16th International Seminar on High Energy Physics QUARKS-2010

Kolomna, Russia, 6-12 June, 2010

Density perturbations in braneworld cosmology and primordial black holes

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Plan of the talk

- Primordial black holes
- Cosmological models with extra dimensions
- Equations for evolution of density perturbations
- Numerical scheme of perturbations calculation
- Enhancement factors for density perturbations
- Conclusions

Introduction

It is well known that in case if primordial density perturbations in the early Universe have rather large amplitude on some length scale, a non-negligible amount of primordial black holes (PBHs) can be produced through the gravitational collapse of this inhomogeneities. PBHs are a useful source of information about the very early Universe, and even their non-observation can give rather useful data for cosmology.

In many cases, the limits on the abundance of PBHs and corresponding cosmological constraints from PBHs have been obtained using simplest (“Standard”) cosmological model, assuming, in particular, the 4-dimensional FLRW space-time.

However, being a probe for the early Universe and very high-energy phenomena in it, PBHs (and limits derived from them) are sensitive also to such assumptions.

Braneworld cosmology

In this talk we consider the possibility of PBH production in a particular theory with extra spatial dimension – a well-known Randall-Sundrum (1999) one-brane model.

In this model, our Universe is considered as a "brane" embedded in a 5-dimensional anti-de Sitter (AdS) "bulk". The bulk has a curvature of scale ℓ related to the bulk cosmological constant as

$$\Lambda_5 = -\frac{6}{\ell^2} \; .$$

An Einstein-Hilbert 5D gravitational action is

$$S_{\rm grav} = \frac{1}{2\kappa_5^2} \int d^4x dy \sqrt{-{}^{(5)}g} \left[{}^{(5)}R - 2\Lambda_5 \right] \; ,$$

where gravitational coupling constant is related to the 5D Planck mass:

$$\kappa_5^2 = 8\pi G_5 = \frac{8\pi}{M_5^3} \; .$$

The 5D Einstein equation is

$$^{(5)}G_{AB} \equiv {}^{(5)}R_{AB} - \frac{1}{2} \cdot {}^{(5)}R^{(5)}g_{AB} = \Lambda_5 \cdot {}^{(5)}g_{AB} + \kappa_5^2 \cdot {}^{(5)}T_{AB}$$.

The static weak field limit gives the Poisson equation, from which the gravitational potential, on scales much smaller than the size of the extra dimension, is

$$V(r) \sim \frac{\kappa_5^2}{r^2}$$.

Thus, Newton's law is modified on scales $r < \ell$.

The AdS metric in Gaussian normal coordinates takes the form

$$^{(5)}ds^2 = e^{-2|y|/\ell}\eta_{\mu\nu}dx^\mu dx^\nu + dy^2$$

(brane is at *y=0*), the exponential "warp" factor reflects the confining role of the cosmological constant.

In Poincare coordinates, the metric can be rewritten in a conformally flat form:

$$ {}^{(5)}ds^2 = \frac{\ell^2}{z^2}\left[\eta_{\mu\nu}dx^\mu dx^\nu + dz^2\right] = \frac{\ell^2}{z^2}\left(-d\tau^2 + \delta_{ij}dx^i dx^j + dz^2\right) $$,

where $z = \ell e^{y/\ell}$, and FLRW metric on the brane is recovered, with $a(\eta) = \ell / z_b(\eta)$ and $d\eta^2 = d\tau_b^2 - dz_b^2$.

The 5D energy-momentum tensor entering Einstein equation is

$$ {}^{(5)}T_{AB} = {}^{(5)}T_{AB}^{\rm bulk} + {}^{(5)}T_{AB}^{\rm brane}\delta(y) $$.

The bulk part is assumed to be zero and

$$ T_{\mu\nu}^{\rm brane} = T_{\mu\nu} - \sigma g_{\mu\nu} $$,

where $T_{\mu\nu}$ is for particles and fields confined on the brane, and σ is the brane tension.

The induced field equation on the brane, for empty bulk, is (Shiromizu, Maeda, Sasaki, 2000)

$$(*) \qquad G_{\mu\nu} = -\Lambda g_{\mu\nu} + \kappa^2 T_{\mu\nu} + 6\frac{\kappa^2}{\lambda} S_{\mu\nu} - \mathcal{E}_{\mu\nu} \; .$$

We see that there are 2 corrections to the usual 4D equation: 1) A quadratic one

$$S_{\mu\nu} = \frac{1}{12} T T_{\mu\nu} - \frac{1}{4} T_{\mu\alpha} T^{\alpha}{}_{\nu} + \frac{1}{24} g_{\mu\nu} \left[3 T_{\alpha\beta} T^{\alpha\beta} - T^2 \right] , \quad T = T^{\mu}_{\mu} \, ,$$

which is negligible for $\rho \ll \sigma$ but dominant if $\rho \gg \sigma$.

and 2) the Weyl tensor projected on the brane, which includes the corrections from 5D graviton effects,

$$\mathcal{E}_{\mu\nu} = {}^{(5)}C_{ACBD}\, n^C n^D g_\mu{}^A g_\nu{}^B \; .$$

It follows from (*) that

$$\kappa^2 = \frac{1}{6} \sigma \kappa_5^4 \quad \text{and} \quad \Lambda = \frac{1}{2}(\Lambda_5 + \kappa^2 \sigma) \; .$$

As usual, we assume that $\kappa^2 = 8\pi G$ and $\Lambda = 0$. In this case all the model parameters are expressed through ℓ and κ^2:

$$\Lambda_5 = -\frac{6}{\ell^2} \;,\; \kappa_5^2 = \kappa^2 \ell \;,\; \sigma = \frac{6}{\kappa^2 \ell^2} .$$

From table-top experiments (Hoyle et al, PRL 86, 1418, 2001; Long et al, Nature, 421, 922, 2003) it is known that

$$\ell \lesssim 0.1\mathrm{mm} .$$

One also needs the junction conditions on the brane, which are written as (W. Israel, Nuovo Cim. B44, 1, 1966)

$$K_{\mu\nu} = -\frac{1}{2}\kappa_5^2 \left[T_{\mu\nu} + \frac{1}{3}(\sigma - T)\, g_{\mu\nu} \right] ,$$

where the extrinsic curvature tensor is given by

$$K_{AB} = g_A{}^C \, {}^{(5)}\nabla_C n_B .$$

Friedman equation and horizon mass in RS model

The quadratic correction to the Einstein equation leads to the corresponding modification of the Friedman equation:

$$H^2 = \frac{8\pi G}{3}\rho\left(1+\frac{\rho}{2\sigma}\right) \quad .$$

Its solution for a radiation-dominated state on the brane is

$$a(t) = a_{eq}\frac{t^{1/4}(t+t_c)^{1/4}}{t_{eq}^{1/2}}, \quad H(t) = \frac{2t+t_c}{4t(t+t_c)}, \quad \rho(t) = \frac{3}{32\pi t(t+t_c)}; \quad t_c = \ell/2$$

and at late times (low energy density) the 4D cosmology is recovered.

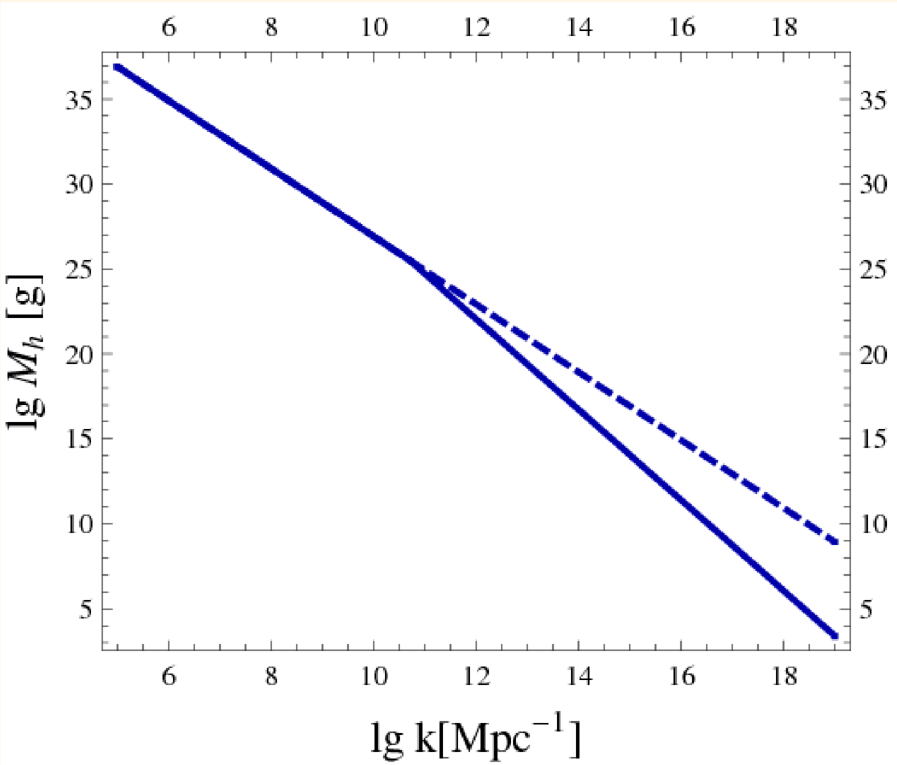
The transition between the high energy (HE) and low energy (LE) regimes happens at the “critical” epoch, at which $H\ell = 1$ and the mode entering the horizon at this time (in this case, $k_c = a_c H_c$) is given by

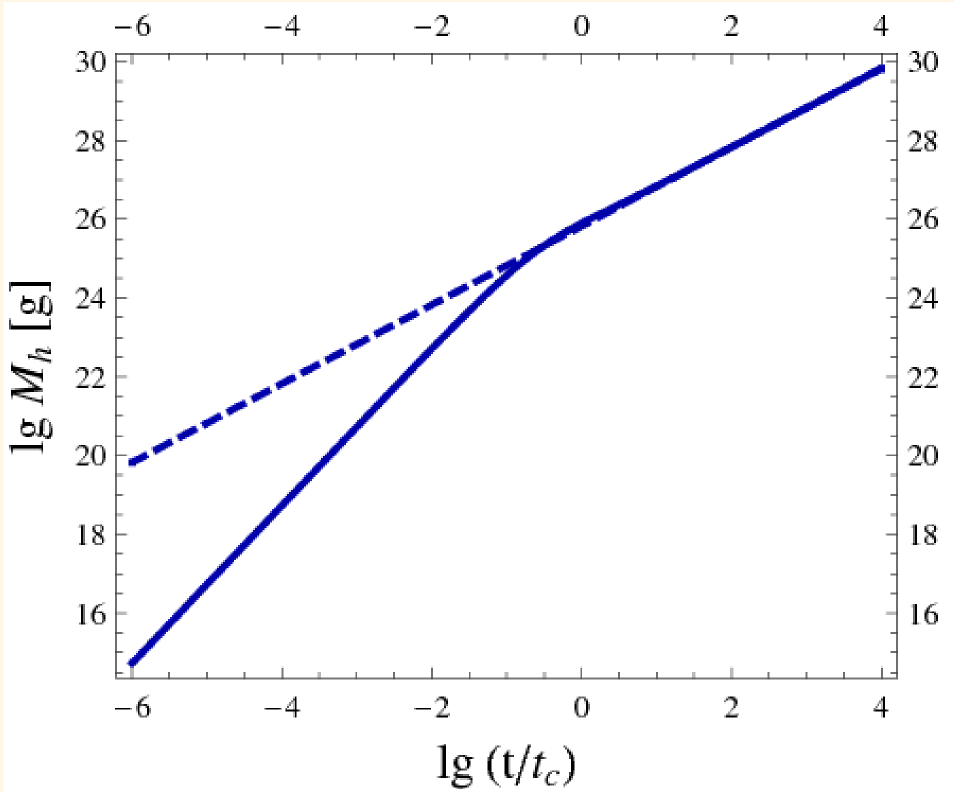
$$k_c^{-1} = 1.4\times 10^{12}\left(\frac{\ell}{0.1\mathrm{mm}}\right)^{1/2}\left(\frac{g_{c*}}{100}\right)^{1/12}\mathrm{m} \ .$$

The horizon mass in this model is $M_h(t) \equiv \frac{4\pi\rho}{3H^3} = \frac{8t^2(t+t_c)^2}{(2t+t_c)^3}$.

If we assume that $\ell \approx 0.1\mathrm{mm}$ then the horizon (and PBH) mass corresponding to the critical epoch is *~10*^{25} *g*.

The dependence of horizon mass on time and *k* is shown in the figures (dashed curves are for the case of 4D cosmology).





Characteristics of 5D black holes

We assume that for sufficiently small scales, $r \ll \ell$, the black hole metric is a 5D Schwarzschild solution (Myers & Perry, 1986)

$$ds^2 = -\left[1 - \left(\frac{r_s}{r}\right)^2\right] dt^2 + \left[1 - \left(\frac{r_s}{r}\right)^2\right]^{-1} dr^2 + r^2 d\Omega_3^2$$

$$r_s = \sqrt{\frac{8}{3\pi}} \left(\frac{\ell}{\ell_4}\right)^{1/2} \left(\frac{M_{BH}}{M_4}\right)^{1/2} \ell_4; \;\; T_{BH} = \frac{1}{2\pi r_s} \, .$$

The rate of a black hole mass change is given by the sum of evaporation and accretion terms (e.g., Majumdar, 2002):

$$\dot{M}_{BH} = 4\pi r_s^2 (-g_{*,br} \sigma_{SB} T_{BH}^4 + \rho_R), \;\; \rho_R = \frac{3M_4^2}{16\pi t \ell} \, .$$

Consideration shows that for $t<t_c$ accretion dominates over evaporation, and after t_c evaporation dominates (Clancy et al, 2003).

The BH mass at the onset of the evaporation (t=t_c), if the age of the black hole is equal to the age of the Universe, is

$$M^*_{BH}(t_0) \sim 5 \times 10^9 \left(\frac{\ell}{0.1 \text{ mm}} \right)^{-1/2} \text{g} \quad ,$$

and, in general,

$$M^*_{BH}(t) \sim \left(\frac{t}{t_4} \right)^{1/2} \quad .$$

Note that in the standard 4D case, the initial mass of PBHs that evaporate today is about $M^* \sim 5 \times 10^{14}$ g.

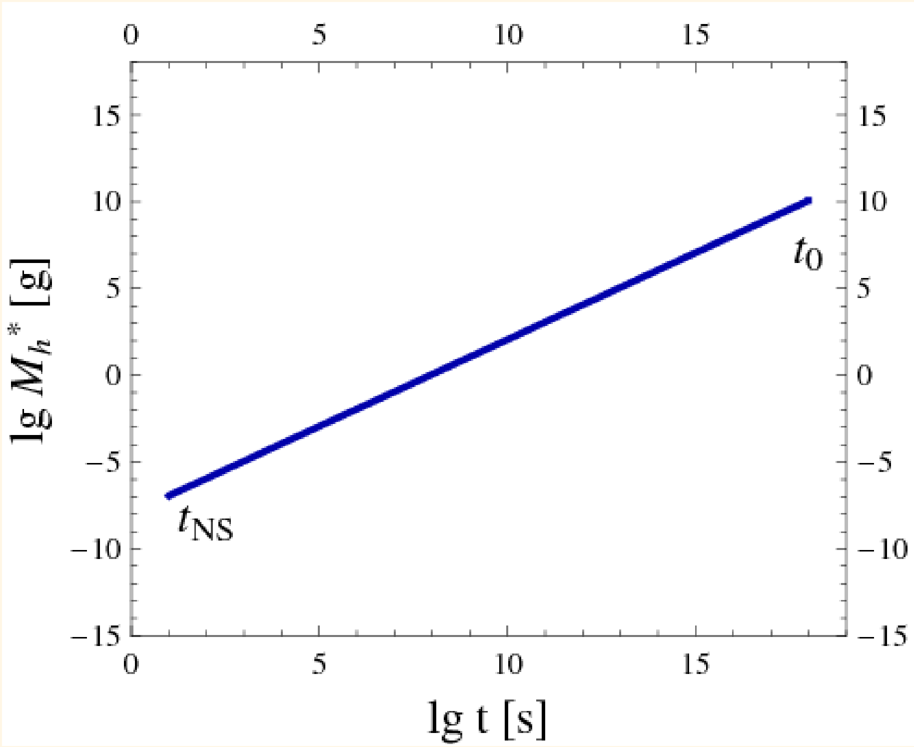


Figure: Black hole mass M_{BH}* vs. the moment of time at which the black hole evaporates.

RS cosmology and spherical collapse

The equation for density contrast in the RS model, including only the HE quadratic contributions is

$$\ddot{\Delta}_k + H\dot{\Delta}_k + \left[-\frac{16\pi}{3M_4^2}\rho - \frac{32\pi^2}{M_5^6}\rho^2 + \frac{1}{3}\left(\frac{k}{a}\right)^2\right]\Delta_k = 0$$

.

The Jeans length is given by

$$\frac{a}{k_J} = \begin{cases} \left(\frac{4\sqrt{\pi\rho}}{M_4}\right)^{-1}, & \text{low } \rho; \\ \left(\frac{4\sqrt{6}\pi\rho}{M_5^3}\right)^{-1}, & \text{high } \rho. \end{cases}$$

The critical overdensity for HE regime is (Kawasaki, 2004)

$$\delta_c \approx 0.1$$,

which is lower than classical result $\delta_c \approx 1/3$ (Carr, 1975).

Scalar perturbations

For considering the perturbation problem, one must write the perturbed metric on the brane and in the bulk, the perturbed equations of motion and the perturbed junction conditions.

It has been shown in (Mukohyama, 2000) that scalar-type perturbations in the bulk can be written in terms of the gauge-invariant master variable $\Omega = \Omega(\tau, z)$, which satisfies the equation

$$0 = -\frac{\partial^2 \Omega}{\partial \tau^2} + \frac{\partial^2 \Omega}{\partial z^2} + \frac{3}{z}\frac{\partial \Omega}{\partial z} + \left(\frac{1}{z^2} - k^2\right)\Omega$$.

The projection of the Weyl tensor in perturbed 4D Einstein equations can be parameterized as

$$\begin{aligned} \delta\mathcal{E}_0{}^0 &= \kappa_4^2 \delta\rho_{\mathcal{E}} Y, \\ \delta\mathcal{E}_i{}^0 &= \kappa_4^2 k Y_i \delta q_{\mathcal{E}}, \\ \delta\mathcal{E}_i{}^j &= -\kappa_4^2 (\tfrac{1}{3}\delta\rho_{\mathcal{E}} Y \delta_i{}^j + k^2 \delta\pi_{\mathcal{E}} Y_i{}^j). \end{aligned}$$

$$Y = e^{i\mathbf{k}\cdot\mathbf{x}}, \qquad Y_i = -\frac{1}{k}\partial_i Y,$$

$$Y_{ij} = \frac{1}{k^2}\partial_i\partial_j Y + \frac{1}{3}\delta_{ij} Y.$$

Here, $\delta\rho_{\mathcal{E}}, \delta q_{\mathcal{E}}, \delta\pi_{\mathcal{E}}$ are Kaluza-Klein degrees of freedom (effects of bulk geometry on the brane). These terms appear in equation for gravitational potentials, in particular

$$\Phi = \frac{4\pi G\rho a^2}{k^2}\left[\left(1+\frac{\rho}{\sigma}\right)\Delta + \Delta_{\mathcal{E}}\right] ,$$

$$\Phi + \Psi = -8\pi G a^2 \delta\pi_{\mathcal{E}} .$$

When perturbing junction conditions, we obtain *1)* a boundary condition for master variable on the brane,

$$\left[\partial_n \Omega + \frac{1}{\ell}\left(1+\frac{\rho}{\sigma}\right)\Omega + \frac{6\rho a^3}{\sigma k^2}\Delta\right]_{\rm b} = 0 ;$$

2) expressions for the KK terms through Ω, in particular

$$\kappa_4^2 \delta\pi_{\mathcal{E}} = \frac{1}{2\ell a^3}\left[\frac{1}{3}k^2\Omega - H\partial_u\Omega - \frac{3(\rho+p)}{\ell\sigma}\partial_n\Omega + \partial_u^2\Omega\right]_{\rm b}$$

where $\partial_u = \frac{\partial}{\partial t} = \frac{1}{a}\frac{\partial}{\partial\eta}$, $\partial_n = \frac{1}{a}\left(-H\ell\frac{\partial}{\partial\tau} + \sqrt{1+H^2\ell^2}\frac{\partial}{\partial z}\right)$;

3) the equation for comoving density contrast $\Delta = (\delta\rho/\rho)_{\delta q=0}$.

In the radiation-dominated brane Universe ($p = \rho/3$) it is

$$\ddot{\Delta}_k + H\dot{\Delta}_k + \left[\frac{1}{3}\left(\frac{k}{a}\right)^2 - \frac{4\rho}{\sigma\ell^2} - \frac{18\rho^2}{\sigma^2\ell^2}\right]\Delta_k = \frac{4k^4}{9\ell a^5}\Omega_{\rm b}$$

Thus, equations for gauge invariant quantities on the brane include the bulk master variable.

In the HE regime, when

$$\rho \gg \sigma, \quad H\ell \gg 1, \quad t \ll t_c \quad ,$$

the following approximations can be used:

$$\rho \approx \sigma H \ell, \quad \partial_n \approx -\partial_u = -\partial_t \quad .$$

One can show that in this case a decoupled 3-rd order equation for Δ exists (Cardoso, 2007) and at leading order in $(k\eta)$ the solution (dominant super-horizon growing mode) is

$$\Delta \sim (k\eta)^2 \sim a^6, \quad \Omega_b \sim (k\eta)^3 \sim a^9 \quad .$$

On the other hand, in sub-horizon regime, Δ undergoes harmonic oscillations, as in 4D case. The curvature perturbation invariant,

$$\zeta = \left[\frac{1}{3} - \frac{3\rho a^2(w\sigma - \sigma - \rho)}{k^2\ell^2\sigma^2}\right]\frac{\Delta}{1+w} + \frac{Ha}{k^2(1+w)}\frac{d\Delta}{d\eta} + \frac{k^2}{6\ell a^3}\Omega_b \quad ,$$

is conserved in the super-horizon regime, also as in 4D.

It appears, however (Cardoso et al, 2007), that for large $k>k_c$ the amplitudes of scalar perturbations (Δ or ζ) increase during horizon crossing, and the degree of enhancement increases with k. There are 2 sources for this enhancement:

(1) $O(\rho^2/\sigma^2)$ - corrections to the perturbation equations for Δ and other variables;

(2) The effect of the bulk degrees of freedom (Kaluza-Klein modes) that are expressed through Ω .

Intermediate ("effective") case: only (1) is included, the calculation is much simplified – only 1 ODE to solve.

In the regime $t>>t_c$ (or in the absence of extra dimensions), General Relativity is recovered.

If including (1)+(2), we need to solve a *coupled* system of 2-nd order PDE for Ω and ODE for Δ .

Numerical scheme

For the numerical calculation, a pseudo-spectral method was employed. We do a following change in the variables:

$\Omega(\tau, z) \to \Omega(t, \xi)$, $\xi = \dfrac{2z - (z_{reg} + z_b(t))}{z_{reg} - z_b(t)}$, $-1 \le \xi \le 1$,

and equation for master variable is rewritten as

$$\frac{\partial^2 \Omega}{\partial t^2} + K_{t\xi} \frac{\partial^2 \Omega}{\partial t \partial \xi} + K_{\xi\xi} \frac{\partial^2 \Omega}{\partial \xi^2} + K_t \frac{\partial \Omega}{\partial t} + K_\xi \frac{\partial \Omega}{\partial \xi} + K\Omega = 0 .$$

To solve this equation using difference method (such as Adams-Bashforth-Moulton scheme) the variable χ is introduced:

$$\frac{\partial \Omega}{\partial t} = \chi - K_{t\xi} \frac{\partial \Omega}{\partial \xi} \equiv F(\chi, \Omega'_\xi;\ t, \xi);$$

$$\frac{\partial \chi}{\partial t} = G(\chi, \Omega, \Omega'_\xi, \Omega''_{\xi\xi};\ t, \xi) .$$

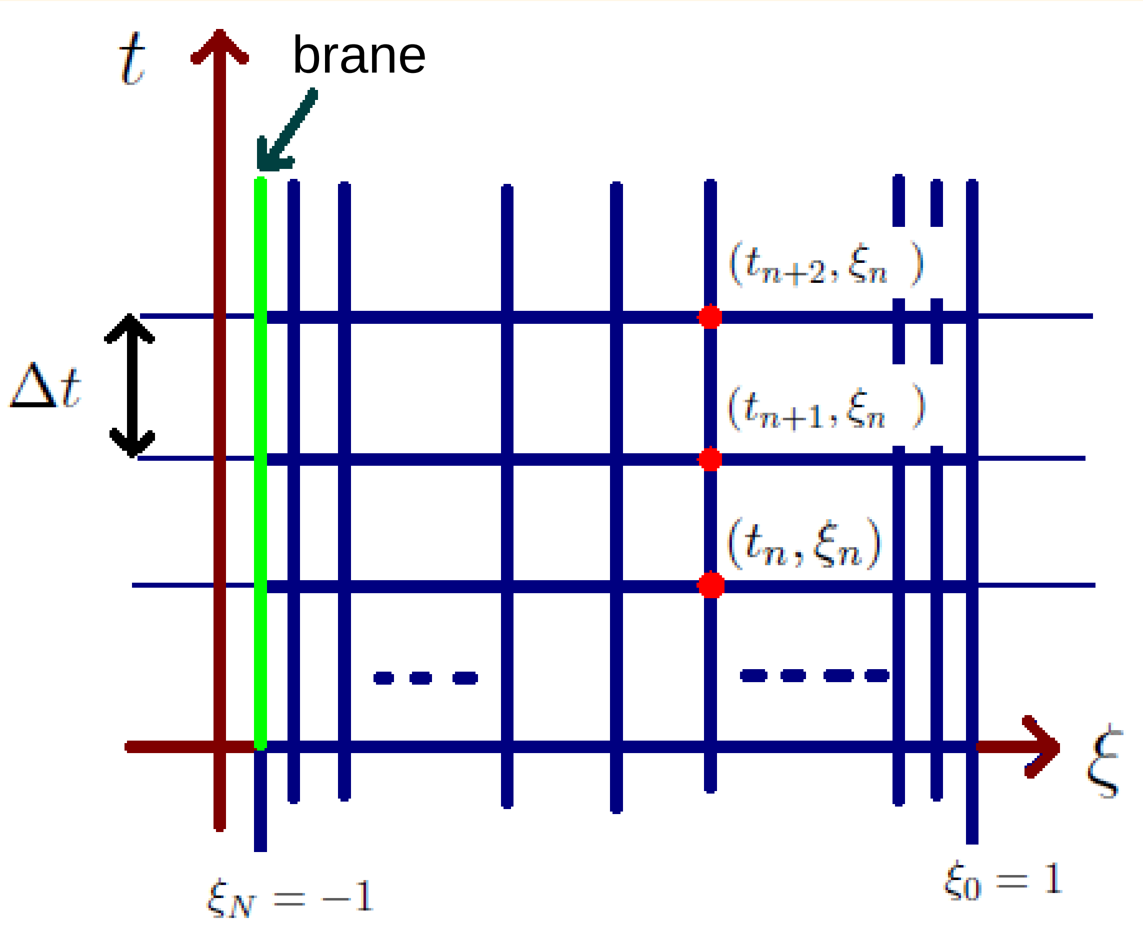
For the ξ - axis, the transformation over Chebyshev polynomials is done at each time step so that PDEs reduce to the system of ODEs.

At each point (t_n, ξ_n), the following quantities are known:

$$\chi, \Omega, \Omega'_\xi, \Omega''_{\xi\xi}, F, G \ ,$$

also known are Chebyshev transforms $\tilde{\chi}_n, \tilde{\Omega}_n, (\tilde{\Omega}'_\xi)_n, (\tilde{\Omega}''_{\xi\xi})_n, \tilde{F}_n, \tilde{G}_n$.

The grid based on Gauss-Lobatto points is used: $\xi_n = \cos\left(\frac{\pi n}{N}\right)$.



Equations solved are:

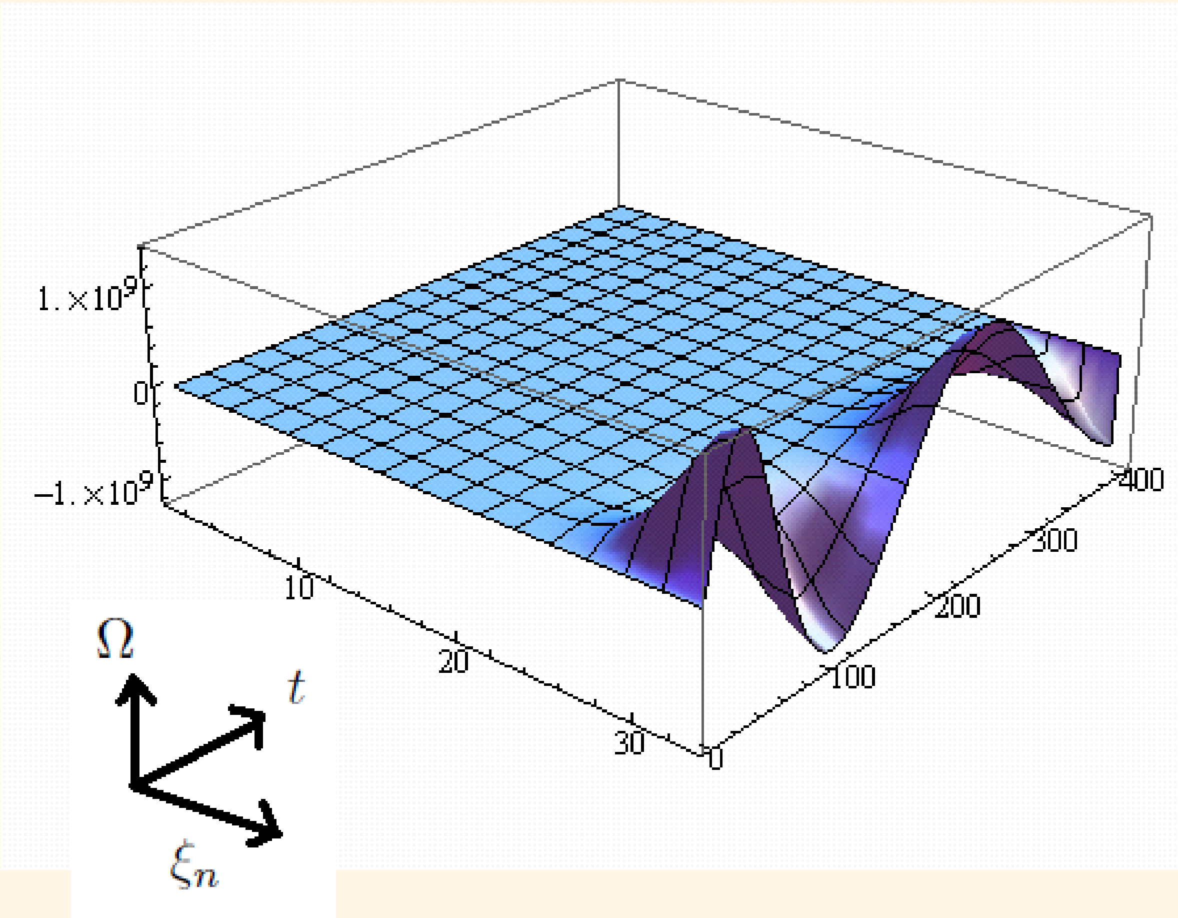
$$\frac{d\tilde{\Omega}_n}{dt} = \tilde{F}_n(t); \ \frac{d\tilde{\chi}_n}{dt} = \tilde{G}_n(t).$$

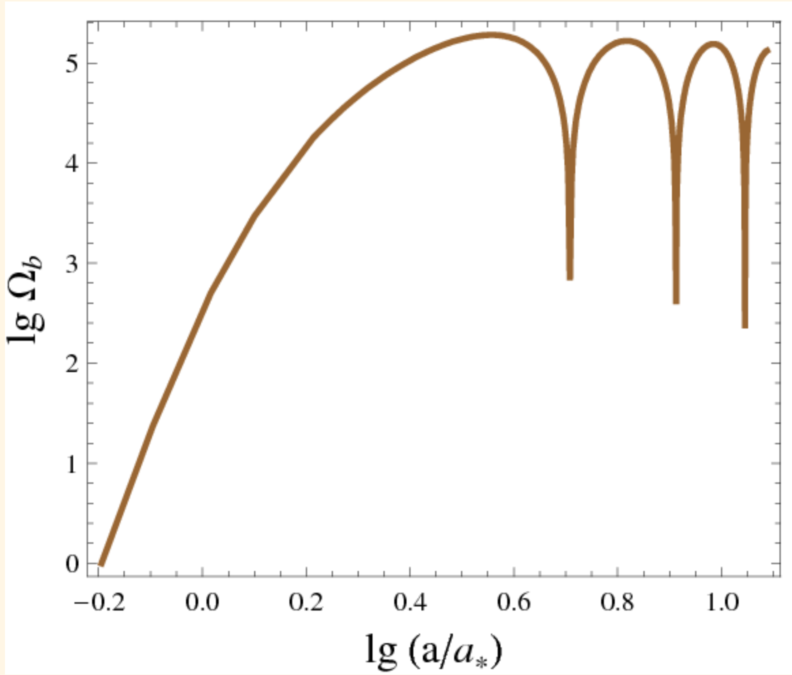
Boundary conditions are imposed on

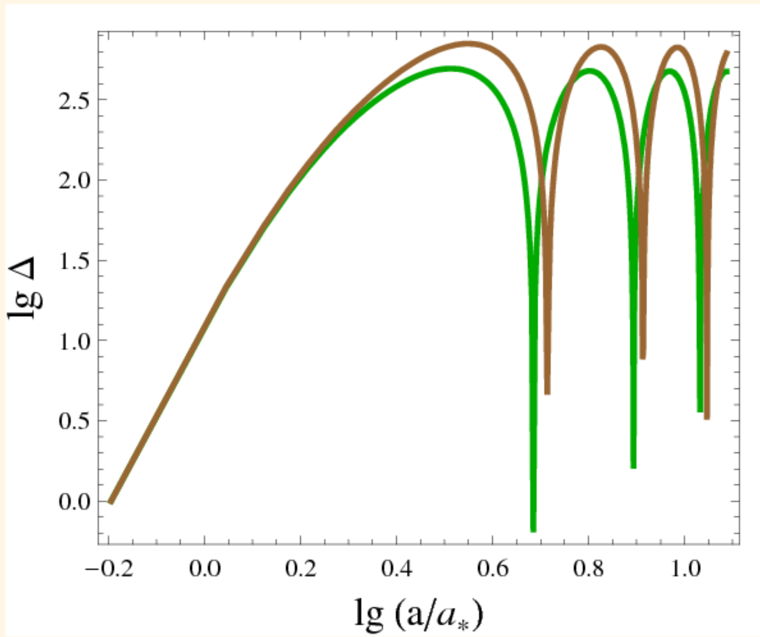
$$\tilde{\Omega}_N, \tilde{\Omega}_{N-1} \ .$$

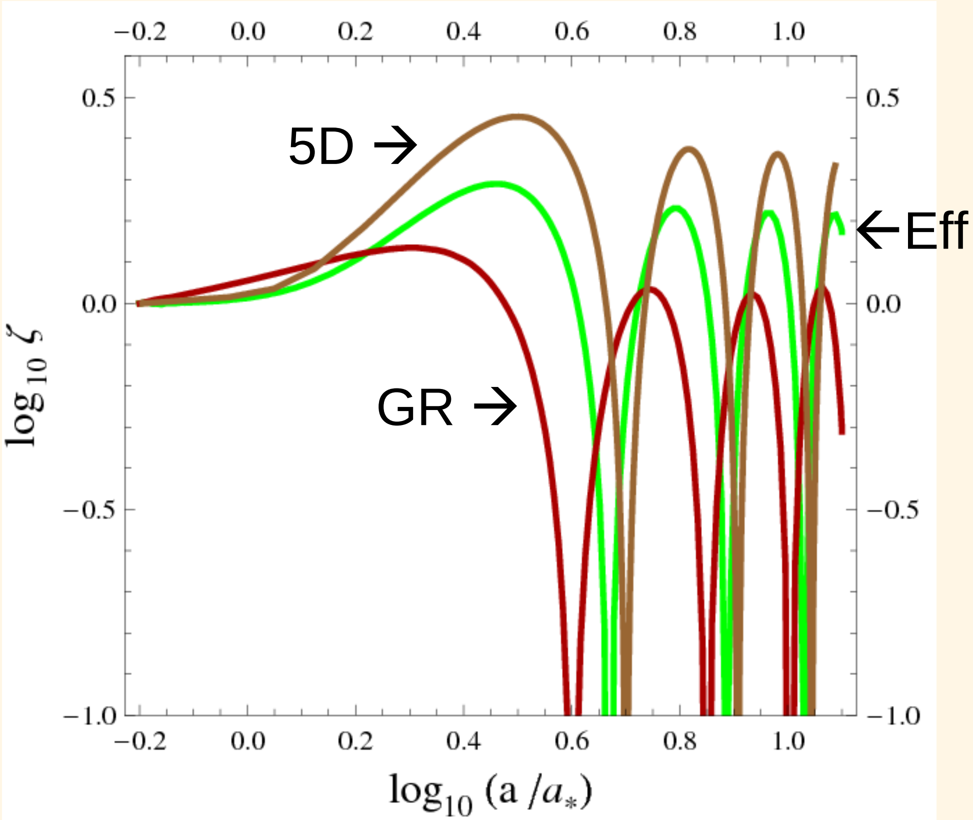
For points on the brane, Δ is also evaluated.

The typical results of the calculations are shown in the figures. Here, we have taken $k=3k_c$.

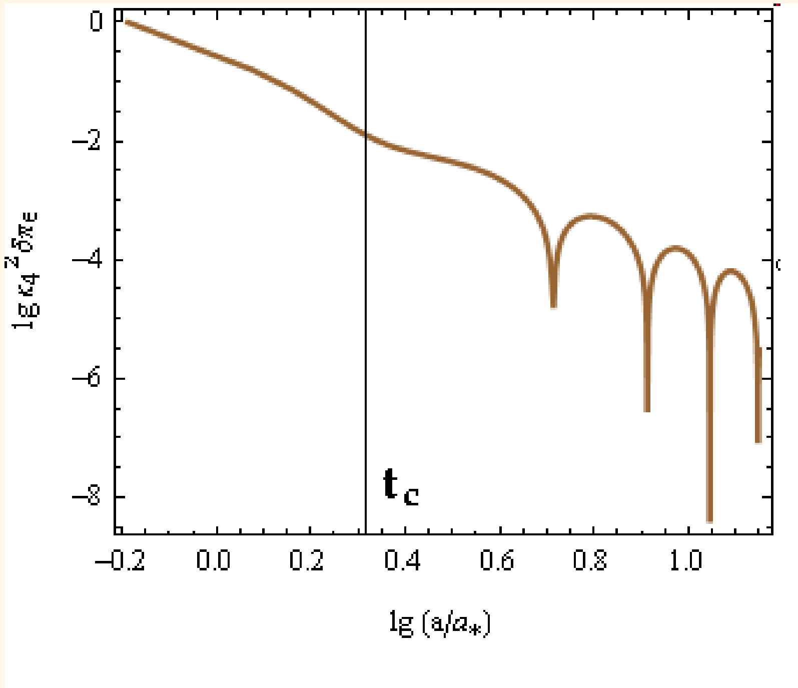








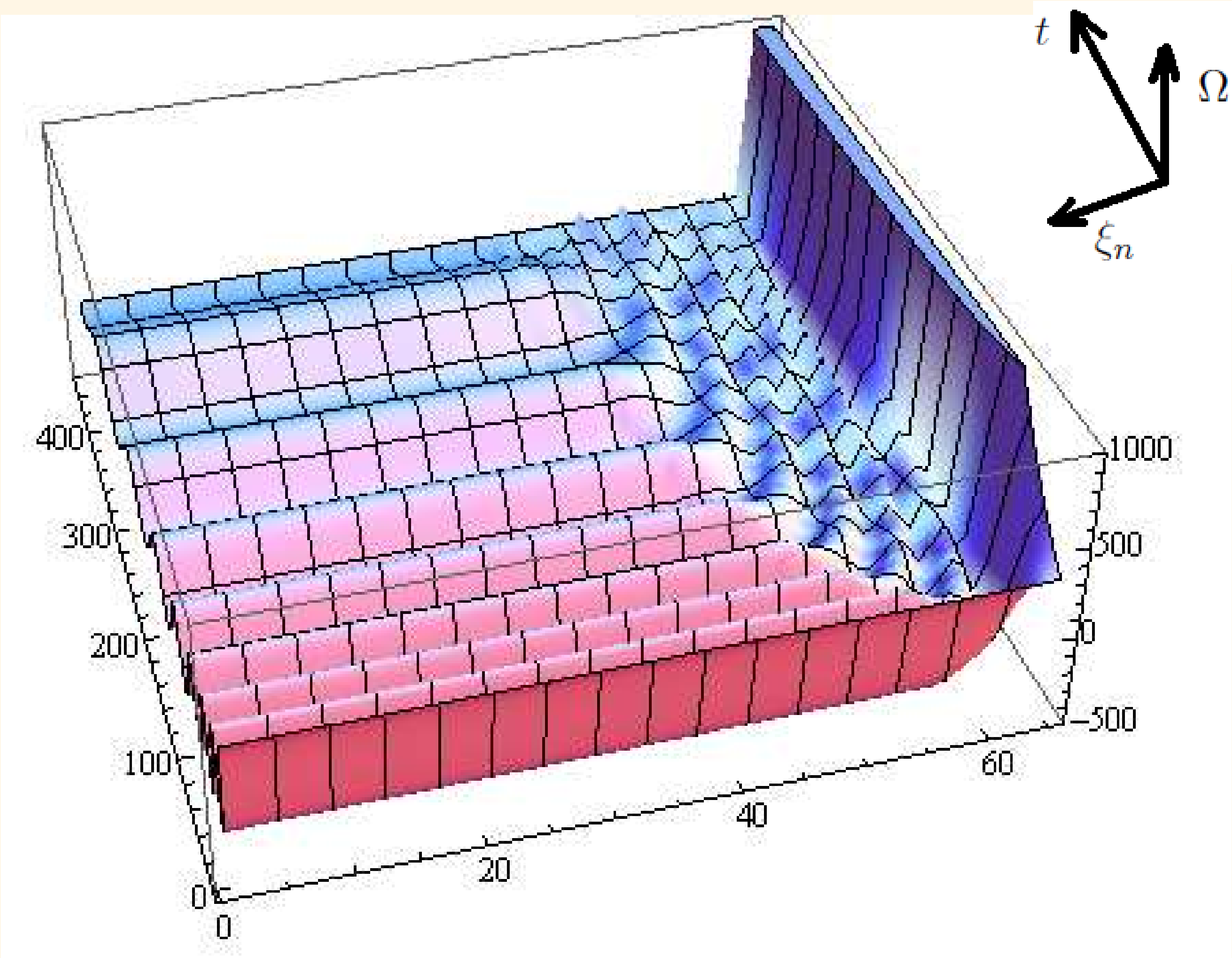
On the upper figure, the comparison of three calculations is given: GR, Effective theory and full 5D calculation. The enhancement of the amplitude is seen after horizon-crossing.



Lower figure: Weyl tensor anisotropic stress calculation.

a_* represents the moment of horizon entry for the mode.

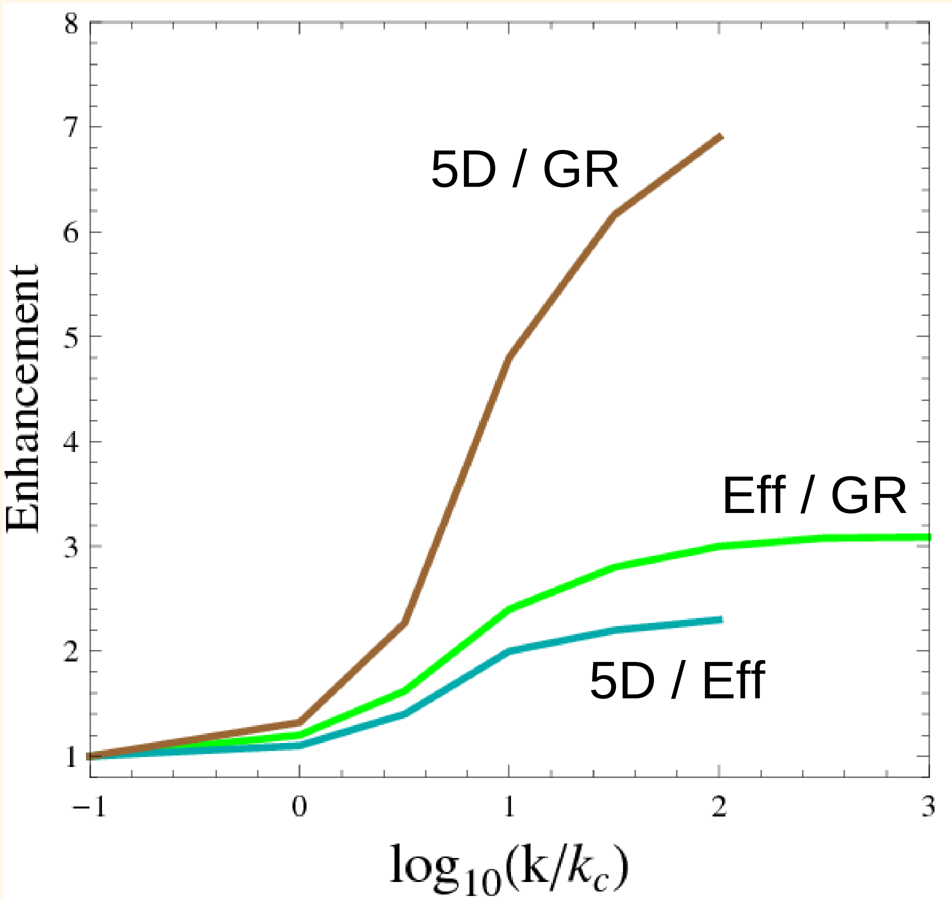
For rather large values of $k > 100\,k_c$ the numerical calculation is very resource-consuming.



Enhancement factors

The enhancement factors have been calculated using 5D calculation and effective theory. In case of effective theory, the enhancement reaches an asymptotic value of *~3* at $k \sim 100k_c$.

We see additional enhancement from including KK terms, similar to one found by Cardoso et al, 2007.



Due to limitations of computing resources, we have been able to make calculations in 5D case only for limited range of *k*.

To explore PBH production for PBH masses $\sim 10^9\ g$ (such PBHs evaporate today), we need $k > 10^6\ k_c$, which is currently an unsolved problem.

Conclusions

Primordial black holes can be used to probe perturbations in our Universe at very small scales, currently inaccessible to other types of experiments.

In theoretical models with extra dimensions, such as the Randall-Sundrum braneworld model, several aspects of PBH formation and evolution are different: the formation threshold is lower, accretion and evaporation obey different laws compared to 4D case.

Furthermore, the evolution of the density perturbations in such models is affected by the bulk degrees of freedom. We have shown that taking this effect into account leads to additional enhancement of perturbation amplitude which increases the PBH formation probability.