

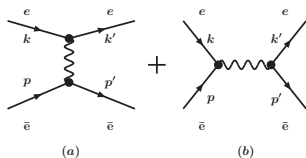
# Bound states in QED

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Quarks-2010

- Bound states is not correctly formulated problem in QFT
- Standard approaches  $\Rightarrow$  reduce to non-relativistic Schr.-eq.
  - 1 Bethe-Salpeter eq., Quasipotential eq.;
  - 2 Effective non-relativistic Lagrangians
- TIME should be excluded to get potential picture.
- Relativistic QFT  $\Rightarrow$  Functional integral approach
  - 1 Formally exact representation for any Green functions;;
  - 2 Standard using  $\Rightarrow$  representation of perturbation series.
- Our results
  - 1 Definition and calculation of bound states.
  - 2 Role of TIME in bound states formation
- Relativistic QED
  - 1 Bound states does exist
  - 2 Role of TIME is important  $\Rightarrow$  corrections of  $\alpha^{\frac{2}{3}}$ ;
  - 3 ortho- para- positronium mass difference is not described correctly.
- QED is not suited to describe bound state problem correctly

## Breit potential - electron+positron



♣ Born approximation  $\mathbf{k} \rightarrow 0$   $A(\mathbf{k}) \sim \tilde{U}(\mathbf{k}) \Rightarrow U(r)$

♣  $e$  and  $\bar{e}$   $\rightarrow$  Solutions of the Dirac equation.

$$U_a(r) = -\frac{\alpha}{r} + \frac{2\pi}{3} \frac{\alpha}{m^2} (\boldsymbol{\sigma}_- \boldsymbol{\sigma}_+) \delta(\mathbf{r}), \quad U_b(r) = \frac{\pi}{2} \frac{\alpha}{m^2} (\boldsymbol{\sigma}_- \boldsymbol{\sigma}_+) \delta(\mathbf{r}),$$

$$\Delta = \epsilon_{ortho} - \epsilon_{para} = \frac{7}{12} \alpha^4 m = \left[ \left( \frac{1}{3} \right)_{(a)} + \left( \frac{1}{4} \right)_{(b)} \right] \alpha^4 m$$

$$\Delta_{exp} = 203.38910 \text{ GHz} = 8.4115 \cdot 10^{-4} \text{ eV} = \frac{7}{12} \alpha^4 m_e \cdot 0.99512\dots$$

## QFT

$$\text{Current} \quad \mathbf{J}_\Gamma(x) = (\bar{\psi}(x)\Gamma\psi(x))$$

$$\begin{aligned} G_\Gamma(x) &= \langle 0 | \mathbf{J}_\Gamma(x) \mathbf{J}_\Gamma(0) | 0 \rangle \\ &= \sum_{\text{particles}} \langle 0 | \mathbf{J}_\Gamma(x) | n \rangle \langle n | \mathbf{J}_\Gamma(0) | 0 \rangle + \sum_{\text{photons}} \langle 0 | \mathbf{J}_\Gamma(x) | n \rangle \langle n | \mathbf{J}_\Gamma(0) | 0 \rangle \\ &= \sum_{\text{particles}} e^{-E_n|x|} |\langle 0 | \mathbf{J}_\Gamma(0) | n \rangle|^2 + \sum_{\text{photons}} e^{-E_n|x|} |\langle 0 | \mathbf{J}_\Gamma(0) | n \rangle|^2 \\ &\implies e^{-E_{\min}|x|} + \frac{1}{|x|^2}, \quad |x| \rightarrow \infty \end{aligned}$$

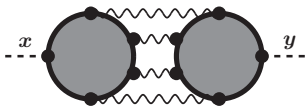
## QED: Lagrangian and bound states

$$\mathcal{L}(x) = -\frac{1}{4} (\partial_\mu A_\nu(x) - \partial_\nu A_\mu(x))^2 + (\bar{\psi}(x)[i(\hat{p} + e\hat{A}(x)) - m]\psi(x))$$

$$G_\Gamma(x-y) = \langle 0 | \mathbf{J}_\Gamma(x) \mathbf{J}_\Gamma(y) | 0 \rangle = \int D A D \psi D \bar{\psi} e^{\int dx \mathcal{L}(x)}$$



$$B(x-y) \sim e^{-M|x-y|}$$



$$H(x-y) \sim \frac{1}{|x-y|^2}$$

Mass of the bound state  $M = - \lim_{|x| \rightarrow \infty} \frac{1}{|x|} \ln B(x)$

$$B_{\Gamma}(x) = \int \frac{DA}{C} e^{-\frac{1}{2}(A_{\mu} D_{\mu\nu}^{-1} A_{\nu}) + T[A]} \cdot \text{Tr}[\Gamma S(x, 0|A) \Gamma S(0, x|A)]$$



$$D_{\mu\nu}(k) = \frac{1}{k^2} \left[ \delta_{\mu\nu} + d(k^2) \frac{k_{\mu} k_{\nu}}{k^2} \right]$$



$$S(x, 0|A) = \frac{1}{i(\hat{p} + e\hat{A}(x)) - m} \delta(x)$$

$$= [i(\hat{p}_x + e\hat{A}(x)) + m] \int_0^{\infty} \frac{ds}{8\pi^2 s^2} e^{-\frac{1}{2} \left[ m^2 s + \frac{x^2}{s} \right]}$$

$$\cdot \int \frac{D\eta}{C} e^{-\int_0^s dt \frac{\dot{\eta}^2(t)}{2}} e^{+ie \int_0^s dt \dot{z}_{\mu}(t) A_{\mu}(z(t))} \text{Tr}_t \left\{ e^{\frac{\varepsilon}{4} \int_0^s dt \sigma_{\mu\nu}(t) F_{\mu\nu}(z(t))} \right\},$$

$$z(t) = x \frac{t}{s} + \eta(t), \quad \eta(0) = \eta(s) = 0$$

$$X = |x| \rightarrow \infty \quad \Longrightarrow \quad s = \frac{X}{m}$$

$$S(x, 0|A) \Rightarrow \frac{\text{const}}{X^{\frac{1}{2}}} (1 + \gamma_0) e^{-mX} \cdot S(X|A)$$

$$S(X|A) = \int \frac{D\eta}{C} e^{-\int_0^X d\tau \frac{m\dot{\eta}^2(\tau)}{2} + ie \int_0^X d\tau \dot{z}_\mu(\tau) A_\mu(z(\tau))} R[z],$$

$$R[z] = T_\tau \left\{ e^{\frac{e}{4m} \int_0^X d\tau \sigma_{\mu\nu}(\tau) F_{\mu\nu}(z(\tau))} \right\} = e^{\frac{e}{4m} (\sigma_{\mu\nu} F_{\mu\nu}[z])}$$

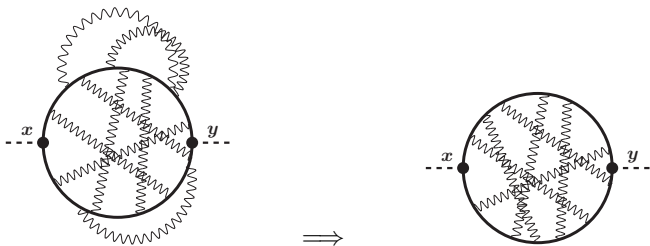
$$z(\tau) = n\tau + \eta(\tau) = \begin{cases} \eta(\tau), \\ \tau + \eta_4(\tau). \end{cases}$$

$$\mathbf{B}_\Gamma(X) \sim e^{-2mX} \iint \frac{D\eta_1 D\eta_2}{C} e^{-\int_0^X d\tau \frac{m}{2} [\dot{\eta}_1^2(\tau) + \dot{\eta}_2^2(\tau)]} \mathcal{F}_\Gamma[X, \eta_1, \eta_2]$$

$$\mathcal{F}_\Gamma[X, \eta_1, \eta_2]$$

$$= \int \frac{DA}{C} e^{-\frac{1}{2}(A_\mu D_{\mu\nu}^{-1} A_\nu) + ie \int_0^X d\tau \dot{z}_\mu^{(1)}(\tau) A_\mu(z^{(1)}(\tau)) + ie \int_0^X d\tau \dot{z}_\mu^{(2)}(\tau) A_\mu(z^{(2)}(\tau))}$$

$$\cdot \frac{1}{4} \text{Tr} \left[ \Gamma (1 + \gamma_0) e^{\frac{e}{4m} (\sigma_{\mu\nu} F_{\mu\nu} [z^{(1)}])} \Gamma (1 - \gamma_0) e^{\frac{e}{4m} (\sigma_{\mu\nu} F_{\mu\nu} [z^{(2)}])} \right]$$





$$\frac{1}{4} \text{Tr} \left[ \Gamma (1 + \gamma_0) e^{\frac{e}{4m} (\sigma_{\mu\nu} F_{\mu\nu} [z^{(1)}])} \Gamma (1 - \gamma_0) e^{\frac{e}{4m} (\sigma_{\mu\nu} F_{\mu\nu} [z^{(2)}])} \right]$$

$$\Rightarrow \Sigma_{\Gamma}^{(0)} \cdot e^{\frac{e^2}{(4m)^2} \int dk \mathcal{K}(k) \Delta_{\Gamma}(k)} + O(e^4)$$

$J^P$	$\Gamma$	$\Sigma_{\Gamma}^{(0)}$	$\Delta_{\Gamma}(k)$
$P(0^-)$	$i\gamma_5$	$-2$	$-2 \cdot \frac{k^2}{k^2 + k_4^2}$
$V(1^-)$	$\gamma_j$	$-2\delta_{ij}$	$\frac{2}{3} \cdot \frac{k^2}{k^2 + k_4^2}$
$S(0^+)$	$I$	$0$	$0$
$A(1^+)$	$\gamma_5 \gamma$	$0$	$0$

## Functional integral problem

$$d\sigma_{\eta_1\eta_2} = \frac{D\eta_1 D\eta_2}{C} e^{-\int_0^X dt \frac{m}{2} [\dot{\eta}_1^2(t) + \dot{\eta}_2^2(t)] + W}, \quad J_0(X) = \int d\sigma_{\eta_1\eta_2} = J_0 e^{\epsilon_0 X}$$

$$W[X; \eta_1, \eta_2] = e^2 \iint_0^X d\tau_1 d\tau_2 \int \frac{dk}{(2\pi)^4} \cdot \frac{e^{ik(\tau_1 - \tau_2)}}{k^2} e^{ik(\eta_1(\tau_1) - \eta_2(\tau_2))}$$

$$\epsilon_\Gamma = \lim_{X \rightarrow \infty} \frac{e^2}{4m^2} \frac{1}{X} \iint_0^X d\tau_1 d\tau_2 \int \frac{dk \Delta_\Gamma(k) e^{ik_4(\tau_1 - \tau_2)}}{(2\pi)^4} \cdot \left\langle e^{ik(\eta_1(\tau_1) - \eta_2(\tau_2))} \right\rangle_{\eta_1\eta_2}$$

$$\left\langle e^{ik(\eta_1(\tau_1) - \eta_2(\tau_2))} \right\rangle_{\eta_1\eta_2} = \frac{1}{J_0(X)} \iint d\sigma_{\eta_1\eta_2} e^{ik(\eta_1(\tau_1) - \eta_2(\tau_2))}$$

## Coulomb and linear potentials

$$\left\{ \begin{array}{l} \left[ -\frac{1}{2m} \frac{d^2}{dx^2} - \frac{\alpha}{|x|} \right] \Psi = -\epsilon \Psi \\ \mathbf{x} = \frac{\mathbf{y}}{\alpha}, \quad \epsilon = \varepsilon \alpha^2 \\ \left[ -\frac{1}{2m} \frac{d^2}{dy^2} - \frac{1}{|x|} \right] \Psi = -\varepsilon \Psi \end{array} \right.$$

$$\left\{ \begin{array}{l} \left[ -\frac{d^2}{ds^2} - \alpha |s| \right] \Phi = E \Phi \\ s = \frac{v}{\alpha^{\frac{1}{3}}}, \quad E = \mathcal{E} \alpha^{\frac{2}{3}} \\ \left[ -\frac{d^2}{dv^2} - |v| \right] \Phi = \mathcal{E} \Phi \end{array} \right.$$

## Functional integral and $\alpha$

$$Y = \alpha^2 m X, \quad \tau = \frac{v}{\alpha^2 m}, \quad k_4 = \alpha^2 m q, \quad \mathbf{k} = \alpha m \mathbf{q}.$$

$$\eta(t) = \frac{1}{\alpha m} \xi(v), \quad \eta(t) = \frac{1}{\alpha m} \xi(v).$$

$$d\sigma = \frac{D\xi_1 D\xi_2 D\xi_1 D\xi_2}{C} e^{-\frac{1}{2} \int_0^Y dv \left[ \dot{\xi}_1^2(v) + \dot{\xi}_2^2(v) + \xi_1^2(\tau) + \xi_2^2(v) \right]} + W[\xi_1, \xi_2, \xi_1, \xi_2; \alpha]$$

$$W[\xi_1, \xi_2, \xi_1, \xi_2; \alpha]$$

$$= \int_0^Y \int_0^Y dv_1 dv_2 \iint \frac{d\mathbf{q} d\mathbf{q}}{4\pi^3} \frac{e^{i\mathbf{q}(v_1 - v_2)}}{\mathbf{q}^2 + \alpha^2 \mathbf{q}^2} \cdot e^{i\alpha \mathbf{q}(\xi_1(v_1) - \xi_2(v_2)) + i\mathbf{q}(\xi_1(v_1) - \xi_2(v_2))}$$

$$\xi_{1,2}(\tau) = \mathbf{R}(\tau) \pm \frac{1}{2}\boldsymbol{\rho}(\tau), \quad \xi_{1,2}(\tau) = R(\tau) \pm \frac{1}{2}\rho(\tau),$$

$$\begin{aligned}
 W[\boldsymbol{\rho}, \rho; \alpha] &= \int_0^Y \int_0^Y dv_1 dv_2 \iint \frac{d\mathbf{q} dq}{4\pi^3} \frac{e^{iq(v_1-v_2) - \frac{1}{4}[\alpha^2 q^2 + \mathbf{q}^2]|v_1-v_2|}}{\mathbf{q}^2 + \alpha^2 q^2} \\
 &\cdot e^{-\frac{i\alpha q}{2}(\rho(v_1)+\rho(v_2)) - \frac{iq}{2}(\boldsymbol{\rho}(v_1)+\boldsymbol{\rho}(v_2))} \\
 &= \int_0^Y \frac{dv}{|\boldsymbol{\rho}(v)|} - \frac{\alpha}{8} \int_0^Y dv |\rho(v)| + O(\alpha^2)
 \end{aligned}$$

$$d\sigma = \frac{D\rho}{C\rho} e^{-\int_0^Y dv \left[ \frac{1}{4}\dot{\rho}^2(v) - \frac{1}{|\boldsymbol{\rho}(v)|} \right]} \cdot \frac{D\rho}{C\rho} e^{-\int_0^Y dv \left[ \frac{1}{4}\dot{\rho}^2(v) + \frac{\alpha}{8} |\rho(v)| \right]}$$

## Orto-para- mass difference

$$\delta M = \frac{1}{3} \cdot \alpha^4 m \cdot \Delta(\alpha),$$

$$\Delta(\alpha) = \lim_{Y \rightarrow \infty} \frac{1}{Y} \int_0^Y \int_0^Y dv_1 dv_2 \iint \frac{d\mathbf{q} d\mathbf{q}'}{2\pi^3} \cdot \frac{\mathbf{q}^2 e^{iq(v_1 - v_2) - \frac{1}{4}[\alpha^2 q^2 + \mathbf{q}'^2]|v_1 - v_2|}}{\mathbf{q}^2 + \alpha^2 q^2}$$

$$\cdot \left\langle e^{-\frac{iq}{2}(\rho(v_1) + \rho(v_2))} \right\rangle_{\rho} \left\langle e^{-\frac{i\alpha q}{2}(\rho(v_1) + \rho(v_2))} \right\rangle_{\rho}$$



$$\left\langle e^{i\frac{k}{2}[\rho(\tau_1) + \rho(\tau_2)]} \right\rangle_{\rho} \Rightarrow \sum_{n\ell} e^{-|\tau_1 - \tau_2|(E_n - E_{00})} (-1)^{\ell} (2\ell + 1) \mathbf{C}_{n\ell}^2 \left( \frac{k}{2} \right).$$

$$\left\langle e^{-i\frac{q}{2}(\rho(\tau_1) + \rho(\tau_2))} \right\rangle_{\rho} \Rightarrow \sum_{\kappa} e^{-|\tau_1 - \tau_2|(E_{\kappa} - E_0)} (-1)^{\kappa} \left| \mathcal{A}_{\kappa} \left( \alpha^{\frac{2}{3}} \mathbf{q} \right) \right|^2$$

$$\Delta(\alpha) = \sum_{n=1}^{\infty} \sum_{\ell=0}^{n-1} \sum_{\kappa=0}^2 (-1)^{\ell+\kappa} (2\ell+1) \frac{32}{\pi^2} \int_0^{\infty} \int_0^{\infty} \frac{dkdq k^4}{k^2 + \alpha^2 q^2}$$

$$\cdot \frac{k^2 + \alpha^2 q^2 + 1 - \frac{1}{n^2} + \alpha^{\frac{2}{3}}(\epsilon_{\kappa} - \epsilon_0)}{\left(k^2 + \alpha^2 q^2 + 1 - \frac{1}{n^2} + \alpha^{\frac{2}{3}}(\epsilon_{\kappa} - \epsilon_0)\right)^2 + 16q^2} \mathbf{C}_{nl}^2 \left(\frac{k}{2}\right) \left| \mathcal{A}_{\kappa} \left(\alpha^{\frac{2}{3}} q\right) \right|^2$$



$$\delta M = \left(\frac{7}{12}\right) \frac{1}{3} \cdot \alpha^4 m \cdot \Delta(\alpha) = \begin{cases} \Delta(0) = 1 \\ \Delta(\alpha) = 1.0000 \\ \Delta(\alpha) = 0.9528 \approx 1 - \frac{5}{4}\alpha^{\frac{2}{3}} \end{cases}$$

$$\Delta_{exp} = 1 - \frac{2}{3}\alpha = 0.9951$$

## Conclusion

- The functional approach is the best mathematical representation to preserve the gauge invariance.
- The lowest approximation of this functional representation is pure non-relativistic Feynman representation of the non-relativistic Schrödinger equation.
- Any regular series for next corrections over  $\alpha$  do not exist and these corrections can not be reduced to some terms into non-relativistic potential in the Schrödinger picture.
- The "non-physical" time coordinate plays important role in the relativistic QED and leads to nonanalytic corrections of the order  $\alpha^{\frac{2}{3}}$ .
- Non-relativistic QED and Relativistic QED give the different numbers for ortho- para- mass difference.
- Relativistic QED is not suited to describe real bound states.
- Non-relativistic QED and Relativistic QED are different theories.
- Dirac (1981): Today QFT is not THEORY, but set of rules to calculate amplitudes.



## Appendix

$$H = - \left( \frac{\partial}{\partial \mathbf{x}} \right)^2 + U(|\mathbf{x}|), \quad H\Psi_n(\mathbf{x}) = E_n\Psi_n(\mathbf{x}).$$

$$\begin{aligned} G_X(\mathbf{x}, \mathbf{x}') &= e^{-HX} \delta(\mathbf{x} - \mathbf{x}') = \int_{\boldsymbol{\rho}(0)=\mathbf{x}', \boldsymbol{\rho}(X)=\mathbf{x}} \frac{D\boldsymbol{\rho}}{C} e^{-\int_0^X d\tau [\frac{1}{4}\dot{\boldsymbol{\rho}}^2(\tau) + U(|\boldsymbol{\rho}(\tau)|)]} \\ &= \sum_n \Psi_n(\mathbf{x}) e^{-E_n X} \Psi_n^+(\mathbf{x}'). \end{aligned}$$

The Green function satisfies the correlation for  $t > t'' > t'$

$$G_X(\mathbf{x}, \mathbf{x}') = \int d\mathbf{y} G_{X-t''}(\mathbf{x}, \mathbf{y}) G_{t''}(\mathbf{y}, \mathbf{x}')$$

## Average

$$\begin{aligned} I &= \left\langle e^{i\frac{k}{2}(\boldsymbol{\rho}(\tau_1)+\boldsymbol{\rho}(\tau_2))} \right\rangle = \int d\sigma e^{i\frac{k}{2}(\boldsymbol{\rho}(\tau_1)+\boldsymbol{\rho}(\tau_2))} \\ &= \int d\rho_1 \int d\rho_2 G_{X-\tau_1}(0, \rho_1) e^{i\frac{k}{2}\rho_1} G_{\tau_1-\tau_2}(\rho_1, \rho_2) e^{i\frac{k}{2}\rho_2} G_{\tau_2}(\rho_2, 0) \\ &= \sum_{n_1 n_2 n_3} \iint d\rho_1 d\rho_2 e^{-E_{n_1}(X-\tau_1)-E_{n_2}(\tau_1-\tau_2)-E_{n_3}\tau_2} \Psi_{n_1}(0) \Psi_{n_3}^*(0) \\ &\quad \cdot \Psi_{n_1}^*(\rho_1) e^{i\frac{k}{2}\rho_1} \Psi_{n_2}(\rho_1) \Psi_{n_2}^*(\rho_2) e^{i\frac{k}{2}\rho_2} \Psi_{n_3}(\rho_2) \quad (n_1 = n_3) \\ &\Rightarrow e^{-E_0 X} |\Psi_0(0)|^2 \sum_n e^{-(E_n-E_0)|\tau_1-\tau_2|} \\ &\quad \cdot \left( \int d\rho_1 \Psi_0^*(\rho_1) e^{i\frac{k}{2}\rho_1} \Psi_n(\rho_1) \right) \left( \int d\rho_2 \Psi_n^*(\rho_2) e^{i\frac{k}{2}\rho_2} \Psi_0(\rho_2) \right) \\ &= e^{-E_0 X} |\Psi_0(0)|^2 \sum_n e^{-(E_n-E_0)|\tau_1-\tau_2|} C_{0n} \left( \frac{\mathbf{k}}{2} \right) C_{0n}^* \left( -\frac{\mathbf{k}}{2} \right) \end{aligned}$$

## "Time" potential

$$H = P^2 + \frac{\alpha}{8}|\rho|; \quad \left[ -\frac{d^2}{d\rho^2} + \frac{\alpha}{8}|\rho| \right] \Phi_n(\rho) = \mathcal{E}_n \Phi_n(\rho)$$

$$\rho = \frac{2v}{\alpha^{1/3}}, \quad \mathcal{E} = \frac{\alpha^{2/3}}{4}\epsilon$$

$$\left[ -\frac{d^2}{dv^2} + v \right] Y(v, \epsilon) = \epsilon Y(v, \epsilon), \quad v \in [0, \infty).$$

$$Y(v, \epsilon) = \begin{cases} \pi \sqrt{\frac{\epsilon-v}{3}} \left[ J_{1/3} \left( \frac{2}{3}(\epsilon-v)^{3/2} \right) + J_{-1/3} \left( \frac{2}{3}(\epsilon-v)^{3/2} \right) \right], & v < \epsilon \\ \sqrt{v-\epsilon} K_{1/3} \left( \frac{2}{3}(v-\epsilon)^{3/2} \right), & v > \epsilon \end{cases}$$

The spectrum is defined by the equations

$$\left. \frac{d}{dv} Y(v, \epsilon_{2n}) \right|_{v=0} = 0 \quad \text{for even states } n \rightarrow 2n$$
$$Y(0, \epsilon_{2n+1}) = 0 \quad \text{for odd states } n \rightarrow 2n + 1.$$

$$\Phi_n(v) = Y(v, \epsilon_n)$$

$n$	$\epsilon_n$	$N_n = \int_0^{\infty} dv \Phi_n^2(v)$
0	1.0188	8.655
1	2.3381	14.558
2	3.2482	16.886
3	4.0879	19.097
4	4.8201	20.652