Generalized Super-Landau Models

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QUARKS-2010, Kolomna, June 06 - 12, 2010

(with A. Beylin, T. Curtright, L. Mezincescu, P.K. Townsend)

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Motivations

- Landau model (Landau, 1930): a charged particle moving on a plane orthogonal to a constant uniform magnetic flux.
- A spherical L model (Haldane, 1983): a charged particle on $S^2 \sim SU(2)/U(1)$ in the Dirac monopole background.
- Superextensions - non-relativistic particles moving on supergroup manifolds with S^2 or its planar limit as a 'body'.

Minimal superextensions of the S^2 Haldane model:

1. L. problem on the $(2+2)$ supersphere $SU(2|1)/U(1|1)$;
2. L. problem on the $(2+4)$ superflag $SU(2|1)/[U(1)xU(1)]$,

L. Mezincescu, P.K. Townsend, E.I., hep-th/0311159, 0404108; A. Beylin, T. Curtright, L. Mezincescu, P.K. Townsend, E.I., 0806.4716[hep-th]

Their large S^2 radius limits (planar super Landau models):

K. Hasebe, hep-th/0503162; L. Mezincescu, P.K. Townsend, E.I., hep-th/0510019; E.I., 0705.2249[hep-th].

- Most surprising feature of the super planar L. problem: the presence of hidden world-line $N = 2$ supersymmetry.
- One starts with a model invariant under some target supersymmetry and, as a *gift*, finally finds the existence of the $d = 1, N = 2$ supercharges which square on the Hamiltonian of the system.
- Thus, the super planar L. models simultaneously provide a class of the supersymmetric quantum mechanics models. SQM models (Witten, 1983) have a plenty of applications in diverse domains.
- A natural extension of the super planar L. models: to take the fundamental notion of the world-line $N = 2$ supersymmetry as the primary one and to construct a most general $N = 2$ SQM model involving the standard super planar L. models as a particular case.
- Such a construction has been recently given in A. Beylin, T. Curtright, L. Mezincescu, E.I., 1003.0218 [hep-th], JHEP 1004:091,2010, starting from$d = 1, N = 2$ superfield formalism.

Planar bosonic Landau model

- Lagrangian and Hamiltonian:

$$L_b = |\dot{z}|^2 - i\kappa\left(\dot{z}\bar{z} - \dot{\bar{z}}z\right) = |\dot{z}|^2 + \left(A_z\dot{z} + A_{\bar{z}}\dot{\bar{z}}\right) ,$$

$$A_z = -i\kappa\bar{z}, A_{\bar{z}} = i\kappa z, \quad \partial_{\bar{z}}A_z - \partial_z A_{\bar{z}} = -2i\kappa .$$

$$H_b = \frac{1}{2}\left(a^\dagger a + a a^\dagger\right) = a^\dagger a + \kappa ,$$

$$a = i(\partial_{\bar{z}} + \kappa z), \quad a^\dagger = i(\partial_z - \kappa\bar{z}), \quad [a, a^\dagger] = 2\kappa .$$

- Invariances:“magnetic translations” and $2D$ rotations:

$$P_z = -i(\partial_z + \kappa\bar{z}), P_{\bar{z}} = -i(\partial_{\bar{z}} - \kappa z), F_b = z\partial_z - \bar{z}\partial_{\bar{z}},$$

$$[P_z, P_{\bar{z}}] = 2\kappa,\ [H, P_z] = [H, P_{\bar{z}}] = [H, F_b] = 0.$$

Wave functions

- Lowest Landau level (LLL), $H\Psi_{(0)} = \kappa\Psi_{(0)}$:

$$a\Psi_{(0)}(z,\bar{z}) = 0 \Leftrightarrow (\partial_{\bar{z}} + \kappa z)\Psi_{(0)} = 0 \rightarrow \Psi_{(0)} = e^{-\kappa|z|^2}\psi_{(0)}(z)$$

- n-th LL:

$$\Psi_{(n)}(z,\bar{z}) = [i(\partial_z - \kappa\bar{z})]^n e^{-\kappa|z|^2}\psi_{(n)}(z), \; H\Psi_{(n)} = \kappa(2n+1)\Psi_{(n)}$$

Each LL is infinitely degenerate due to $(P_z, P_{\bar{z}})$ invariance: infinite-dimensional irreps, with the basis $z^m, m > 0$.

- Invariant norm:

$$||\Psi_{(n)}||^2 \sim \int dz d\bar{z} e^{-2\kappa|z|^2}\overline{\psi}_{(n)}(\bar{z})\psi_{(n)}(z) < \infty$$

for any monomial $\psi_{(n)}(z) \sim z^m$.

Generalization to S^2

- An S^2 analog of the planar Lagrangian L_b is

$$L_b = \frac{1}{(1+r^2|z|^2)^2}|\dot{z}|^2 + is\frac{1}{1+r^2|z|^2}\left(\dot{z}\bar{z} - \dot{\bar{z}}z\right),$$

  The 2nd term is the $d = 1$ WZ term on the coset $SU(2)/U(1)$, r the "inverse" radius of S^2.

- The w. f. are finite-dimensional $SU(2)$ irreps, s being their "spin". The LLL w.f. is determined by the covariant analyticity condition on S^2

$$\nabla_{\bar{z}}\Psi_{(0)} = 0, \quad \nabla_{\bar{z}} = (1 + r^2|z|^2)\partial_{\bar{z}} + U(1) \text{ connection}$$

- Each LL is finitely degenerated since w. f. are $SU(2)$ irreps. The limit $r \to 0$ yields the planar Landau model.

Superextensions: worldline vs target space susy

Super-Landau models are quantum-mechanical models for a charged particle on a homogeneous supermanifold, such that the 'bosonic' truncation is either Landau's original model for a charged particle on a plane or Haldane's spherical version of it. There are two approaches.

- Worldline SUSY:

$$t \Rightarrow (t, \theta, \bar{\theta}), \quad z, \bar{z} \Rightarrow \mathcal{Z}(t, \theta, \bar{\theta}), \ \bar{\mathcal{Z}}(t, \theta, \bar{\theta}) \, ,$$

$$z, \bar{z} \Rightarrow (z, \bar{z}, \psi, \bar{\psi}, \ldots) - -\text{worldline supermultiplet}$$

  A version of Supersymmetric Quantum Mechanics.

- Target space SUSY:

$$\text{group manifold}: \ (z, \bar{z}) \Rightarrow \text{supergroup manifold}: \ (z, \bar{z}, \zeta, \bar{\zeta})$$

$$(P_z, P_{\bar{z}}, F_b, \kappa) \Rightarrow (P_z, P_{\bar{z}}, \Pi_\zeta, \Pi_{\bar{\zeta}}, F_b, F_f, \kappa, \ldots),$$

$$\Pi_\zeta = \partial_\zeta + \kappa\bar{\zeta}, \ \Pi_{\bar{\zeta}} = \partial_{\bar{\zeta}} + \kappa\zeta, \ F_f = \zeta\partial_\zeta - \bar{\zeta}\partial_{\bar{\zeta}}, \ \{\Pi_\zeta, \Pi_{\bar{\zeta}}\} = 2\kappa$$

  Geometrical meaning: 2 dim. plane is extended to (2+2) dim. superplane.

Superplane Landau model

Planar Super Landau models are the large radius limits (contractions) of the supersphere and superflag Landau models. One makes explicit the S^2 radius R, properly rescales Hamiltonians and sendS $R \to \infty$. The supersphere $SU(2|1)/U(1|1)$ goes into an $(2+2)$ dim. superplane.

- The superplane Landau model Lagrangian and Hamiltonian:

$$L = L_f + L_b = |\dot{z}|^2 + \dot{\zeta}\dot{\bar{\zeta}} - i\kappa\left(\dot{z}\bar{z} - \dot{\bar{z}}z + \dot{\zeta}\bar{\zeta} + \dot{\bar{\zeta}}\zeta\right),$$

$$H = a^\dagger a - \alpha^\dagger \alpha =$$

$$= \partial_{\bar{\zeta}}\partial_{\zeta} - \partial_z\partial_{\bar{z}} + \kappa\left(\bar{z}\partial_{\bar{z}} + \bar{\zeta}\partial_{\bar{\zeta}} - z\partial_z - \zeta\partial_{\zeta}\right) + \kappa^2\left(z\bar{z} + \zeta\bar{\zeta}\right)$$

- Invariances: $P_z, P_{\bar{z}}, \Pi_{\zeta}, \Pi_{\bar{\zeta}}$ and the new generators

$$Q = z\partial_{\zeta} - \bar{\zeta}\partial_{\bar{z}}\,,\; Q^\dagger = \bar{z}\partial_{\bar{\zeta}} + \zeta\partial_z\,,\; C = z\partial_z + \zeta\partial_{\zeta} - \bar{z}\partial_{\bar{z}} - \bar{\zeta}\partial_{\bar{\zeta}}$$

- They generate $ISU(1|1)$, contraction of $SU(2|1)$:

$$\{Q, Q^\dagger\} = C\,,\; [Q, P] = i\Pi\,,\; \{Q^\dagger, \Pi\} = iP$$

Norms and hidden worldline supersymmetry

- The natural $ISU(1|1)$-invariant inner product

$$< \phi|\psi >= \int d\mu\, \overline{\phi\left(z, \bar{z}; \zeta, \bar{\zeta}\right)}\, \psi\left(z, \bar{z}; \zeta, \bar{\zeta}\right),\; d\mu = dzd\bar{z}d\zeta d\bar{\zeta},$$

leads to negative norms for some component wave functions.

- All norms can be made positive by introducing the "metric" operator:

$$G = \frac{1}{\kappa}\left[\partial_\zeta \partial_{\bar{\zeta}} + \kappa^2 \bar{\zeta}\zeta + \kappa\left(\zeta\partial_\zeta - \bar{\zeta}\partial_{\bar{\zeta}}\right)\right],\; << \phi|\psi >>\sim \int d\mu\, \overline{(G\phi)}\, \psi\,.$$

- H commutes with G, so $H = H^\dagger = H^\ddagger$. However, the hermitian conjugation properties of the operators which do not commute with G, change. Let $\mathcal{O}$ be generator of some symmetry, such that $[H, \mathcal{O}] = 0$. Then

$$\mathcal{O}^\ddagger \equiv G\mathcal{O}^\dagger G = \mathcal{O}^\dagger + G\mathcal{O}^\dagger_G\,,\; \mathcal{O}_G \equiv [G, \mathcal{O}]\,, \qquad (1)$$

and O_G is another operator such that $[H, \mathcal{O}_G] = 0$. The symmetry generators that do not commute with G thus generate, in general, additional "hidden" symmetries.

Norms and hidden worldline supersymmetry (cont.)

- In our case G commutes with all $ISU(1|1)$ generators, except Q, $Q^\dagger \Rightarrow$

$$Q^\ddagger = Q^\dagger - \frac{i}{\kappa} S\,, \; S = i\left(\partial_z\partial_{\bar\zeta} + \kappa^2\bar z\zeta - \kappa\bar z\partial_{\bar\zeta} - \kappa\zeta\partial_z\right),$$

$$[H, S] = [H, S^\ddagger] = 0\,.$$

- The operators $S, S^\ddagger, H$ form $\mathcal{N} = 2, d = 1$ superalgebra

$$[H, S] = [H, S^\ddagger] = 0\,,$$

$$S = a^\dagger\alpha\,,\; S^\ddagger = a\alpha^\ddagger\,,\; \{S, S^\ddagger\} = 2\kappa H\,, \{S, S\} = 0 = \{S^\ddagger, S^\ddagger\}\,.$$

- The LLL ground state is annihilated by $S, S^\ddagger$

$$S\psi^{(0)} = S^\ddagger\psi^{(0)} = 0,$$

and so it is $\mathcal{N} = 2$ SUSY singlet. Hence $\mathcal{N} = 2$ SUSY is unbroken and all higher LL form irreps of this SUSY.

Superfield formulation (E.I., 2008)

- $\mathcal{N}$=2, d=1 superspace in the left-chiral basis: $(t + i\theta\bar\theta, \theta, \bar\theta)$.
- The basic objects are $\mathcal{N}$=2, d=1 chiral bosonic and fermionic superfields $\Phi = z(t) + \theta\chi(t)$, $\Psi = \bar\zeta(t) + \theta h(t)$, with $\chi(t)$ and $h(t)$ auxiliary fields:
$$\bar D\Phi = \bar D\Psi = 0\,.$$
- The superfield action yielding the superplane model action is:
$$S = \int dt d^2\theta \left\{\Phi\bar\Phi + \Psi\bar\Psi + \rho\left[\Phi D\Psi - \bar\Phi\bar D\bar\Psi\right]\right\} \qquad \rho = 1/(2\sqrt{\kappa})$$
- The fields h and χ are eliminated as $\chi = 2i\rho\,\dot\zeta\,, \quad h = -2i\rho\,\dot{\bar z}$. Then the action in terms of physical fields is
$$S \Rightarrow \int dt\left[i\kappa\left(z\dot{\bar z} - \bar z\dot z + \zeta\dot{\bar\zeta} - \dot\zeta\bar\zeta\right) + \left(\dot z\dot{\bar z} + \dot\zeta\dot{\bar\zeta}\right)\right].$$
- The idea (A.Beylin, T. Curtright, L. Mezincescu, E.I., 2010): to construct generalized $\mathcal{N} = 2$ supersymmetric L models by passing to the most general $\mathcal{N} = 2$ superfield action, $S \Rightarrow S_{gen}$.

Most general N=2 superfield action

- A generalization of the superplane model action is as follows

$$S_{gen} = \int dt d^2\theta \left\{ K(\Phi, \bar{\Phi}) + V(\Phi, \bar{\Phi}) \Psi \bar{\Psi} + \rho \left(\Phi D \Psi - \bar{\Phi} \bar{D} \bar{\Psi} \right) \right\} = \int dt \mathcal{L} \, .$$

  Involves two independent superfield potentials, $K(\Phi, \bar{\Phi}), \; V(\Phi, \bar{\Phi})$, goes into the superplane action when $K \Rightarrow \Phi\bar{\Phi}, \; V \Rightarrow 1$.

- After eliminating auxiliary fields in $\Phi = z + \ldots$, $\Psi = \psi + \ldots$, and setting $4\rho^2 = 1$, the component Lagrangian reads

$$\mathcal{L}_{comp} = V^{-1} \, \dot{z}\dot{\bar{z}} + i \left(\dot{z} K_z - \dot{\bar{z}} K_{\bar{z}} \right) + \; \psi - \text{terms} \, .$$

- Introducing $Z^A = (z, \psi)$, it is rewritten as

$$\mathcal{L} = \dot{Z}^A \dot{\bar{Z}}^{\bar{B}} g_{\bar{B}A} + \left(\dot{Z}^A \mathcal{A}_A + \dot{\bar{Z}}^{\bar{B}} \mathcal{A}_{\bar{B}} \right) \, ,$$

  where, e.g. $g_{\bar{z}z} = V^{-1} \left(1 - \psi\bar{\psi} \, \frac{V_z V_{\bar{z}}}{K_{z\bar{z}} V} \right), g_{\bar{z}\psi} = -\frac{V_{\bar{z}}}{K_{z\bar{z}} V} \bar{\psi}$, etc,

$$\mathcal{A}_z = i(K_z + \psi\bar{\psi} \, V_z) \, , \; \mathcal{A}_\psi = iV\bar{\psi} \, , etc$$

  The potential V defines a Kähler metric, while K - background super gauge field.

Hamiltonian and $\mathcal{N} = 2$ supercharges

- Classical Hamiltonian

$$H_{class} = \mathcal{P}_A\, g^{A\bar{B}}\, \mathcal{P}_{\bar{B}}\,, \quad \mathcal{P}_A = P_A - \mathcal{A}_A\,, P_A = \frac{\partial L}{\partial \dot{Z}^A}\,.$$

- Supercharges

$$Q = \frac{1}{i}\mathcal{P}_z\mathcal{P}_\psi, \quad \bar{Q} = \frac{1}{i}\mathcal{P}_{\bar{\psi}}\mathcal{P}_{\bar{z}},$$

$$\{Q, Q\}_{PB} = \{\bar{Q}, \bar{Q}\}_{PB} = 0, \quad \{Q, \bar{Q}\}_{PB} = -2iH_{class}\,.$$

- Quantization:

$$P_A \to -i\partial_A, \qquad P_{\bar{B}} \to -i\partial_{\bar{B}}.$$

- Quantum Hamiltonian:

$$H_q = \mathcal{P}_z V \mathcal{P}_{\bar{z}} + \mathcal{P}_z\mathcal{P}_{\bar{\psi}} V_{\bar{z}}\bar{\psi} - V_z\psi\mathcal{P}_{\bar{z}}\mathcal{P}_\psi + \mathcal{P}_{\bar{\psi}}(K_{z\bar{z}} + \psi\bar{\psi}V_{z\bar{z}})\mathcal{P}_\psi\,,$$

$$\{Q, Q^\dagger\} = 2H_q$$

- With the above definitions of Q and H, the Hermitian properties are specified with respect to the inner product with a unity measure,

$$< f, g >= \int dz d\bar{z} d\psi d\bar{\psi}\, \overline{f(z,\bar{z},\psi,\bar{\psi})}\, g(z,\bar{z},\psi,\bar{\psi})\,, \quad < f, Qg >=< Q^{\dagger} f, g > .$$

- The general wavefunction

$$\Psi\left(z,\bar{z},\psi,\bar{\psi}\right) = f_0(z,\bar{z}) + \psi f_1(z,\bar{z}) + \bar{\psi} f_2(z,\bar{z}) + \bar{\psi}\psi f_3(z,\bar{z}),$$

contains two invariant subspaces of H_q:

$$H_q\,\Psi = \lambda\Psi \;\Rightarrow\; H_q\,\psi f_1 = \lambda_1\,\psi f_1\,, \quad H_q\,\bar{\psi} f_2 = \lambda_2\,\bar{\psi} f_2\,,$$

$$H_q\,(f_0 + \bar{\psi}\psi f_3) = \lambda_3\,(f_0 + \bar{\psi}\psi f_3)\,,\; H_q\,(V^{-1} f_3 + \bar{\psi}\psi V f_0) = \lambda_4\,(V^{-1} f_3 + \bar{\psi}\psi V f_0),$$

yielding two sets of eigenvalues problems

$$-\left(\partial_{\bar{z}} - K_{\bar{z}}\right) V \left(\partial_z + K_z\right) f_1 = \lambda_1\, f_1,\;\; -\left(\partial_z + K_z\right) V \left(\partial_{\bar{z}} - K_{\bar{z}}\right) f_2 = \lambda_2\, f_2,$$

and

$$-\left(\partial_z + K_z\right)\left(\partial_{\bar{z}} - K_{\bar{z}}\right) V f_0^L = \lambda_3' f_0^L\,,\;\; -\left(\partial_{\bar{z}} - K_{\bar{z}}\right)\left(\partial_z + K_z\right) V f_0^H = \lambda_4' f_0^H$$

with $f_0 = f_0^L + f_0^H$, $f_3 = V\left(f_0^L - f_0^H\right)$.

- The functions (f_1, f_0^L) and (f_2, f_0^H) form two irreducible $N = 2$ multiplets and possess the same spectrum.

- The inner product of two wave functions contains states with negative norms, like in the super plane model

$$< f, g >= \int dz\, d\bar{z}\, d\psi\, d\bar{\psi}\, (\bar{\Psi}_f \Psi_g)$$

$$= \int dz\, d\bar{z} \left(\bar{f}_1 g_1 - \bar{f}_2 g_2 + 2V \bar{f}_0^L g_0^L - 2V \bar{f}_0^H g_0^H \right).$$

- This is cured by introducing the appropriate metric operator, $< f, g > \;\Rightarrow\; << f, g >>=< Gf, g >$,

$$G = \frac{[\mathcal{P}_{\bar{\psi}}, \mathcal{P}_\psi]}{2V} + 2\left(\psi \frac{\partial}{\partial \psi} - \bar{\psi} \frac{\partial}{\partial \bar{\psi}} \right), \; [G, Q] = [G, Q^\dagger] = [G, H_q] = 0\,.$$

With respect to the new product $<< \, , \, >>$, all norms are positive.

Integrability

- In which cases the above eigenvalue problems can be fully solved and the whole spectrum can be found?
- The salient features of the quantum theory is that the world-line $\mathcal{N}=2$ supersymmetry implies the factorization property for the component Hamiltonians (modulo constant shifts). These Hamiltonians live on a curved 2-dimensional manifold $M_2 \sim (z, \bar{z})$ and involve couplings to background magnetic field. The factorizable Hamiltonians of this sort were studied by Ferapontov and Veselov, JMP 42 (2001) 590. They found that a sufficient condition for such systems to be integrable is the existence of an infinite sequence of factorization chains, which amounts to determining infinite sequences of eigenvalues and eigenvectors of the corresponding Hamiltonians. They proved that this is the case, iff **i)** The Gauss curvature $\mathbb{K}$ of M_2 is a constant:

$$\mathbb{K} = 2g^{z\bar{z}}\partial_z\partial_{\bar{z}} \ln g^{z\bar{z}} = const,$$

and **ii)** The corresponding magnetic field is also a constant over M_2:

$$g^{z\bar{z}}\left[\bar{\nabla}_{\bar{z}}, \nabla_z\right] = c = const\,.$$

- In our case these conditions, with $\mathbb{K} \neq 0$, amount to the relation:

$$K = \frac{1}{2}\left(1 + \frac{c}{\mathbb{K}}\right) \ln V\,.$$

Integrability (cont.)

- The constant Gauss curvature $\mathbb{K}$ is known to be associated with only three types of the manifolds M_2: **a)** 2-plane with $\mathbb{K} = 0, g^{z\bar{z}} = const$; **b)** 2-sphere with $\mathbb{K} > 0, g^{z\bar{z}} \sim (1 + z\bar{z})^2$ and **c)** hyperboloid with $\mathbb{K} < 0, g^{z\bar{z}} \sim (1 - z\bar{z})^2$. The L model corresponding to the case **a)** is just super plane model. As an example of non-trivial curved solvable Landau model with world-line $N = 2$ suprsymmetry we considered the option **b)**.

Lagrangian

- The $\mathbb{CP}^1$ model corresponds to the subclass of the generic $\mathcal{N}=2$ superfield action, with

$$K(\Phi,\bar\Phi) = -N\ln\left(1+\Phi\bar\Phi\right), \quad V(\Phi,\bar\Phi) = \left(1+\Phi\bar\Phi\right)^2.$$

Here N is quantized by the standard cohomology arguments, $N \in (\mathbb{N}, \mathbb{N}+\frac{1}{2})$.

- The superfield action is invariant under SU(2) transformations

$$\delta\Phi = \varepsilon + i\beta\,\Phi + \bar\varepsilon\,\Phi^2, \quad \delta\Psi = -\left(i\beta + 2\bar\varepsilon\,\Phi\right)\Psi.$$

- The component Lagrangian:

$$\mathcal{L}_{su(2)} = \frac{\dot z\dot{\bar z}}{(1+z\bar z)^2} + N^{-1}(1+z\bar z)^2\left[1+2N^{-1}\psi\bar\psi\,(1+z\bar z)^2\right]\nabla\psi\nabla\bar\psi$$

$$-i\left[\frac{N-2\psi\bar\psi\,(1+z\bar z)^2}{1+z\bar z}\left(\dot z\bar z - \dot{\bar z}z\right) - (1+z\bar z)^2\left(\dot\psi\bar\psi - \psi\dot{\bar\psi}\right)\right],$$

$$\nabla\psi = \dot\psi + 2\frac{\dot z\bar z}{1+z\bar z}\,\psi, \quad \nabla\bar\psi = \dot{\bar\psi} + 2\frac{\dot{\bar z}z}{1+z\bar z}\,\bar\psi.$$

Eigenvalue problems

- It will be convenient to use the manifestly SU(2) covariant inner product

$$<< f, g >>= \int \frac{dz\, d\bar{z}}{(1+z\bar{z})^2} \left[\bar{f}_1 g_1 + \bar{f}_2 g_2 + 2(1+z\bar{z})^2 \left(\bar{f}_0^L g_0^L + \bar{f}_0^H g_0^H\right)\right].$$

- The corresponding eigenvalue equations:

$$-V\nabla_{\bar{z}}^{(N+1)}\nabla_z^{(N+1)} f_1 = \lambda_1 f_1, \quad -V\nabla_z^{(N-1)}\nabla_{\bar{z}}^{(N-1)} f_2 = \lambda_2 f_2,$$

$$-\nabla_{\bar{z}}^{(N-1)} V \nabla_z^{(N-1)} f_0^H = \lambda_3 f_0^H, \quad -\nabla_z^{(N+1)} V \nabla_{\bar{z}}^{(N+1)} f_0^L = \lambda_4 f_0^L,$$

with $V = (1+z\bar{z})^2, \quad \nabla_z^{(N)} = \partial_z - N\frac{\bar{z}}{1+\bar{z}z}, \; \nabla_{\bar{z}}^{(N)} = \partial_{\bar{z}} + N\frac{z}{1+\bar{z}z}.$

- Ground states are defined by equations

$$\nabla_z^{(N+1)} f_1 = \nabla_{\bar{z}}^{(N-1)} f_2 = \nabla_z^{(N-1)} f_0^H = \nabla_{\bar{z}}^{(N+1)} f_0^L = 0\,.$$

Both the ground states and excited LL states should be normalizable with respect to the above norm. This is very stringent requirement. It implies the wave functions to carry irreducible SU(2) multiplets with spins related to the number N.

Ground states

- For $N = 0$ there are two normalizable singlet ground states
$$f_0^{H,0}(z,\bar{z}) = \frac{f_0^{H,0}}{1+\bar{z}z}, \quad f_0^{L,0}(z,\bar{z}) = \frac{f_0^{L}}{1+\bar{z}z}.$$
where $f_0^{H,0}$ and $f_0^{L,0}$ are constants. Thus in this case the ground states are SU(2) singlets.
- For $N = \frac{1}{2}$, one has normalizable doublet ground states:
$$f_0^{L,0}(z,\bar{z}) = \frac{A+Bz}{(1+\bar{z}z)^{\frac{3}{2}}},$$
the constants A and B thus forming spin $1/2$ multiplet of SU(2).
- For $N \geqslant 1$, one has the following set of the ground states:
$$f_2^{0}(z,\bar{z}) = \frac{f_2^{0}(z)}{(1+\bar{z}z)^{N-1}}, N_{max} = 2(N-1), f_0^{L,0}(z,\bar{z}) = \frac{f_0^{L,0}(z)}{(1+\bar{z}z)^{N+1}}, N_{max} = 2N.$$
$f_2^0(z)$ and $f_0^{L,0}(z)$ are z-polynomials of the maximum degree N_{max}. Thus the ground states carry SU(2) spins $N-1$ and N.
- All ground states are singlets under the $\mathcal{N} = 2$ SUSY transformations, i.e. for $N \geq 0$ $\mathcal{N} = 2$ SUSY is unbroken.

Excited states

- For the first $\mathcal{N}=2$ multiplet (f_1, f_0^L) one has the full set of eigenvalues

$$E_\ell = \ell(\ell + 2N + 1), \quad \ell = 0, 1, 2, \ldots$$

$$f_1^1 = \tilde{f}_1^1, \; f_1^\ell = \nabla_z^{(N+3)} \cdots \nabla_z^{(N+2\ell-1)} \tilde{f}_1^\ell, \ell > 1, \; f_0^{L,\ell} = \nabla_z^{(N+1)} \hat{f}_1^\ell, \; \ell \geq 1$$

$$\tilde{f}_1^\ell = \frac{\tilde{f}_1^\ell(z)}{(1 + \bar{z}z)^{N+1}}, \quad \tilde{\hat{f}}_1^\ell = \frac{\tilde{\hat{f}}_1^\ell(z)}{(1 + \bar{z}z)^{N+1}}$$

  The polynomials $\tilde{f}_1^\ell(z)$ and $\tilde{\hat{f}}_1^\ell(z)$ both carry spins $(N + \ell)$. This two-fold degeneracy is related to $\mathcal{N}=2$ SUSY which mixes these two states.
- Situation with the second $\mathcal{N}=2$ multiplet (f_2, f_0^H) is more complicated and requires a separate analysis for $N \geq 1$ and $0 \leq N < 1$. For these two cases we have, respectively, the following sequences of the eigenvalues

$$E_\ell^H = \ell(\ell+2N-1), \quad \ell = 0, 1 \ldots, \; E_\ell = (\ell+1)(\ell-2N+2), \quad \ell = 0, 1 \ldots .$$

- For $N = 0$ in the second case the system reveals a four-fold degeneracy (like in the superplane Landau model).
- For $N = \frac{1}{2}$, there is no ground state for the second $\mathcal{N}=2$ multiplet and $\mathcal{N}=2$ SUSY looks as spontaneously broken in this sector. However, no actual breaking occurs because there is a singlet ground state in the first multiplet (f_1, f_0^L) at $N = \frac{1}{2}$.

Summary and outlook

- The world-line $\mathcal{N}=2$ SUSY defines a general family of quantum super L. models in terms of two independent potentials generating a Kähler metric and coupling to magnetic field. The Hamiltonians are factorized, which allows for a general definition of ground states.
- Due to non-canonical second-order kinetic terms for fermions, the states at each excited LL are grouped into two irreducible $\mathcal{N}=2$ multiplets. This is in contrast to the models with the first-order fermionic kinetic terms, where such states span a single $\mathcal{N}=2$ multiplet.
- The appearance of the negative norms, like in other super L. models, can be evaded by redefining the inner product.
- In the $\mathbb{CP}^1$ model, the eigenvalues and eigenfunctions are split into two sequences corresponding to two super monopole systems, one with the charge $2N$ and the other with the charge $2(N-1)$. $\mathcal{N}=2$ SUSY is unbroken for any strength of the monopole, in contrast to the "minimal" $\mathcal{N}=2$ $\mathbb{CP}^1$ model in which it is spontaneously broken at $N=1/2$.
- Possible physical applications: supersymmetric versions of the Quantum Hall Effect? Relations to integrable structures in $\mathcal{N}=4$ SYM and string theory? Extension to higher $\mathcal{N}$ world-line SUSY?

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