Wave function of gravitating shell

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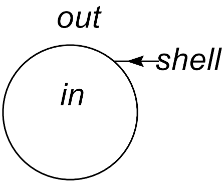
Introduction

We describe a simple model of the spherical dust shell quantization. It is well known, that a mass of gravitating system at the space infinity, $m_{\rm out}$, is a total energy, which conserved during the dynamical evolution of the shell. For this reason it is natural to equate a hamiltonian of the system with a total mass of the shell. The discrete spectrum is calculated and exact analytical solution in the form of the Meiksner polynom is found for the wave function of a thin gravitating shell in the Reissner - Nordström geometry. It's shown, that extreme state in the quantum spectrum of gravitating shell is absent similar to the case of extreme black hole. This model were considered previously by Berezin, Kozimirov, Kuzmin, Tkachev, Phys. Lett. B 212, 415 (1988) and Berezin Phys. Lett. B 241, 194 (1990).

The general equations

Einstein equations $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi T_{\mu\nu}$ with the tensor energy — momentum equals $T_\mu^\nu = \tilde{T}_\mu^\nu + S_\mu^\nu \delta(n)$, where $\delta(n)$ is δ function, S_μ^ν is tensor energy — momentum on the shell and $\tilde{T}_\mu^\nu$ is the regular part of T_μ^ν.

shell



Einstein equations for the thin shell

$$\{K_0^0\} S_0^0 + 2\{K_2^2\} S_2^2 + [T_n^n] = 0,$$
$$\frac{dS_0^0}{d\tau} + \frac{2\dot{\rho}}{\rho}(S_0^0 - S_2^2) + [T_0^n] = 0,$$
$$[K_2^2] = 4\pi S_0^0,$$
$$[K_0^0] + [K_2^2] = 8\pi S_2^2.$$

K_{ij} is the outer curvature tensor, $[T] = T_{out} - T_{in}$, $\{T\} = T_{out} + T_{in}$.

A simple model

Equation of motion for dust shell in the Reissner - Nordström geometry

$$\sigma_{\rm in}\sqrt{\dot\rho^2+1-\frac{2m_{\rm in}}{\rho}+\frac{Q_{\rm in}^2}{\rho^2}}-\sigma_{\rm out}\sqrt{\dot\rho^2+1-\frac{2m_{\rm out}}{\rho}+\frac{Q_{\rm out}^2}{\rho^2}}=\frac{M}{\rho},$$

where $\sigma_{\rm in,out}=\pm 1$, $m_{\rm in}$, $m_{\rm out}$, $Q_{\rm in}$, $Q_{\rm out}$ — respectively masses and charges of a black hole inside and outside of the shell and M — mass of the shell.

We rewrite this equation in the form ($x = M\rho$)

$$H=m_{\rm out}=\sigma_{\rm in}\sqrt{\dot x^2+M^2\left(1-\frac{2m_{\rm in}M}{x}+\frac{Q_{\rm in}^2M^2}{x^2}\right)}+m_{\rm in}+M\frac{Q_{\rm out}^2-Q_{\rm in}^2-M^2}{2x}$$

Wave equation

Canonical impulse

$$p = \sigma_{\rm in} \ln \left[\frac{\dot{x}}{M} + \sqrt{\frac{\dot{x}^2}{M^2} + \left(1 - \frac{2m_{\rm in}M}{x} + \frac{Q_{\rm in}^2 M^2}{x^2}\right)} \right].$$

Commutation relation $[p, x] = -i$ and wave equation $H\phi(x) = m_{\rm out}\phi(x)$ is

$$\phi(x-i) + \left(1 - \frac{2m_{\rm in}M}{x} + \frac{Q_{\rm in}^2 M^2}{x^2}\right)\phi(x+i) - \frac{M^2 - Q_{\rm out}^2 + Q_{\rm in}^2}{x}\phi(x) = 2E\phi(x),$$

where $E = (m_{\rm out} - m_{\rm in})/M$.
Boundary conditions: $\phi^{2l}(0) = 0, \quad l = 0, 1 \ldots$[†].

† *Hajicek, Comm. Math. Phys. 150, 545 (1992).*

The first case: $m_{\mathrm{in}} = Q_{\mathrm{in}} = 0$.

Using transformation $x \to -ix$, the wave equation becomes

$$\phi(x+1) + \phi(x-1) - i\frac{M^2 - Q_{\mathrm{out}}^2}{x}\phi(x) = 2E\phi(x).$$

Solution is Meiksner polynoms:

$$\phi_n(x) = C(x)\frac{\beta^x x}{\beta^{2x+2n}}\triangle^n\left[\frac{\beta^{2x}\Gamma(x)}{\Gamma(x+1-n)}\right],$$

where

$$\beta = E + \sqrt{E^2 - 1}, \quad \triangle f(x) = f(x+1) - f(x), \quad C(x) = C(x+1),$$

$\Gamma(x)$ — gamma function.

Inverse Fourier transform:

$$C(x) = \sum_{k=-\infty}^{\infty} c_k \exp(2\pi ikx).$$

Wave function is orthogonal function in terms of $x_i = x$ and $x_{i+1} = x_i + 1$ for $0 < \beta < 1$:

$$\sum_{x_i=0}^{\infty} \phi_n(x_i)\phi_m(x_i)\rho(x_i) = \delta_{nm} d_n^2,$$

where weight function $\rho(x) = 1/\big(xC^2(x)\big)$ and

$$d_n^2 = \frac{\Gamma(n)\Gamma(n+1)}{\beta^{2n}}.$$

Mass spectrum:

$$E_n^2 = \left(\frac{m_{\rm out}}{M}\right)^2 = 1 - \frac{(M^2 - Q_{\rm out}^2)^2}{4n^2}.$$

Extreme case

In the case of extreme black hole, when $m_{\rm out} = Q_{\rm out}$, the mass spectrum is degenerate, $E = 1$.
Wave function is

$$\phi(x) = \sum_{k=0}^{\infty} c_k \exp(-2\pi kx) + ix \sum_{k=0}^{\infty} d_k \exp(-2\pi kx),$$

The second case: $Q_{\rm in} = 0$

Using transformation $x \to ix$, the wave equation becomes

$$\phi(x-1) + \left(1 + \frac{2m_{\rm in}Mi}{x}\right)\phi(x+1) + i\,\frac{M^2 - Q_{\rm out}^2}{x}\,\phi(x) = 2E\phi(x).$$

Solution is

$$\phi_n(x) = C(x)\frac{\tilde{\beta}^x\Gamma(x+1)}{\tilde{\beta}^{2x+2n}\Gamma(\gamma+x)}\,\triangle^n\left[\frac{\tilde{\beta}^{2x+2n}\Gamma(\gamma+x)}{\Gamma(x+1-n)}\right],$$

where

$$\tilde{\beta} = E - \sqrt{E^2-1}, \quad \gamma = i2m_{\rm in}M, \quad C(x) = C(x+1).$$

Wave functions are orthogonal:

$$\sum_{x_i=0}^{\infty} \phi_n(x_i)\phi_m(x_i)\rho(x_i) = \delta_{nm} d_n^2,$$

where

$$\rho(x) = \frac{\Gamma(\gamma + x)}{\Gamma(1+x)\Gamma(\gamma)C^2(x)},$$

$$d_n^2 = \frac{n!\Gamma(n+\gamma)}{\tilde{\beta}^{2n}(1-\tilde{\beta}^2)^{\gamma}\Gamma(\gamma)}.$$

The mass spectrum:

$$i(M^2 - Q_{\rm out}^2 + 2m_{\rm in}M\tilde{\beta}) = 2n\sqrt{E_n^2 - 1}.$$

Mass spectrum is imaginary, because the hamiltonian is not hermitian.
Hamiltonian would be hermitian after transformation:

$$\frac{1}{x}\exp\left(i\frac{\partial}{\partial x}\right)\to\frac{1}{2}\left[\frac{1}{x}\exp\left(i\frac{\partial}{\partial x}\right)+\exp\left(-i\frac{\partial}{\partial x}\right)\frac{1}{x}\right].$$

The transformed wave equation is

$$\phi(x+i)+\phi(x-i)-m_{\rm in}M\left[\frac{\phi(x+i)}{x}+\frac{\phi(x-i)}{x-i}\right]-\frac{M^2-Q_{\rm out}^2}{x}\,\phi(x)=2E\phi(x).$$

Approximate solution

When mass of the shell M is very small parameter the solution is

$$\phi(x) = \phi_0(x) + My(x),$$

where

$$\phi_0(x) = C_1(x)\beta^x + C_2(x)\tilde{\beta}^x,$$

$$y(x) = \frac{im_{\mathrm{in}}}{2\sqrt{E^2-1}}\Bigg\{ - C_1(x)\Big[\frac{\beta^{x+1}}{x}[2F(x,1,1+x,\beta^2) - 1]$$

$$+\beta^{x-1}\left[\beta^2\Psi(x+1) + \Psi(x)\right]\Big]$$

$$+C_2(x)\Big[\frac{\tilde{\beta}^{x+1}}{x}[2F(x,1,1+x,\tilde{\beta}^2) - 1] + \tilde{\beta}^{x-1}[\tilde{\beta}^2\Psi(x+1)+\Psi(x)]\Big]\Bigg\},$$

The spectrum mass of thin shell is continuous.

Conclusions

- Exact analytical solution in the form of the Meiksner polynom is found for the wave function of a thin gravitating shell in the Reissner - Nordström geometry.
- Discrete spectrum is calculated
- It is shown that extreme state in the quantum spectrum of gravitating shell is absent, similar to the case of extreme black hole.