Adler Functions, DIS sum rules and Crewther Relations

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in collaboration to **P. Baikov and J. Kühn**

based on recent publication:

"Adler Function, Bjorken Sum Rule, and the Crewther Relation to Order α_s^4",Phys.Rev.Lett.104:132004,2010; arXiv:1001.3606v1

+

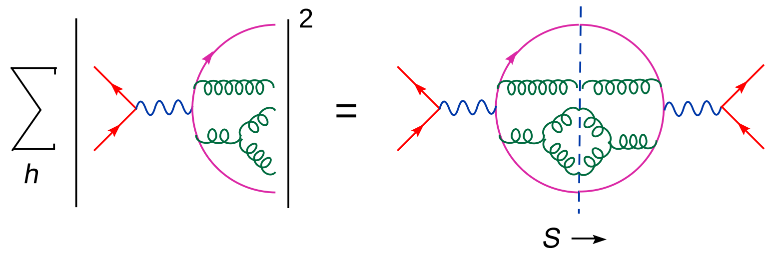
P. Baikov, K. Ch. and J. Kühn (in preparation)

QUARKS 2010, Kolomna

- the current status of R(s)/D(Q) (including the singlet conribution) at order $({\alpha_s}^4)$
- two aspects of the reliability problem: values of masters and the correctness of the reduction procedure
- DIS sum rules and the generalized Crewther relation /**R. Crewther (1972); D. Broadhurst and A. Kataev (1993)**/
- results for the Bjorken sum rule and the (non-singlet) Adler function for a generic gauge group in $\mathcal{O}({\alpha_s}^4)$ (NEW!)
- successfull test of the both results (and the quenched QED β function) with the (generalized) Crewther relation (NEW!)
- results for the Gross-Llewellyn Smith sum rule in $\mathcal{O}({\alpha_s}^4)$ for a generic gauge group and constraints for the singlet part of the Adler function (NEW!)
- Conclusions and Perspectives

$$\boldsymbol{R(s) = \sigma_{tot}(e^+e^- \to hadrons)/\sigma(e^+e^- \to \mu^+\mu^-)}$$

R(s) is related (via unitarity) to the correlator of the EM quark currents:



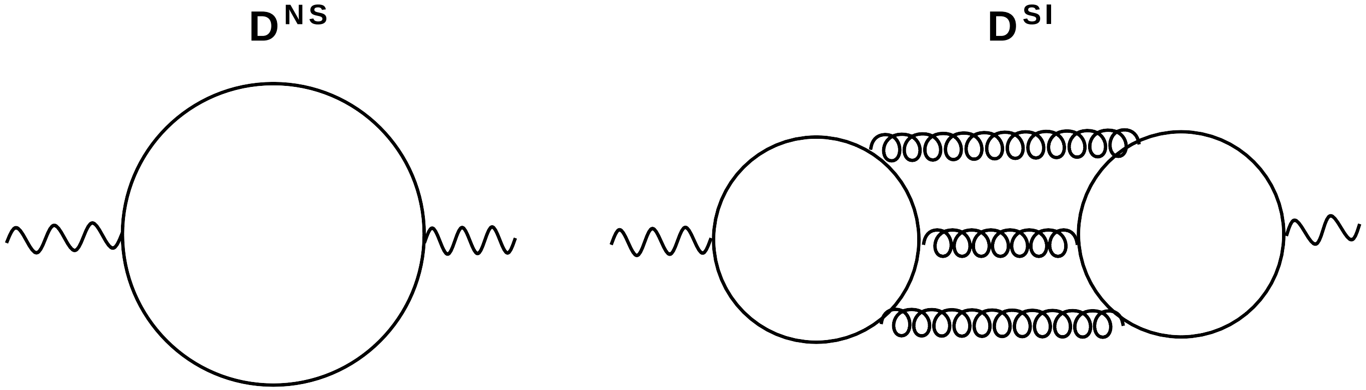
$$R(s) \approx \Im\,\Pi(s - i\delta)$$

$$3Q^2\Pi(q^2 = -Q^2) = \int e^{iqx}\langle 0|T[\ j^v_\mu(x) j^v_\mu(0)\ ]|0\rangle dx$$

$$R(s) \leftrightarrow D(Q) \quad \Longleftarrow \text{Adler function}$$

$$D(Q^2) = Q^2 \frac{d}{d\,Q^2}\Pi(q^2) = Q^2 \int \frac{R(s)}{(s+Q^2)^2}\,ds$$

$$D = 1 + \sum_{i \geq 1} d_i\, a_s(Q)^i, \quad (a_s \equiv \alpha_s/\pi, \mu = Q)$$



$$j^v_\mu = \sum_f \overline{\psi}_f \gamma_\mu \psi_f, \qquad D = n_f \, dR \, \left(D^{NS} + D^{NS}\right),$$

$$D^{NS} = 1 + \sum_{i \geq 1} d_i \, a_s(Q)^i \qquad D^{SI} = \sum_{i \geq 3} d_i^{SI} \, a_s(Q)^i$$

with $d_3^{SI} \approx n_f \dfrac{d_F^{abc} d_F^{abc}}{d_R}$ and $(a_s \equiv \alpha_s/\pi, \mu = Q)$

Status of D(Q) ($\overline{\text{MS}}$-scheme, $\mu^2 = Q^2$) since 1991 till 2008

$$d_2 = -\frac{3}{32}C_F^2 + C_F T_f\left[\zeta_3 - \frac{11}{8}\right] + C_F C_A\left[\frac{123}{32} - \frac{11\zeta_3}{4}\right] \Longleftarrow \boxed{\textbf{QUARKS 1980}}$$

$$d_3 = -\frac{69}{128}C_F^3 + C_F^2 T_f\left[-\frac{29}{64} + \frac{19}{4}\zeta_3 - 5\zeta_5\right]$$

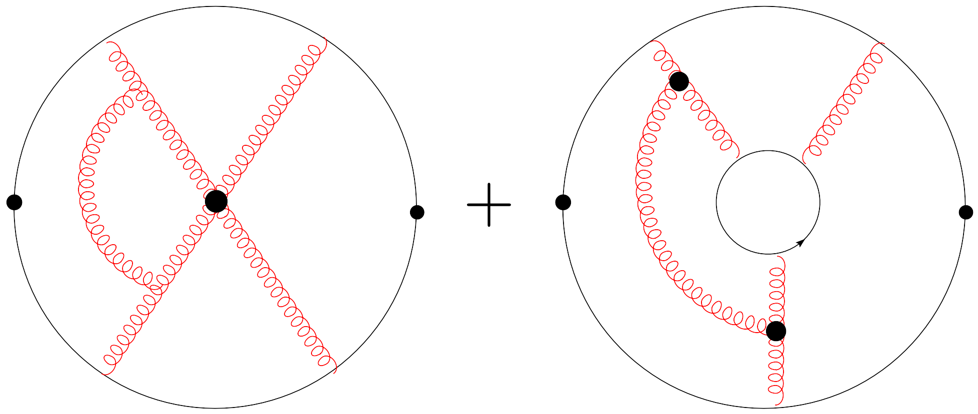
$$+C_F T_f^2\left[\frac{151}{54} - \frac{19}{9}\zeta_3\right] + C_F^2 C_A\left[-\frac{127}{64} - \frac{143}{16}\zeta_3 + \frac{55}{4}\zeta_5\right]$$

$$+C_F T_f C_A\left[-\frac{485}{27} + \frac{112}{9}\zeta_3 + \frac{5}{6}\zeta_5\right] + C_F C_A^2\left[\frac{90445}{3456} - \frac{2737}{144}\zeta_3 - \frac{55}{24}\zeta_5\right],$$

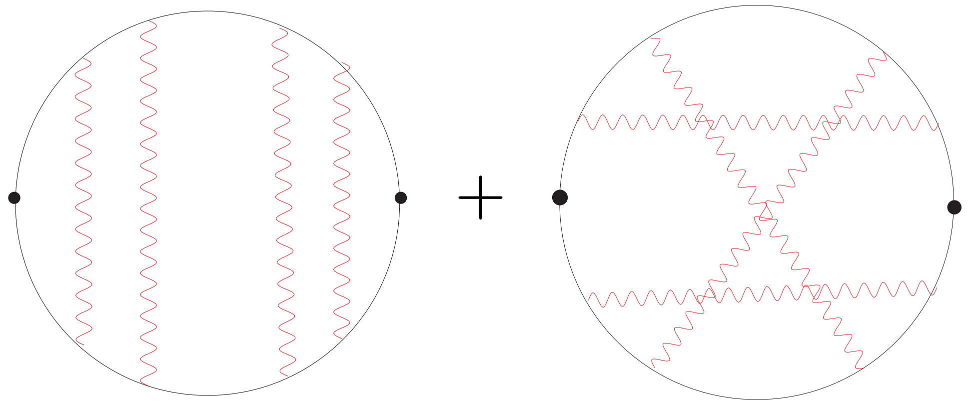
$$d_3^{SI} = n_f \frac{d_F^{abc} d_F^{abc}}{d_R}\left(d_{3,1}^{SI} \equiv \frac{11}{192} - \frac{1}{8}\zeta_3\right)$$

/Gorishnii, Kataev, Larin, (1991); /Surguladze, Samuel, (1991) (both in Feynman gauge); K.Ch, (1997) $\leftarrow$ in general covariant gauge

R(s) at five loops is conributed by $\approx 17 \cdot 10^3$ of nonabelian or/and non-quenched diagrams like



as well as 2671 purely abelian quenched diagrams like



masslessness $\longleftrightarrow$ simplicity:

5-loop R(s) is reducible*

to 4-loop massless propagators ($\equiv$ p-integrals)

$\longleftarrow$

main object to compute

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* **(i) the same is true for massive corrections like $\mathrm{m_q^2/s}$, etc.**
/J. Kühn, K.Ch (91,94)/

Tool Box *

- **IRR** / **Vladimirov, (78)/** **+** **IR R^* -operation /K. Ch., Smirnov (1984)/** **+** **resolved combinatorics /K. Ch., (1997)/**
- **reduction to Masters: "direct and automatic" construction of CF's through $1/D$ expansion—made with BAICER—within the Baikov's representation for Feynman integrals**[1]
- **all 4-loop master p-integrals are known analytically** **/P. Baikov and K.Ch. (2004)/**
- **computing time and required resources: could be huge (the price for full automatization)**; to cope with it we use parallel FORM /Vermaseren, Retey, Fliegner, Tentyukov, ...(2000 – . . .)

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* NO IBP identities are ever used at any step!

[1] Baikov, Phys. Lett. B385 (1996) 403; B474 (2000) 385; Nucl.Phys.Proc.Suppl.116:378-381,2003

QUARKS-2008

$$d_4 = n_f^3 \left[-\frac{6131}{5832} + \frac{203}{324}\zeta_3 + \frac{5}{18}\zeta_5\right]$$ ("renormalon" chain /M. Beneke 1993/)

$$+ n_f^2 \left[\frac{1045381}{15552} - \frac{40655}{864}\zeta_3 + \frac{5}{6}\zeta_3^2 - \frac{260}{27}\zeta_5\right]$$ /Baikov, Kühn, K.Ch. (2002)/

$$+ \boldsymbol{n_f \left[-\frac{13044007}{10368} + \frac{12205}{12}\zeta_3 - 55\,\zeta_3^2 + \frac{29675}{432}\zeta_5 + \frac{665}{72}\zeta_7\right]}$$

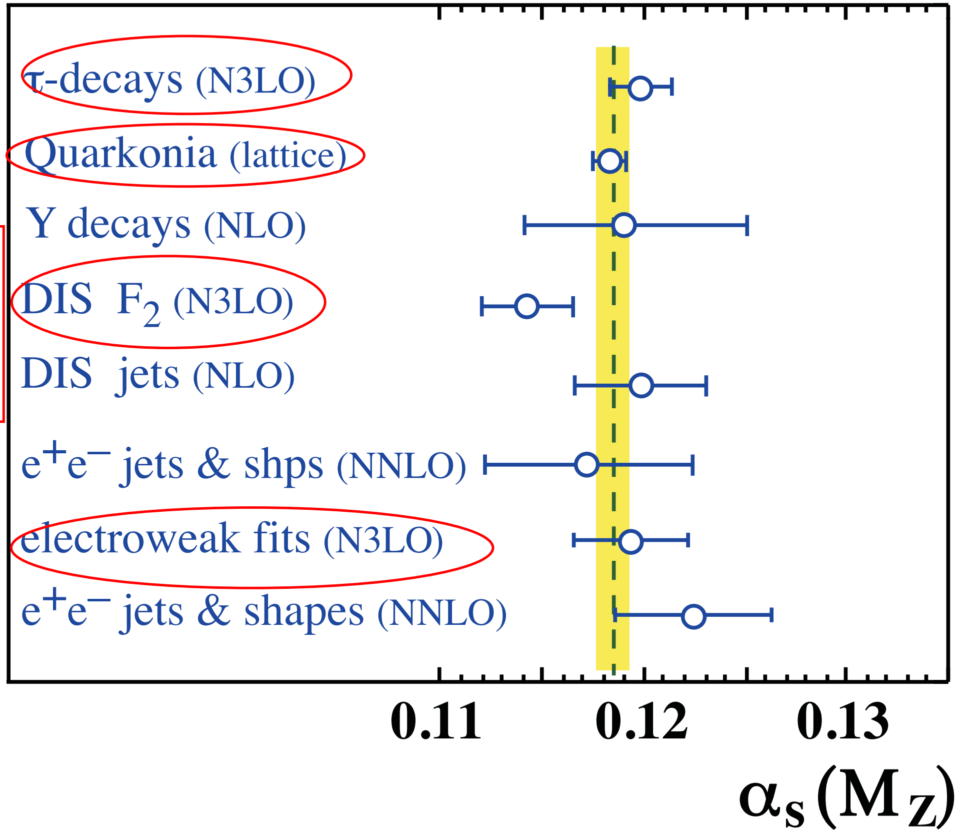
$$+ \boldsymbol{\left[\frac{144939499}{20736} - \frac{5693495}{864}\zeta_3 + \frac{5445}{8}\zeta_3^2 + \frac{65945}{288}\zeta_5 - \frac{7315}{48}\zeta_7\right]}$$

just three numbers are still missing:

$$d_4^{SI} = n_f \frac{d_F^{abc} d_F^{abc}}{d_R} \left(C_F\, d_{4,1}^{SI} + C_A\, d_{4,2}^{SI} + T_F\, d_{4,3}^{SI}\right)$$

World Summary of α_s 2009:

NNNLO to D(Q) and R(S)



$\rightarrow\ \alpha_s(M_Z) = 0.1184 \pm 0.0007$

(Bethke, arXiv:0908.1135)

How reliable are our results?

History of $R(s)$ teaches us to be cautious:

$\sim$ 20 years ago A. Kataev (with S. Gorishny and S. Larin) first produced a severly wrong result for the $\mathcal{O}(\alpha_s^3)$ term (corrected only by three years later!) in $R(s)$

We do want to avoid such a debacle at order α_s^4

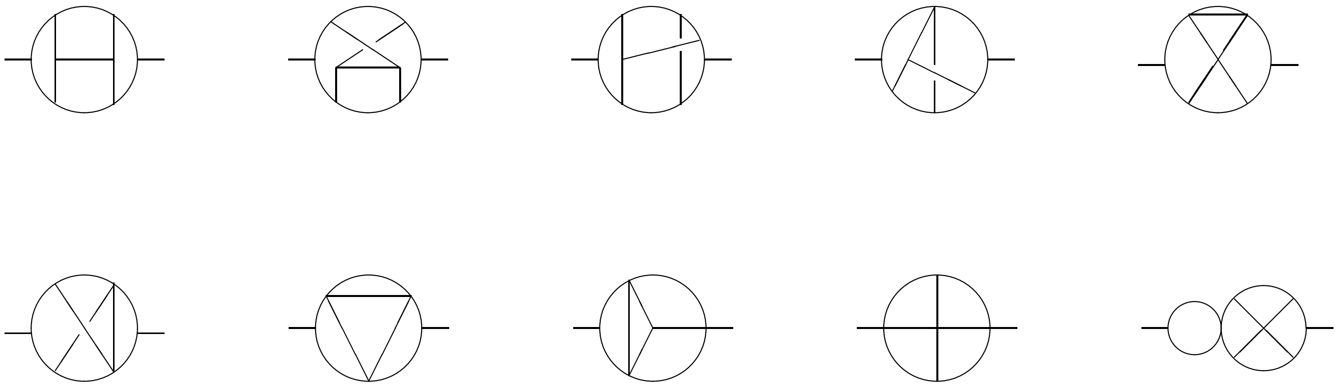
TWO things must be checked:

A. the masters

B. reduction to masters

recently both check of results have been SUCCESSFULLY done !!!:

A. all non-trivial master integrals



have been successfully reproduced (with at least 3-digit accuracy) by numerical integration (with the use of quite sophisticated sector technique) by A. Smirnov and M. Tentyukov (arXiv:1004.1149[hep-ph])

B. By computing $\mathcal{O}(\alpha_s{}^4)$ corrections to the polarized Bjorken sum rule and the Adler function for the general gauge group we have checked that the generalized Crewther relation

—which in order α_s^4 provide as as many as **SIX!**—

—**HIGHLY NON-TRIVIAL** constraints—

is indeed **exactly** fulfilled by our result.

DIS Sum Rules

- the polarized Bjorken sum rule ($a_s \equiv \frac{\alpha_s}{\pi}$)

$$\boldsymbol{B}jp(Q^2) = \int_0^1 [g_1^{ep}(x,Q^2) - g_1^{en}(x,Q^2)]dx = \frac{1}{6}|\frac{g_A}{g_V}|C^{Bjp}(a_s)$$

Coefficient function $C^{Bjp}(a_s)$ is fixed by OPE of two non-singlet vector currents (up to power suppressed corrections)

$$i\int TV^a_\alpha(x)V^b_\beta(0)e^{iqx}dx|_{q^2\to\infty} \approx C^{Q,abc}_{\alpha\beta\rho}A^c_\rho(0) + \ldots \qquad (1)$$

where

$$C^{Q,abc}_{\alpha\beta\rho} \sim i d^{abc}\epsilon_{\alpha\beta\rho\sigma}\frac{q^\sigma}{Q^2}C^{Bjp}(a_s)$$

and $Q^2 = -q^2$

- the Gross-Llewellyn Smith sum rule

$$GLS(Q^2) = \frac{1}{2}\int_0^1 F_3^{\nu p+\overline{\nu}p}(x,Q^2)dx = 3\,C_{GLS}(a_s)$$

the function $C_{GLS}(a_s)$ comes from operator-product expansion of the axial and vector non-singlet currents

$$i\int T A_\mu^a(x) V_\nu^b(0) e^{iqx} dx|_{q^2\to\infty} \approx C_{\mu\nu\alpha}^{V,ab} V_\alpha(0) + \ldots$$

where $C_{\mu\nu\alpha}^{V,ab} \sim i\delta^{ab}\epsilon_{\mu\nu\alpha\beta}\frac{q^\beta}{Q^2}C_{GLS}(a_s)$

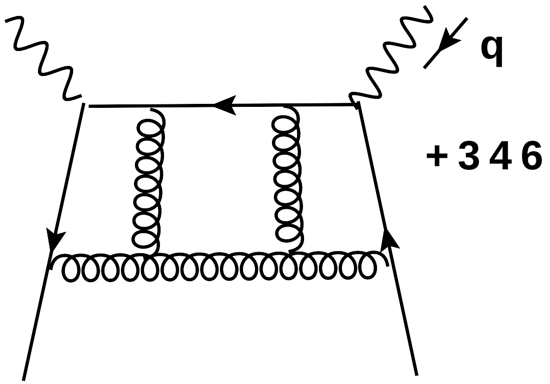
Note that both sum rules are unambiguous

/modulo higher twists!/

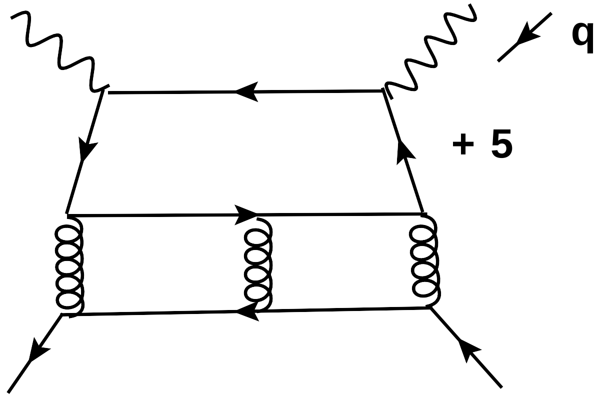
predictions of QCD which *in principle* could be confronted with experimental data

As is well-known, the evaluation of L-loop corrections to a CF of OPE could be done in terms of massless L-loop propagators (S. Gorisny, S. Larin and F. Tkachov (1982)) $\Longrightarrow$ **one could use techniques developed for** $R(s)$

Bjp *and* GLS



GLS *only*



At order α_s^3 both CF's were computed in early nineties.The next order is contributed by about 54 thousand of 4-loop diagrams . . . (cmp. to $\approx$ 20 thousand of 5-loop diagrams contributing to $R(s)$ at the same order)

The Crewther relation states that in the conformal invariant limit ($\beta \equiv 0$) $C_{Bjp}(a_s)$ is related to the (nonsinglet) Adler function via the following beautiful equality

$$\boldsymbol{C_{Bjp}(a_s(Q^2))D^{NS}(a_s(Q^2))|_{c-i} = 1}$$

its generalization for real QCD reads:

$$\boldsymbol{C^{Bjp}(a_s)\, D^{NS}(a_s) = 1 + \frac{\beta(a_s)}{a_s}\Big[K_1\, a_s + K_2\, a_s^2 + K_3\, a_s^3 + \dots\Big]}$$

Note that similar relation connects also the CF of the Gross-Llewellyn Smith sum rule to the full Adler function;

Main ingredients of the derivation: the AVV 3-point function and constraints on it from (approximate) conformal invariance + Adler-Bardeen anomaly theorem

Crewther Relation: (short) bibliography

discovered: R.J. Crewther, *Phys. Rev. Lett.* **28**, 1421 (1972).

S.L. Adler, C.G. Callan, D.J. Gross and R. Jackiw, *Phys. Rev.* D **6**, 2982 (1972).

generalized for "real" QCD:

D.J. Broadhurst and A.L. Kataev, *Phys. Lett.* B **315**, 179 (1993).

further developed:

G.T. Gabadadze and A.L. Kataev,*JETP Lett.* **61**, 448 (1995).)

S.J. Brodsky, G.T. Gabadadze, A.L. Kataev and H.J. Lu, *Phys. Lett.* B **372**, 133 (1996).

proven:

R.J. Crewther, *Phys. Lett.* B **397**, 137 (1997).

V. M. Braun, G. P. Korchemsky and D. Müller, Prog. Part. Nucl. Phys. **51**, 311 (2003)

discovered: R.J. Crewther, *Phys. Rev. Lett.* **28**, 1421 (1972).
S.L. Adler, C.G. Callan, D.J. Gross and R. Jackiw, *Phys. Rev.* D **6**, 2982 (1972).

"downgraded" (after 20(!) years) from ideal, conformal-invariant paradise to the dirty world of real QCD by **David Broadhurst and Andrey Kataev**:
D.J. Broadhurst and A.L. Kataev, *Phys. Lett.* B **315**, 179 (1993).

further developed: ...
proven: ...

Note that

the generalization would not be possible without analytical $\mathcal{O}(\alpha_s^3)$ calculations of $D(q)$ and C^{Bjp}; the latter would not be possible without dedicated people:

(S. Gorishny, A. Kataev, S. Larin; M. Samuel, L. Surguladze; J. Vermaseren, F. Tkachov)

and without dedicated tools:

SCHOONSCHIP / M. Veltman/ and FORM 2 / J. Vermaseren/

Which exactly constraints come from the Crewther relation?

$$C^{Bjp}(a_s)\,D^{NS}(a_s) = 1 + \frac{\beta(a_s)}{a_s}\Big[K_1\,a_s + K_2\,a_s^2 + K_3\,a_s^3 + \dots\Big]$$

If it is valid at order a_s^n, then at the next order a_s^{n+1}, we have

$$(d_{n+1} + C^{Bjp}_{n+1} + \text{interference terms})\,a_s^{n+1} = -\beta_0\,a_s\Big[K_n\,a_s^n\Big]$$

$\alpha_s^1 : (d_1 - C_1) : C_F \Longleftrightarrow K_0 \equiv 0 \longleftarrow$ one constraint

$\alpha_s^2 : (d_2 - C_2) : C_F^2, T\,C_F, C_F\,C_A \Longleftrightarrow K_1 : C_F \longleftarrow$ two constraints

$$\alpha_s^3 : (d_3 - C_3) : C_F^3\,, C_F^2 C_A\,, C_F C_A^2\,, C_F^2 T\,, C_F C_A T\,, C_F T^2$$

$$\Updownarrow$$

$K_2 : C_F^2, C_F C_A, C_F T \longleftarrow$ three constraints

At last, at $\mathcal{O}(\alpha_s^4)$ there exist exactly 12 color strtuctures:

$$C_F^4\,,\; C_F^3 C_A\,,\; C_F^2 C_A^2\,,\; C_F C_A^3\,,\; C_F^3 T_F n_f\,,\; C_F^2 C_A T_F n_f\,,$$

$$C_F C_A^2 T_F n_f\,,\; C_F^2 T_F^2 n_f^2\,,\; C_F C_A T_F^2 n_f^2\,,\; C_F T_F^3 n_f^3\,, d_F^{abcd} d_A^{abcd}\,, n_f d_F^{abcd} d_F^{abcd}$$

while the coefficient K_3 is contributed by only **6** color structures:

$$C_F T^2\,, C_F\, C_A^2\,, C_F^2\, T\,, C_F\, C_A\, T\,, C_F^2\, C_A\,, C_F^3$$

Thus, we have 12-6 $=$ **6** constraints on the difference

$$\boldsymbol{d_4 - (C^{Bjp})_4}$$

3 of them are very simple: the above difference cannot contain

$$C_F^4,\;\; d_F^{abcd} d_A^{abcd} \quad n_f d_F^{abcd} d_F^{abcd}$$

remaining three are a bit more complicated

Results for the Adler $D^{NS}(a_s)$-function and the CF $C^{Bjp}(a_s)$ of the Bjorken sum rule for the polarized DIS are available since the last fall

The ones for the CF $C_{GLS}(a_s)$ have been just completed!

<u>All for a generic gauge group at $\boldsymbol{\mathcal{O}(\alpha_s{}^4)}$</u>

providing, thus, a beautiful opportunity to check our reduction machine!

the check is OK!

Comments:

The Crewther test is highly non-trivial:

- four-loop box-type diagrams (in propagator kinematics) versus five loop propagators

- **No IR-trickery is neccessary** in calculation of $C_{Bjp}(a_s)$

- As a result we he have been able to check that $C_{Bjp}(a_s)$ is indeed gauge-independent (the Adler finction was computed in the simplest, Feynman gauge only!)

- in the course of our calculations we have had to extend the Larin treatment of Hooft-Veltman γ_5 at 4-loop level (a natural object for the dim. reg., which really appears in the course of calculations, is $\gamma^{[\mu\nu\alpha]}$ instead of $\gamma_5\gamma^\mu$ with anticommuting γ_5; the mismatch should be corrected by the Larin factor)

$$C^{Bjp} = 1 \quad - \quad a_s + a_s^2\left[-4.583 + 0.3333\,n_f\right] + a_s^3\left[-41.44 + 7.607\,n_f - 0.1775\,n_f^2\right]$$

$$+ \quad a_s^4\left[-479.4 + 123.4\,n_f - 7.697\,n_f^2 + 0.1037\,n_f^3\right]$$

$$C^{Bjp}(n_f = 3) = 1 - 1.a_s - 3.583a_s^2 - 20.22a_s^3 \; \boxed{\text{-175.7}} \; a_s^4$$

FAC/PMS prediction due to Kataev ans Starshenko (almost 15 years old!) for the $\mathcal{O}(\alpha_s^4)$ term at $n_f = 3$ is $\boxed{-130\ a_s^4}$

$$C^{Bjp}(n_f = 4) = 1 - a_s - 3.25a_s^2 - 13.85a_s^3 \;\; -102.4\,a_s^4$$

$$C^{Bjp}(n_f = 5) = 1 - a_s - 2.917a_s^2 - 7.84a_s^3 \;\; -41.96\,a_s^4$$

$$C^{Bjp}(n_f = 6) = 1 - a_s - 2.583a_s^2 - 2.185a_s^3 \;\; +6.2\,a_s^4,$$

K&S for $n_f = 6$: $\boxed{+22\ a_s^4}$

	d_4	$(1/C^{Bjp})_4$
C_F^4	$\frac{4157}{2048} + \frac{3}{8}\,\zeta_3$	$\frac{4157}{2048} + \frac{3}{8}\,\zeta_3$
$n_f \frac{d_F^{abcd} d_F^{abcd}}{d_R}$	$-\frac{13}{16} - \,\zeta_3 + \frac{5}{2}\,\zeta_5$	$-\frac{13}{16} - \,\zeta_3 + \frac{5}{2}\,\zeta_5$
$\frac{d_F^{abcd} d_A^{abcd}}{d_R}$	$\frac{3}{16} - \frac{1}{4}\,\zeta_3 - \frac{5}{4}\,\zeta_5$	$\frac{3}{16} - \frac{1}{4}\,\zeta_3 - \frac{5}{4}\,\zeta_5$
$C_F T_f^3$	$-\frac{6131}{972} + \frac{203}{54}\,\zeta_3 + \frac{5}{3}\,\zeta_5$	$-\frac{605}{972}$
$C_F^2 T_f^2$	$\frac{5713}{1728} - \frac{581}{24}\,\zeta_3 + \frac{125}{6}\,\zeta_5 + 3\,\zeta_3^2$	$\frac{869}{576} - \frac{29}{24}\,\zeta_3$
$C_F T_f^2 C_A$	$\frac{340843}{5184} - \frac{10453}{288}\,\zeta_3 - \frac{170}{9}\,\zeta_5 - \frac{1}{2}\,\zeta_3^2$	$\frac{165283}{20736} + \frac{43}{144}\,\zeta_3 - \frac{5}{12}\,\zeta_5 + \frac{1}{6}\,\zeta_3^2$
$C_F^3 T_f$	$\frac{1001}{384} + \frac{99}{32}\,\zeta_3 - \frac{125}{4}\,\zeta_5 + \frac{105}{4}\,\zeta_7$	$-\frac{473}{2304} - \frac{391}{96}\,\zeta_3 + \frac{145}{24}\,\zeta_5$
$C_F^2 T_f C_A$	$\frac{32357}{13824} + \frac{10661}{96}\,\zeta_3 - \frac{5155}{48}\,\zeta_5 - \frac{33}{4}\,\zeta_3^2 - \frac{105}{8}\,\zeta_7$	$-\frac{17309}{13824} + \frac{1127}{144}\,\zeta_3 - \frac{95}{144}\,\zeta_5 - \frac{35}{4}\,\zeta_7$
$C_F T_f C_A^2$	$-\frac{(\cdots)}{(\cdots)} + \frac{8609}{72}\,\zeta_3 + \frac{18805}{288}\,\zeta_5 - \frac{11}{2}\,\zeta_3^2 + \frac{35}{16}\,\zeta_7$	$-\frac{(\cdots)}{(\cdots)} - \frac{59}{64}\,\zeta_3 + \frac{1855}{288}\,\zeta_5 - \frac{11}{12}\,\zeta_3^2 + \frac{35}{16}$
$C_F^3 C_A$	$-\frac{253}{32} - \frac{139}{128}\,\zeta_3 + \frac{2255}{32}\,\zeta_5 - \frac{1155}{16}\,\zeta_7$	$-\frac{8701}{4608} + \frac{1103}{96}\,\zeta_3 - \frac{1045}{48}\,\zeta_5$
$C_F^2 C_A^2$	$-\frac{592141}{18432} - \frac{43925}{384}\,\zeta_3 + \frac{6505}{48}\,\zeta_5 + \frac{1155}{32}\,\zeta_7$	$-\frac{435425}{55296} - \frac{1591}{144}\,\zeta_3 + \frac{55}{9}\,\zeta_5 + \frac{385}{16}\,\zeta_7$
$C_F C_A^3$	$\frac{(\cdots)}{(\cdots)} - \frac{(\cdots)}{(\cdots)}\,\zeta_3 - \frac{77995}{1152}\,\zeta_5 + \frac{605}{32}\,\zeta_3^2 - \frac{385}{64}\,\zeta_7$	$\frac{(\cdots)}{(\cdots)} - \frac{(\cdots)}{(\cdots)}\,\zeta_3 - \frac{12545}{1152}\,\zeta_5 + \frac{121}{96}\,\zeta_3^2 - 3$

NEW: results for the GLS sum rule at $\mathcal{O}(\alpha_s^4)$

$$C_{GLS}^{SI} = C_{BJp} + (C_{GLS}^{SI} = a_s^3\, c_3^{SI} + a_s^4\, c_4^{SI})$$

$$c_3^{SI} = n_f \frac{d_F^{abc} d_F^{abc}}{d_R} \left(c_{3,1}^{SI} \equiv -\frac{11}{192} + \frac{1}{8}\zeta_3 \right)$$

$$c_4^{SI} = n_f \frac{d_F^{abc} d_F^{abc}}{d_R} (C_F c_{4,1}^{SI} + C_A c_{4,2}^{SI} + T_F c_{4,3}^{SI})$$

$$c_{4,1}^{SI} = \frac{37}{128} + \frac{1}{16}\zeta_3 - \frac{5}{8}\zeta_5 \qquad c_{4,2}^{SI} = -\frac{481}{1152} + \frac{971}{1152}\zeta_3 - \frac{295}{576}\zeta_5 + \frac{11}{32}\zeta_3^2$$

$$c_{4,3}^{SI} = \frac{119}{1152} - \frac{67}{288}\zeta_3 + \frac{35}{144}\zeta_5 - \frac{1}{8}\zeta_3^2$$

Numerically, $C_{GLS}^{SI} \ll C_{GLS}^{NS}$ as expected ($\overline{\text{MS}}$-scheme):

$$
\begin{aligned}
C_{GLS}^{NS} = 1 \quad & - \quad a_s + a_s^2 \left[-4.583 + 0.3333\, n_f\right] \\
& + \quad a_s^3 \left[-41.44 + 7.607\, n_f - 0.1775\, n_f^2\right] \\
& + \quad a_s^4 \left[-479.4 + 123.4\, n_f - 7.697\, n_f^2 + 0.1037\, n_f^3\right]
\end{aligned}
$$

$$
C_{GLS}^{SI} = 0.4132\, n_f\, a_s^3 + a_s^4\, n_f\, (5.80157 - 0.233185\, n_f)
$$

Note significant cancellations for $n_f = 3, 4$ between n_f^0 and n_f^1 contributions into the non-siglet part at order a_s^4

Crewther relation between $D = D^{NS} + D^{SI}$ and C_{GLS}

$$\left(D^{NS} + d_3^{SI} a_s^3 + d_4^{SI} a_s^4\right) \left(C_{GLS}^{NS} + c_3^{SI} a_s^3 + c_4^{SI} a_s^4\right) = 1 + \frac{\beta(\alpha_s)}{\alpha_s}\left[K^{NS} + a_s^3 K_3^{SI} n_f \frac{d_F^{abc} d_F^{abc}}{d_R}\right] \quad \star$$

with $\frac{\beta(\alpha_s)}{\alpha_s} \equiv -\beta_0 a_s + \ldots, \quad \beta_0 = \frac{11}{12} C_A - \frac{T_f}{3}$

$$d_3^{SI} = n_f \frac{d_F^{abc} d_F^{abc}}{d_R} d_{3,1}^{SI}, \quad d_4^{SI} = n_f \frac{d_F^{abc} d_F^{abc}}{d_R} \left(C_F d_{4,1}^{SI} + C_A d_{4,2}^{SI} + T_F d_{4,3}^{SI}\right)$$

$$c_3^{SI} = n_f \frac{d_F^{abc} d_F^{abc}}{d_R} c_{3,1}^{SI}, \quad c_4^{SI} = n_f \frac{d_F^{abc} d_F^{abc}}{d_R} \left(C_F c_{4,1}^{SI} + C_A c_{4,2}^{SI} + T_F c_{4,3}^{SI}\right)$$

rhs of $\star$ depends on only 1 unknown parameter, K_3^{SI}, thus we have 3-1 =2 constraints on 3 unknown coefficients in d_4^{SI}

Obvious solution of these constraints reads:

$$d_{4,1}^{SI} = -\frac{3}{2}c_{3,1}^{SI} - c_{4,1}^{SI} = -\frac{13}{64} - \frac{\zeta_3}{4} + \frac{5\zeta_5}{8}$$

$$d_{4,2}^{SI} = -c_{4,2}^{SI} + \frac{11}{12}K_{3,1}^{SI}$$

$$d_{4,3}^{SI} = -c_{4,3}^{SI} + \frac{1}{3}K_{3,1}^{SI}$$

CONCLUSION

- The Adler $D^{NS}(a_s)$-function and the CF $C^{Bjp}(a_s)$ of the Bjorken sum rule for the polarized DIS have been both analytically evaluated for a generic gauge group at $\boldsymbol{\mathcal{O}(\alpha_s{}^4)}$
- The generalized Crewther relation puts as many as 6 highly non-tivial constraints on the difference $\boldsymbol{d_4 - (C^{Bjp})_4}$ which **are all** fulfilled!
- CF $C^{GLS}(a_s)$ of the Gross-Llewellyn Smith sum rule has been analytically evaluated for a generic gauge group at $\boldsymbol{\mathcal{O}(\alpha_s{}^4)}$

- At order $\boldsymbol{\mathcal{O}}(\alpha_s{}^4)$ the singlet Adler function D^{SI} is contrubuted by exactly three color structures. All three are fixed by the corresponding Crewther relation up to only **one** still unknown constant

- The full calculation of D^{SI} at order $\boldsymbol{\mathcal{O}}(\alpha_s{}^4)$ is under way and should be finished soon (another non-trivial check of the reduction and the CBK relation!)