Universal first order formulation for general gauge field theories

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Based on:
M.G. and Glenn Barnich, to appear
also on
hep-th/0406192, CMP, (with G. Barnich, A.Semikhatov,I.Tipunin)
hep-th/0605089,
hep-th/0504119, hep-th/0602166 JHEP (with G. Barnich),

Preliminaries:

“Category” of local gauge field theories:
Equations of motion (Lagrangian), gauge generators, etc. involve finite number of space time derivatives. In general, “controllable number” like e.g. in noncommutative theories, String Filed Theory or Higher spin gauge theories.

Depending on a particular question (global symmetries, consistent deformations, existence of Lagrangian, and, in fact, anomalies, counterterms, quantization) one or another equivalent formulation can be more convenient.

Which local gauge field theories are to be consider equivalent?

Theories related through elimination of generalized auxiliary fields

These comprise usual auxiliary fields and the algebraically pure gauge fields (Stückelberg variables).

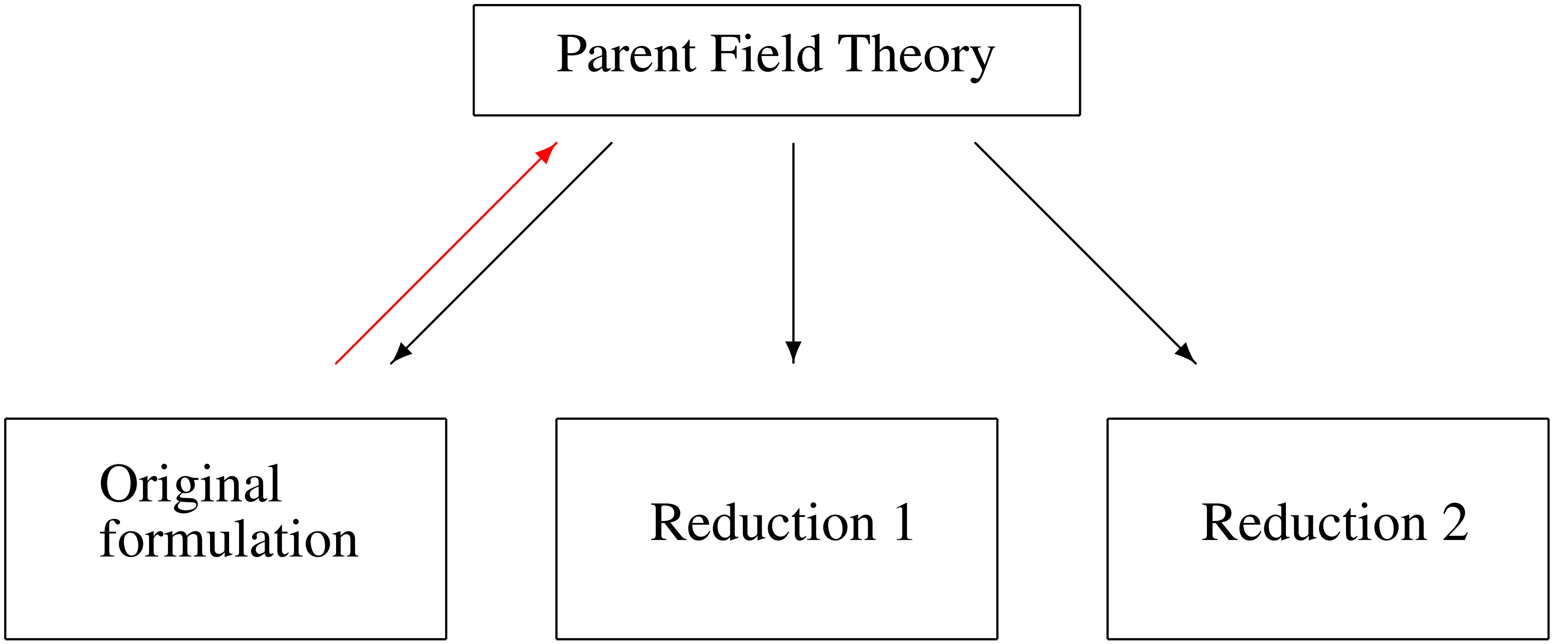
Global symmetries, consistent interactions, etc. are invariant under elimination of generalized auxiliary fields. At the level of formal path integral these can be integrated out/gauge fixed.

Examples:

- Lagrangian and Hamiltonian formulations.
- Gravity as a gauge theory of Lorentz, Poincare, ... groups.
- Gravity with cosmological constant or conformal gravity as a gauge theory of $(A)dS$ or conformal group.
- Vasiliev unfolded (and related) formulations of higher spin gauge fields

Often: new formulation manifests new geometrical structure

It turns out that new formulations can often be arrived at through equivalent reductions of an appropriate *parent* theory:



For free systems: *Barnich, M.G., Semikhatov, Tipunin (2004)*

- red arrow – explicit and constructive
- Systematic way to construct *Vasiliev* unfolded formulations
- Essentially new formulation useful in e.g. generic mixed symmetry fields on flat or AdS spaces, some conformal fields, etc.

Linear parent theory

Linear gauge theories $\cong$ BRST first quantized systems

$\mathcal{H} = \oplus_i \mathcal{H}^i$ - graded vector space. e_A - basis
$\Phi(x)$ – $\mathcal{H}$-valued "wave function"
$\Omega = \Omega^A_B(x, \frac{\partial}{\partial x})$ – BRST operator: $\mathrm{gh}(\Omega) = 1$ and $\Omega\Omega = 0$.

$$\Phi(x) = \ldots + \Phi^{-1} + \Phi^0 + \Phi^1 + \ldots \qquad \mathrm{gh}(\Phi^i) = -i\,.$$

Φ^0 – physical fields, Φ^1 – gauge parameters (ghosts),
Φ^{-1} – antifields, $\ldots$

Equations of motion, gauge symmetries, $\ldots$:

$$\Omega\Phi^{(0)} = 0\,, \qquad \Phi^{(0)} \sim \Phi^{(0)} + \Omega\chi^{(1)}\,, \ldots$$

Extension analogous to that used in Fedosov quantization

symplectic manifolds: *Fedosov (1994)*

For constrained systems: *Batalin, Fradkin, Fradkina (1990)*

Unified description in BRST terms: *Batalin, M.G., Lyakhovich (2001)*

For cotangent bundles: *Bordemann, Neumaier, Waldmann (1997)*

new variables: y^μ

new constraints: $\frac{\partial}{\partial x^\mu} - \frac{\partial}{\partial y^\mu} = 0$

new ghosts: θ^μ *Barnich, M.G., Semikhatov, Tipunin (2004)*

$$\Phi(x) \to \Phi(x, y, \theta)\,, \qquad \Omega \to \Omega^{\mathrm{parent}}$$

$$\Omega^{\mathrm{parent}} = \boldsymbol{d} - \sigma + \bar{\Omega}\,, \qquad \bar{\Omega} = \Omega(x+y, \frac{\partial}{\partial y})$$

$$\boldsymbol{d} = \theta^\mu \frac{\partial}{\partial x^\mu}\,, \qquad \sigma = \theta^\mu \frac{\partial}{\partial y^\mu}$$

$$\text{Fields:} \quad \Psi^A \quad \longrightarrow \quad \Psi^A_{(\mu_1\ldots\mu_k)[\nu_1\ldots\nu_l]}$$

Being almost trivial in the case of non-gauge systems or flat space it quite meaningful for gauge theories or general geometry. For instance higher forms become dynamical.

BRST theory

Parent theory can be generalized for general gauge theories. But one needs a bit more technology.

Batalin-Vilkovisky formalism:

Given equations T_a, gauge symmetries R^i_α, reducibility relations,.... the BRST differential:

$$s = \delta + \gamma + \ldots\,, \qquad s^2 = 0\,, \quad \mathrm{gh}(s) = 1$$

$$\delta = T_a \frac{\partial}{\partial \mathcal{P}_a} + \ldots\,, \qquad \gamma = c^\alpha R^i_\alpha \frac{\partial}{\partial \phi_i} + \ldots\,.$$

δ – (Koszule-Tate) restriction to the stationary surface
γ – implements gauge invariance condition

ϕ^i – fields, c^α – ghosts, $\mathcal{P}_a$ – ghost momenta/antifields, $\ldots$

$$\mathrm{gh}(\phi^i) = 0\,, \quad \mathrm{gh}(c^\alpha) = 1\,, \quad \mathrm{gh}(\mathcal{P}_a) = -1\,, \quad \ldots$$

BRST differential completely defines the theory.
Equations of motion, gauge symmetries can be read off from s:

$$s\mathcal{P}|_{\Psi=0,\mathrm{gh}(\Psi)\neq 0} = 0\,, \quad \delta_\epsilon \phi^i = (s\phi^i)|_{\Psi=0,\mathrm{gh}(\Psi)\neq 0,1,\, c^\alpha=\epsilon^\alpha}\,, \quad \ldots$$

In the context of local gauge field theory:

Jet space: coordinates

$$\Psi^A, \Psi^A_{;\mu}, \Psi^A_{;\mu\nu}, \ldots \qquad (x^\mu \text{ can also be included})$$

Total derivative:

$$\partial_\mu = \frac{\partial}{\partial x^\mu} + \Psi^A_{;\mu}\frac{\partial}{\partial \Psi^A} + \Psi^A_{;\mu\nu}\frac{\partial}{\partial \Psi^A_{;\nu}} + \ldots$$

BRST differential is an evolutionary vector field:

$$[\partial_\mu, s] = 0\,, \qquad s\Psi^A = s^A[\Psi, x]$$

Local functionals:

$$\text{Quotient space:} \qquad f[\Psi] \sim f[\Psi] + \partial_\mu j^\mu[\Psi]$$

More invariant way: $H^n(d = dx^\mu\partial_\mu, \text{local forms})$

$$s\omega^n + d\omega^{n-1} = 0\,, \qquad \omega^n_k \sim \omega^n_k + d\chi^{n-1}_k + s\chi^n_{k-1}$$

In the local field theory – local BRST cohomology encode physically interesting quantities.

Generalized auxiliary fields and equivalent reductions

In the Lagrangian context *(Dresse, Grégoire, Henneaux (1990))*:
–"auxiliary fields for the master action".

At the level of equations of motion: φ^α, v^a, w^a

$$(sw^a)|_{w^a=0} = 0 \quad \Leftrightarrow \quad v^a = V^a[\varphi]$$

v^a, w^a – *generalized auxiliary fields*.

Reduced system:

$$\widetilde{s}\phi^\alpha = s\phi^\alpha|_{w=0,v=V[\phi]}\,, \qquad \widetilde{s}^2 = 0$$

Constraints:

$$w^a = 0, \qquad v^a - V^a[\varphi] = 0$$

Equivalence $=$ Elimination of generalized auxiliary fields

(Local) BRST cohomology are invariant. E.g. observables, global symmetries, consistent interactions, anomalies, etc. are isomorphic. In addition, possible Lagrangians are also the isomorphic.

Parent theory at the nonlinear level

How to pass from the first-quantized to the field theory description?
(replace BRST operator Ω with BRST differential s)

For a BRST first quantized system. Consider $\Phi(x)$ as a "string field"
$\Phi(x) = \Psi^A(x)e_A$ and define

$$s\Phi = \Omega\Phi$$

Here: s acts on Ψ^A while Ω acts on x and e_A.
This is easy to do for $\boldsymbol{d}$ and σ entering Ω^{parent}. Combine fields into:

$$\widetilde{\Psi}^A(x|y,\theta) = \sum_{k,l} \frac{1}{k!l!} \Psi^A_{(\lambda_1\dots\lambda_k)[\nu_1\dots\nu_l]}(x)\theta^{\nu_1}\dots\theta^{\nu_l}y^{\lambda_1}\dots y^{\lambda_k}$$

(variables y, θ are now just "bookkeeping device" for indexes!!!)

$$(\boldsymbol{d})^{\mathrm{F}}\,\widetilde{\Psi} = \boldsymbol{d}\widetilde{\Psi}\,, \qquad (\sigma)^{\mathrm{F}}\,\widetilde{\Psi} = \sigma\widetilde{\Psi}$$

For instance:

$$(\sigma)^{\mathrm{F}}\,\Psi^A_{(0)[\mu]} = \Psi^A_{(\mu)[0]}\,, \qquad (\boldsymbol{d})^{\mathrm{F}}\,\Psi^A_{(0)[\mu]} = \partial_\mu\Psi^A_{(0)[0]}\,,$$

On the space of fields $\Psi^A_{(\lambda_1\dots\lambda_k)[\nu_1\dots\nu_l]}$ define $\bar{s}$ by

$$\bar{s}\Psi^A_{(0)[0]} = s\Psi^A|_{\partial_{(\mu)}\Psi^A \to \Psi^A_{(\mu)[0]}}\,,$$

$$[\bar{s}, \frac{\partial}{\partial x^\mu} + \left(\frac{\partial}{\partial y^\mu}\right)^{\mathrm{F}}] = 0\,, \qquad [\bar{s}, \left(\frac{\partial}{\partial \theta^\mu}\right)^{\mathrm{F}}] = 0$$

Finally:

$$s^{\mathrm{parent}} = (\boldsymbol{d})^{\mathrm{F}} - (\sigma)^{\mathrm{F}} + \bar{s}\,, \qquad \text{Fields: } \Psi^A_{(\mu)[\nu]}(x)$$

In more conventional form:

$$s^{\mathrm{parent}} = \int d^n x d^n \theta \left[\boldsymbol{d}\Psi^a - \theta^\mu \left((\frac{\partial}{\partial y^\mu})^F \Psi\right)^a + s^a(\Psi)\right] \frac{\delta}{\delta \Psi^a(x,\theta)}$$

$\Psi^a = \{\Psi^A_{(\mu)}\}$ – target space coordinates

– generalization of AKSZ sigma model

AKSZ sigma model

Alexandrov, Kontsevich, Schwartz, Zaboronsky(1994)

Consider two Q-manifolds: *A.Schwartz (1992)*

Target space: $\mathcal{M}$, degree $\mathrm{gh}_{\mathcal{M}}$, nilpotent vector field Q

$$Q^2 = 0\,, \qquad \mathrm{gh}_{\mathcal{M}}(Q) = 1$$

Space-time: $\mathcal{X}$, degree $\mathrm{gh}_{\mathcal{X}}$, $\boldsymbol{d}$, $\mathrm{gh}_{\mathcal{X}}(\boldsymbol{d}) = 1$, $\boldsymbol{d}^2 = 0$,
$\boldsymbol{d}$-invariant volume form $d\mu$

Typical example: $\mathcal{X} = \Pi T\mathcal{X}_0$, coordinates x^μ, θ^μ, $n = \dim \mathcal{X}_0$

$$\boldsymbol{d} = \theta^\mu \frac{\partial}{\partial x^\mu}\,, \quad d\mu = dx^0 \ldots dx^{n-1} d\theta^{n-1} \ldots d\theta^0 \equiv d^n x d^n \theta$$

Supermanifold of maps ($\mathcal{M}$-valued fields on $\mathcal{X}$): BRST differential:

$$s = \int d^n x d^n \theta \left[\boldsymbol{d} \Psi^a(x,\theta) + Q^a(\Psi(x,\theta)) \right] \frac{\delta}{\delta \Psi^a(x,\theta)}$$

total ghost degree: $\mathrm{gh}(A) = \mathrm{gh}_{\mathcal{M}}(A) + \mathrm{gh}_{\mathcal{X}}(A)$

Because $s^2 = 0$, $\mathrm{gh}(s) = 1 \quad \Longrightarrow$
local gauge field theory

In terms of jet space:

$$s = (\boldsymbol{d})^{\mathrm{F}} + \bar{Q} \qquad \text{(cf. } s^{\mathrm{parent}} = (\boldsymbol{d})^{\mathrm{F}} - (\sigma)^{\mathrm{F}} + \bar{s}\text{)}$$

– Nonlinear parent theory is a generalization of the AKSZ sigma model with the target space being jet space associated to the starting point system and Q-structure being the starting point BRST differential s.

Features:

- Parent theory is *equivalent to the original* theory through the elimination of the generalized auxiliary fields. More precisely all $\Psi^A_{(\mu)[\nu]}$ are generalized auxiliary save for $\Psi^A_{(0)[0]}$.
- Any collection of *contractible pairs for s* (i.e. w^a, v^a such that $sw^a = v^a$) *are generalized auxiliary fields* for the parent theory. This gives lots of possibilities to construct equivalent reduced theories. For instance using the known results on (local) BRST cohomology.
- If the starting point theory is diffeomorphism invariant and diffeomorphisms are in the generating set of gauge transformations (there is a ghost field ξ^μ and $s = \xi^\mu \partial_\mu + \ldots$) then $(\sigma)^{\mathrm{F}}$ can be absorbed into $\bar{s}$ by the following field redefinition:

$$\xi^\mu_{(0)[\nu]} \to \xi^\mu_{(0)[\nu]} + \delta^\mu_\nu \qquad \text{(originates from } \xi^\mu \to \xi^\mu + \theta^\mu\text{)}$$

In this case parent theory is precisely AKSZ sigma model

- For regular gauge theories $s = \delta + \gamma + \dots$ and cohomology of Koszule-Tate differential δ do not involve antifields. Eliminating contractible pairs for δ one gets rid off all the antifields. In this case $\mathrm{gh}(\Psi^A) \geqslant 0$ after the reduction. If in addition the system is diffeomorphism invariant then

$$s^{\mathrm{parent}} = (\boldsymbol{d})^{\mathrm{F}} + \widetilde{\gamma}\,, \qquad \widetilde{\gamma} = (\gamma \text{ reduced to } \delta \text{ cohomology})$$

Equations of motion: Free Differential Algebra (FDA)

This explains the relation to the *unfolded formalism*

Vasiliev (1989)...(2005)...

Moreover, the above procedure can be seen as systematic way to construct unfolded formulation.

There is also a close relationship to the FDA approach to SUGRA by *d'Auria, P. Fre (1982)...(2008)...*

In pure math FDA were introduced and studied in *Sullivan (1977).*

Somewhat related recent approach "double field theory" by *Hull, Zwiebach (2009).*

- Universality – because all the derivatives are independent fields the parent theory is a kind of universal (hence the talk title...)

Examples

Gravity in the metric formulation

Fields: g^{ab} itself, ghost field ξ^a, antifields g^*_{ab} and ξ^*_a. The BRST differential: $s = \delta + \gamma$

$$\delta g^*_{ab} = \frac{\delta}{\delta g^{ab}} L[g]\,, \qquad \delta \xi^*_c = g^*_{ab}\partial_c g^{ab} + 2\partial_a (g^{ab} g^*_{bc})$$

and

$$\begin{aligned}
\gamma g^{ab} &= L_\xi g^{ab} = \xi^c \partial_c g^{ab} - g^{cb}\partial_c \xi^a - g^{ac}\partial_c \xi^b\,,\\
\gamma \xi^c &= \frac{1}{2}[\xi,\xi]^c = \xi^a \partial_a \xi^c\,,\\
\gamma g^*_{ab} &= -\partial_c (g^*_{ab}\xi^c) - g^*_{ac}\partial_b \xi^c\,,\\
\gamma \xi^*_c &= -\partial_a(\xi^*_c \xi^a) - \xi^*_c \partial_a \xi^a
\end{aligned}$$

Eliminating the antifields and absorbing $(\sigma)^{\mathrm{F}}$ one ends up with

$$\begin{aligned}
\widetilde{\gamma} g^{ab} &= L_\xi g^{ab} = \xi^c \partial_c g^{ab} - g^{cb}\partial_c \xi^a - g^{ac}\partial_c \xi^b\,,\\
\widetilde{\gamma} \xi^c &= \frac{1}{2}[\xi,\xi]^c = \xi^a \partial_a \xi^c\,,
\end{aligned}$$

The parent theory BRST differential reduces to:

$$s^{\text{parent}} = (\boldsymbol{d})^{\text{F}} + \widetilde{\gamma}$$

Physical (vanishing ghost degree) fields:

$$g^{ab}_{(c)} = g^{ab}_{(c)} + \theta^\mu(\ldots) + \ldots\,, \qquad \xi^a_{(b)} = \xi^a_{(b)} + \theta^\mu \xi^a_{\mu(b)} + \ldots$$

To make contact to the literature, let

$y^a, p_b \qquad [y^a, p_b] = \delta^a_b$ – canonically conjugated variables

Generating functions:

$$F = g^{ab}_{(c)} p_a p_b y^{(c)}\,, \qquad A = \theta^\mu \xi^a_{\mu(b)} p_a y^{(b)}$$

A – 1-form with values in the Lie algebra of vector fields
F – 0-form taking values in the respective module.
Equations of motion: *Vasiliev (2005)*

$$dA + \frac{1}{2}[A, A] = 0\,, \quad dF + [A, F] = 0\,, \quad \text{algebraic constraints}$$

Gravity as a gauge theory of diffemorphism algebra

Spin-2 sector of equations in *Fedosov (1994)*. First-quantized interpretation *M.G. (2006).* Linearized version *Barnich, M.G., Semikhatov, Tipunin (2004).*

One can reduce further:
All variables are contractible pairs but
Barnich, Brandt, Henneaux (1995), Brandt (1997)

$$\xi^a, \xi^a_b, \qquad g^{ab}, R^a_{bcd}, D_e R^a_{bcd}, \ldots .$$

The differential:

$$\widetilde{\gamma}\xi^a = \xi^b \xi^A_b, \qquad \widetilde{\gamma}\xi^b_a = \xi^c_a \xi^b_c + \frac{1}{2}\xi^c \xi^d R^b_{cda}, \quad \widetilde{\gamma}(R) = \ldots$$

"Russian formula". *Known in YM context since R.Stora (1982)*
Under usual assumptions one can put $g^{ab} = \eta^{ab}$ and eliminate symmetric part of ξ^a_b. The physical fields are then

$$\xi^a = \xi^a + \theta^\mu e^a_\mu + \ldots, \quad \xi^a_b = \xi^a_b + \theta^\mu \omega^a_{\mu b}, \quad \text{curvatures} + \ldots$$

The equations of motion are

$$\boldsymbol{d}e^a + \omega^b_a e^a = 0, \qquad \boldsymbol{d}\omega^a_b + \omega^c_a \omega^b_c + \frac{1}{2} e^c e^d R^b_{cda} = 0, \qquad \ldots$$

The direct procedure to obtain the remaining equations was in *Vasiliev (2005)*

Relativistic particle

Action

$$S = \frac{1}{2}\int d\tau(\lambda^{-1}\eta_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu} + \lambda m^2)\,.$$

Gauge transformation

$$\delta x^{\mu} = \dot{x}^{\mu}\epsilon\,, \qquad \delta\lambda = \dot{\lambda}\epsilon + \lambda\dot{\epsilon}\,,$$

Ghost variables ξ, antifields $x^*_{\mu}, \lambda^*, \xi^*$. BRST differential

$$sx^{\mu} = \xi\dot{x}^{\mu}\,, \quad s\lambda = \dot{\xi}\lambda + \xi\dot{\lambda}\,, \quad s\xi = 0\,, \quad \ldots$$

For simplicity $\eta_{\mu\nu} = const$ but all works in general.

Because of 1-d diffeomorphisms $(\sigma)^{\mathrm{F}}$ can be absorbed by s:

$$s^{\mathrm{parent}} = (\boldsymbol{d})^{\mathrm{F}} + s\,, \qquad \text{1d AKSZ sigma model}$$

All variables can be eliminated save for *Brandt (97)*

$$x^\mu\,, \quad p_\mu = \lambda^{-1}\dot{x}_\mu - \xi x^*_\mu\,, \quad C = \lambda\xi\,, \quad \mathcal{P} = \lambda^* - \lambda^{-1}\xi\xi^*$$

The reduced differential:

$$\widetilde{s}x^\mu = Cp^\mu\,, \quad \widetilde{s}p_\mu = 0\,, \quad \widetilde{s}C = 0\,, \quad \widetilde{s}\mathcal{P} = \frac{1}{2}(m^2 - p^2)$$

Remarkably:

$$\widetilde{s} = \{\Omega,\,\cdot\,\}\,, \qquad \Omega = -C\frac{1}{2}(p^2 - m^2)$$

$$\{x^\mu, p_\nu\} = \delta^\mu_\nu\,, \qquad \{C, \mathcal{P}\} = 1$$

The resulting 1d AKSZ model is known *M.G., Damgaard (1999)*

In general:
In 1d case parent theory contains the Hamiltonian BFV-BRST formulation as a particular reduction!
Target space is an extended phase space of BFV-BRST formalism.

This suggests the interpretation of the target space in the multidimensinal case as well...

Conclusions

- Because of the generality nearly any gauge theory can easily be used as an example. Moreover, the computations needed for reductions are identical to those for local BRST cohomlogy. Especially in the covariant tensor calculus of *F. Brandt*. Using these results one can explicitly find new forms of
  - YM (and Einstein-YM theory). In this way one can e.g. derive Vasiliev unfolded formulation of YM.
  - 4d minimal SUGRA
  - Bosonic string. In this case it is not surprising that the resulting theory is a gauge theory of the Virasoro algebra
  - Conformal gravity. BRST cohomology computations done in *Boulanger (2004)*
  - ...
- Can be considered as a systematic way to construct unfolded formulation of general gauge theories.

- Generating procedure for new formulations. In particular, those that manifest one or another structure. In some sense parent formulation and its reductions make the gauge and the BRST cohomology structure manifest. For instance, gravity as a gauge theory of diffeomorphism algebra or bosonic string as a gauge theory for Virasoro algebra.

- As a tool to find a relevant geometry. For instance starting from metric gravity one end up with the Cartan formulation and finds relevant curvatures just by trying to compute BRST cohomology.

- Instead of constructing parent formulation for a given theory one can construct theory immediately in the parent or related form. This approached has proved extremely useful in the context of Higher Spin gauge theories already in its linear version. For instance, concise formulations for general mixed symmetry gauge fields on Minkowski or AdS space were constructed *Alkalaev, M.G, Tipunin (2008), Alkalaev, M.G (2009).* (talk by K. Alkalaev at this conference). Classification of global symmetries for bosonic singletons *Bekaert, M.G. (2009).*