') \right) \right) \ln \frac{-k_{\perp}^2}{\mu^2} -$$

$$\frac{\gamma_K(a)}{16} \left( \ln^2 \frac{-k_{\perp}^2}{\mu^2} - \ln^2 \frac{-t_1}{-t_2} - \zeta_2 \right) - \frac{1}{2} \int_0^a \frac{da'}{a'} \ln \frac{a}{a'} \left( \frac{\gamma_K(a')}{4\epsilon^2} + \frac{\beta(a')}{\epsilon} + \delta(a') \right)$$



## 8 Regge factorization violation (BLS)



$$M_{2 \rightarrow 4} |_{s, s_2 > 0; s_1, s_3 < 0} = C \Gamma_1 \left( \frac{-s_1}{\mu^2} \right)^{\omega(t_1)} \Gamma_{21} \left( \frac{-s_2}{\mu^2} \right)^{\omega(t_2)} \Gamma_{32} \left( \frac{-s_3}{\mu^2} \right)^{\omega(t_3)} \Gamma_3,$$

$$C = \exp \left[ \frac{\gamma_K(a)}{4} i\pi \left( \ln \frac{t_1 t_2}{(\vec{k}_1 + \vec{k}_2)^2 \mu^2} - \frac{1}{\epsilon} \right) \right]$$

# 9 Mandelstam cuts in $j_2$ -plane

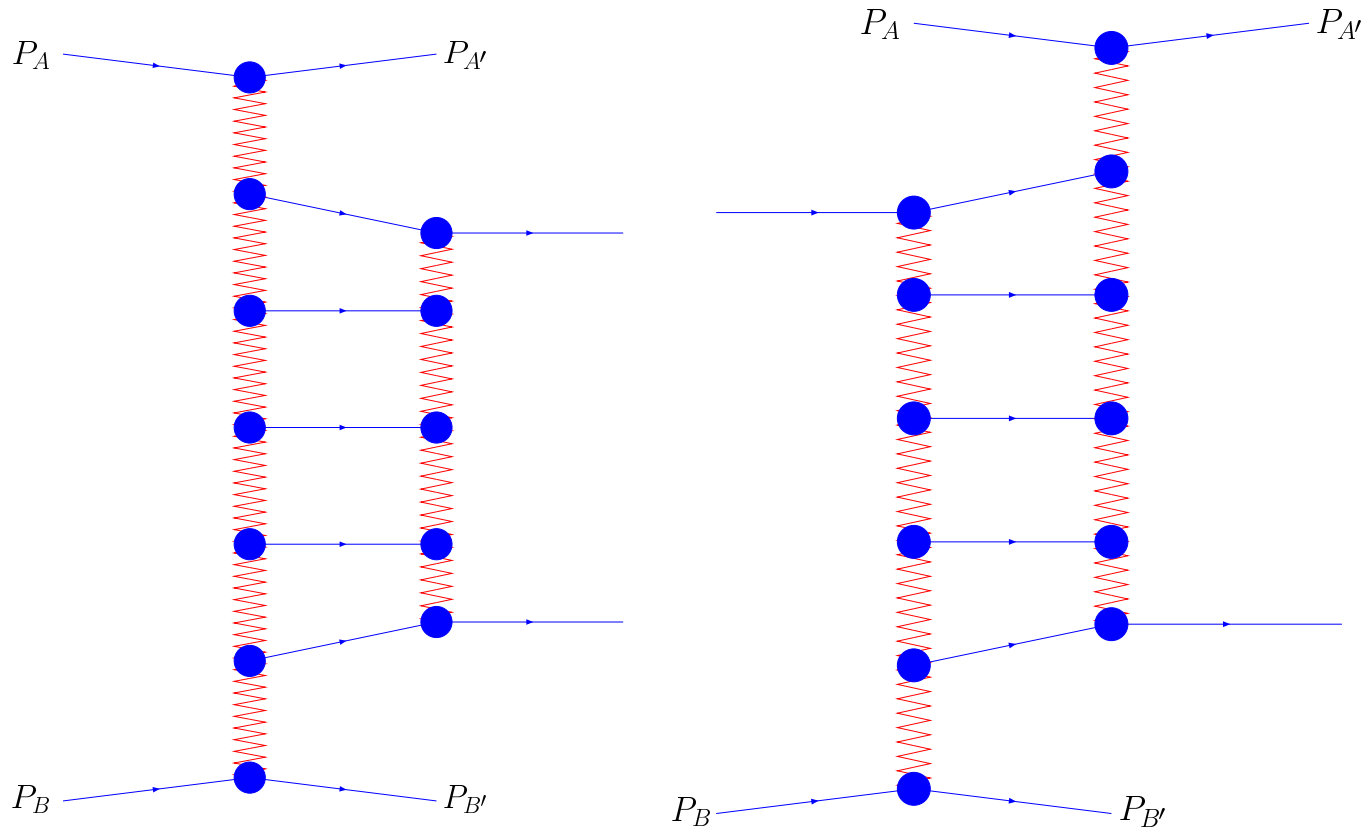


Figure 1: BFKL ladders in  $M_{2 \rightarrow 4}$  and  $M_{3 \rightarrow 3}$

# 10 BFKL equation for octets (BLS)

Regge singularity trajectories

$$\omega(t_2) = -a \left( E + \ln \frac{-t_2}{\mu^2} - \frac{1}{\epsilon} \right), \quad \Delta = -aE$$

BFKL hamiltonian for partial waves  $f_{j_2}$

$$H = \ln \frac{|p_1 p_2|^2}{|p_1 + p_2|^2} + \frac{1}{2} p_1 p_2^* \ln |\rho_{12}|^2 \frac{1}{p_1 p_2^*} + \frac{1}{2} p_1^* p_2 \ln |\rho_{12}|^2 \frac{1}{p_1^* p_2} + 2\gamma$$

Eigenfunctions and eigenvalues

$$\Psi_{n,\nu} = \left( \frac{p_1}{p_2} \right)^{i\nu+n/2} \left( \frac{p_1^*}{p_2^*} \right)^{i\nu-n/2}, \quad E_{n,\nu} = 2\text{Re} \psi(i\nu + \frac{|n|}{2}) - 2\psi(1)$$

Factorization of infrared divergencies in LLA

$$M_{|s,s_2>0;s_1,s_3<0}^{2\rightarrow 4} = (1 + i\delta_{2\rightarrow 4}) M_{2\rightarrow 4}^{BDS},$$

# 11 Möbius and conformal invariances

Analytic result in LLA in the region  $a \ln s_2 \sim 1$

$$\delta_{2 \rightarrow 4} = \frac{a}{2} \sum_{n=-\infty}^{\infty} (-1)^n \int_{-\infty}^{\infty} \frac{d\nu}{\nu^2 + \frac{n^2}{4}} (V^*)^{i\nu - \frac{n}{2}} V^{i\nu + \frac{n}{2}} \left( s_2^{\Delta(\nu, n)} - 1 \right)$$

Duality transformation to the Möbius representation

$$V = \frac{q_3 k_1}{k_2 q_1} \rightarrow \frac{z_{03} z_{0'1}}{z_{0'3} z_{01}}$$

Perturbation theory expansion

$$i\delta_{2 \rightarrow 4} = -2i\pi a^2 \ln s_2 \ln \frac{|k_1 + k_2||q_2|}{|k_2||q_1|} \ln \frac{|k_1 + k_2||q_2|}{|k_1||q_3|} + \dots$$

Functions of 4-dimensional anharmonic ratios

$$i\delta_{2 \rightarrow 4} = \frac{a^2}{4} Li_2(\chi) \ln \frac{\chi t_2 s_{13}}{s_3 t_1} \ln \frac{\chi t_2 s_{02}}{t_3 s_1} + \dots, \quad \chi = 1 - \frac{s s_2}{s_{012} s_{123}}$$

# 12 Factorization and exponentiation

## Factorization hypothesis

$$M_{2\rightarrow 4} = c M_{2\rightarrow 4}^{BDS} = M_{2\rightarrow 4}^{pole} + M_{2\rightarrow 4}^{cut}, \quad c = 1 + i\Delta_{2\rightarrow 4} + \dots$$

BDS ansatz at  $s, s_2 > 0, s_1, s_3 < 0$

$$M_{2\rightarrow 4}^{BDS} = |M_{2\rightarrow 4}^{BDS}| C e^{i\pi(\omega_a + \omega_b)}, \quad \omega_1 \approx \frac{a}{2} \ln \frac{k_1^2 \lambda^2}{q_1^2 q_2^2}, \quad \omega_2 \approx \frac{a}{2} \ln \frac{k_2^2 \lambda^2}{q_2^2 q_3^2}$$

## Regge pole contribution

$$M_{2\rightarrow 4}^{pole} = |M_{2\rightarrow 4}^{BDS}| \left( e^{-i\pi\omega_2} 2 \sin \pi\omega_a \sin \pi\omega_b + e^{-i\pi\omega_2} e^{i\pi(\omega_a + \omega_b)} \right)$$

## Mandelstam cut contribution

$$M_{2\rightarrow 4}^{cut} = i \int_{-i\infty}^{i\infty} \frac{d\omega_{2'}}{2\pi i} (-s_2)^{\omega_{2'}} f_{2\rightarrow 4}(\omega_{2'})$$

Factor  $c$  is not a pure phase

# 13 Multi-gluon states in octet channels

Channels with the Mandelstam cuts constructed from  $n$  gluons

$$s_1, s_2, \dots, s_{n-1}, s_{n+1}, \dots, s_{2n} < 0, s, s_n > 0$$

Schrödinger equation for octet composite states

$$H\Psi = E\Psi, \quad \omega(t) = a \left( -\ln \frac{-t}{\mu^2} + \frac{1}{\epsilon} \right) - \frac{a}{2}E, \quad a = \frac{g^2 N_c}{8\pi^2}$$

Holomorphic separability

$$H = h + h^*, \quad h = \ln \frac{p_1 p_n}{q^2} + \sum_{r=1}^{n-1} h_{r,r+1}^t$$

Helpful operator relation

$$\ln \partial = -\ln x + \frac{1}{2} (\psi(x\partial + 1) + \psi(-x\partial))$$

# 14 Möbius invariance in the $p$ -space

Duality transformation

$$p_k = Z_{k-1,k}, \quad \rho_{k,k+1} = i \frac{\partial}{\partial Z_k} = i \partial_k$$

Consequence of the Möbius invariance in  $Z$ -space

$$h = \ln(Z_1^2 \partial_1) - 2\psi(1) + \ln \partial_{n-1} + \sum_{k=1}^{n-2} h_{k,k+1}, \quad Z_0 = 0, \quad Z_n = \infty,$$

$$h_{1,2} = \ln(Z_{12}^2 \partial_1) + \ln(Z_{12}^2 \partial_2) - 2 \ln Z_{12} - 2\psi(1)$$

First integral of motion

$$A' = Z_1 \prod_{s=1}^{n-2} Z_{s,s+1} \prod_{r=1}^{n-1} \partial_r, \quad [h, A'] = 0$$

# 15 Integrable Heisenberg spin chain

Helpful identity

$$[L_k(u)L_{k+1}(u), h_{k,k+1}] = -i(L_k(u) - L_{k+1}(u))$$

Integrals of motion:  $[D, h] = 0$

$$D(u) = \sum_{k=0}^{n-1} u^{n-1-k} q'_k, \quad q'_k = - \sum_{0 < r_1 < \dots < r_k < n} Z_{r_1} \prod_{s=1}^{k-1} Z_{r_s, r_{s+1}} \prod_{t=1}^k i \partial_{r_t}$$

Sklyanin ansatz and Baxter equation

$$\Omega = \prod_k Q(\hat{u}_k) \Omega_0, \quad \Omega_0 = \prod_{l=1}^{n-1} \frac{1}{|Z_l|^4},$$

$$D(u)Q(u) = (u + i)^{n-1} Q(u + i)$$



# 16 Three-gluon composite state

Wave function in the coordinate representation

$$\Psi = Z_2^{a_1+a_2} (Z_2^*)^{\tilde{a}_1+\tilde{a}_2} \int \frac{d^2 y}{|y|^2} y^{-a_2} (y^*)^{\tilde{a}_2} \left( \frac{y-1}{y-Z_2/Z_1} \right)^{a_1} \left( \frac{y^*-1}{y^*-Z_2^*/Z_1^*} \right)^{\tilde{a}_1}$$

Fourier transformation

$$\Psi(\vec{Z}_1, \vec{Z}_2) = \int d^2 p_1 d^2 p_2 \exp(i\vec{p}_1 \vec{Z}_1) \exp(i\vec{p}_2 \vec{Z}_2) \Psi(\vec{p}_1, \vec{p}_2), \quad E = E(a_1) + E(a_2)$$

Baxter-Sklyanin representation

$$\Psi^t(\vec{p}_1, \vec{p}_2) = P^{-a_1-a_2} (P^*)^{-\tilde{a}_1-\tilde{a}_2} \int d^2 u u \tilde{u} Q(u, \tilde{u}) \left( \frac{p_1}{p_2} \right)^u \left( \frac{p_1^*}{p_2^*} \right)^{u^*}$$

Baxter function

$$Q(u, \tilde{u}) = \frac{\Gamma(-u) \Gamma(-\tilde{u})}{\Gamma(1+u) \Gamma(1+\tilde{u})} \frac{\Gamma(u-a_1) \Gamma(u-a_2)}{\Gamma(1-\tilde{u}+\tilde{a}_1) \Gamma(1-\tilde{u}+\tilde{a}_2)}, \quad \int d^2 u = \int d\nu \sum_n$$

# 17 Discussion

1. Steinmann relations and bootstrap.
2. Integrability of BFKL dynamics in LLA.
3. Remarkable properties of NLLA in  $N = 4$  SUSY.
4. BDS amplitudes in the multi-Regge kinematics.
5. Breakdown of the Regge factorization.
6. Mandelstam cuts in planar amplitudes at  $n \geq 6$ .
7. Integrable open spin chain for color octet channels.