SQM with non-Abelian self-dual fields, harmonic superspace description

Maxim Konyushikhin

(with Evgeny Ivanov and Andrei Smilga)

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Motivation: self-duality equation

$$\mathcal{F}^{a}_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\rho\lambda}\mathcal{F}^{a}_{\rho\lambda}$$

$$(\mathcal{A}_{\alpha\dot{\alpha}})^{i}_{j}(x) \qquad \longleftrightarrow \qquad \left(W^{++}\right)^{i}_{j}\left(x^{+\dot{\alpha}},\varphi\right)$$

$$\left[\mathcal{D}_{\alpha\dot{\alpha}},\mathcal{D}_{\beta\dot{\beta}}\right] = \varepsilon_{\alpha\beta}F_{\dot{\alpha}\dot{\beta}} + \varepsilon_{\dot{\alpha}\dot{\beta}}F_{\alpha\beta}$$

Solution of self-duality equation

Introduce "twistors":

$$\bar{\varphi}^{\alpha}\varphi_{\alpha} = 1, \qquad \varphi_{\alpha} \to \varphi^{+}_{\alpha}, \qquad \bar{\varphi}^{\alpha} \to \varphi^{-\alpha}$$

Make projection:

$$x^{\alpha\dot{\alpha}} \to x^{\pm\dot{\alpha}} = x^{\alpha\dot{\alpha}}\varphi^{\pm}_{\alpha}, \qquad \partial_{\pm\dot{\alpha}} = \frac{\partial}{\partial x^{\pm\dot{\alpha}}}$$

$$\left[\partial_{-\dot{\alpha}} + i\mathcal{A}^{+}_{\dot{\alpha}},\, \partial_{-\dot{\beta}} + \mathcal{A}^{+}_{\dot{\beta}}\right] = 0 \;\longleftrightarrow\; \mathcal{A}^{+}_{\dot{\alpha}} = -ie^{-i\lambda}\partial_{-\dot{\alpha}}e^{i\lambda}$$

Bridge $\lambda\left(x^{\pm\dot{\alpha}}, \varphi\right)$ is a matrix.

$$W^{++} = -ie^{-i\lambda}\partial^{++}e^{i\lambda}, \qquad \partial^{++} = \varphi^{+}_{\alpha}\frac{\partial}{\partial\varphi^{-}_{\alpha}}, \qquad \partial_{-\dot{\alpha}}W^{++} = 0$$

General n-instanton solution, SU(2) gauge group

$$\left(W^{++}\right)^i_j = i \sum_{A=1}^{n} \frac{\rho_A^2}{\left(x_A^{+-}\right)^2} \left(M_A\right)^i_{\ k} \varphi^{+k} \varphi^{+}_{\ell} \left(M_A^\dagger\right)^\ell_{\ j}$$

S.Kalitzin, E.Sokatchev, '87

$$\mathbb{R}^4 \qquad \longrightarrow \qquad \mathbb{R}^4 \times \frac{\mathrm{SU(2)_L}}{\mathrm{U(1)}}$$

Generalizations? Lagrangian?

“Color twistors”

Bosonic model:

$$L_{\text{int}} = i\bar{\varphi}^{\alpha}\dot{\varphi}_{\alpha} + (\mathcal{A}_{\mu})^{i}_{j}\,\varphi_i\bar{\varphi}^j\,\dot{x}_{\mu}$$

A.P.Balachandran, Per Salomonson, Bo-Sture Skagerstam, Jan-Olof Winnberg, ’77

$\mathcal{N} = 4$ SQM:

$$L = \frac{1}{2}\,\dot{x}_{\mu}\dot{x}_{\mu} + i\bar{\psi}^{\dot{\alpha}}\dot{\psi}_{\dot{\alpha}} + i\bar{\varphi}^{\alpha}\left(\dot{\varphi}_{\alpha} + iB\varphi_{\alpha}\right) + kB$$

$$+ (\mathcal{A}_{\mu})^{i}_{j}\,\varphi_i\bar{\varphi}^j\,\dot{x}_{\mu} - \frac{i}{4}\,(\mathcal{F}_{\mu\nu})^{i}_{j}\,\varphi_i\bar{\varphi}^j\,\psi\sigma^{\dagger}_{\mu}\sigma_{\nu}\bar{\psi}$$

E.Ivanov, M.Konyushikhin, A.Smilga, ’09

Harmonic superspace (HSS)

E.Ivanov and O.Lechtenfeld, '03

Harmonics:

$$u^{+\alpha} u^{-}_{\alpha} = 1, \qquad u^{-}_{\alpha} \equiv \left(u^{+\alpha}\right)^{*}, \qquad D^{++} = u^{+}_{\alpha} \frac{\partial}{\partial u^{-}_{\alpha}}$$

Superspace:

$$\left\{ t, \theta_{\alpha}, \bar{\theta}^{\beta}, u^{\pm}_{\gamma} \right\} \qquad \longleftrightarrow \qquad \left\{ t_{\mathrm{A}}, \theta^{\pm}, \bar{\theta}^{\pm}, u^{\pm}_{\alpha} \right\}$$

$$t_{\mathrm{A}} = t + i\left(\theta^{+}\bar{\theta}^{-} + \theta^{-}\bar{\theta}^{+}\right), \qquad \theta^{\pm} = u^{\pm}_{\alpha}\theta^{\alpha}, \qquad \bar{\theta}^{\pm} = u^{\pm}_{\alpha}\bar{\theta}^{\alpha}$$

$$D^{+} = \frac{\partial}{\partial \theta^{-}}, \qquad \bar{D}^{+} = -\frac{\partial}{\partial \bar{\theta}^{-}}$$

Constraints:

$$D^{++}q^{+\dot\alpha} = 0, \qquad q^{+\dot\alpha} = \varepsilon^{\dot\alpha\dot\beta}\widetilde{q^{+\dot\beta}}$$

$$(D^{++} + iV^{++})v^{+} = 0, \qquad V^{++} = \widetilde{V^{++}}$$

Field content:

$$q^{+\dot\alpha}\left(t,\theta,\bar\theta\right) \qquad \longrightarrow \qquad \left\{x^{\alpha\dot\alpha}(t),\, \psi_{\dot\alpha}(t)\right\}$$

$$v^{+}\left(t,\theta,\bar\theta\right) \qquad \longrightarrow \qquad \left\{\varphi_{\alpha},\, \bar\varphi^{\beta}\right\}$$

$$V^{++}_{\rm WZ} \ = \ 2i\,\theta^{+}\bar\theta^{+}B(t)$$

$$S_{\rm int} = -\frac{1}{2}\int dt\, du\, d\bar{\theta}^{+} d\theta^{+} \left\{ K\left(q^{+\dot{\alpha}}, u^{\pm}_{\beta}\right) v^{+}\widetilde{v^{+}} + ikV^{++} \right\}$$

$$= \int dt \left\{ i\bar{\varphi}^{\alpha}\dot{\varphi}_{\alpha} + (\mathcal{A}_{\mu})^{\alpha}_{\beta}\, \varphi_{\alpha}\bar{\varphi}^{\beta}\dot{x}_{\mu} + \ldots \right\}$$

E.Ivanov, M.Konyushikhin, A.Smilga, '09

$$\mathcal{A}^{a}_{\mu} \; = \; -\bar{\eta}^{a}_{\mu\nu}\partial_{\nu} \ln h(x)$$

$$h(x) = \int du\, K(x^{+\dot{\alpha}}, u^{\pm}_{\beta}), \qquad \partial^{2}_{\mu} h(x) = 0$$

Outlook

- A new $\mathcal{N} = 4$ SQM model is constructed
- Superfield description for the particular case of t'Hooft ansatz
- Non-Abelian self-dual gauge fields can be described in terms of functions on extended space
- General correspondence between solutions of self-duality equation and functions on extended space? SU(N)? (work in progress)
- QFT Lagrangian?