Symplectic invariants of Quantum Riemann surfaces (and matrix models)

Leonid Chekhov

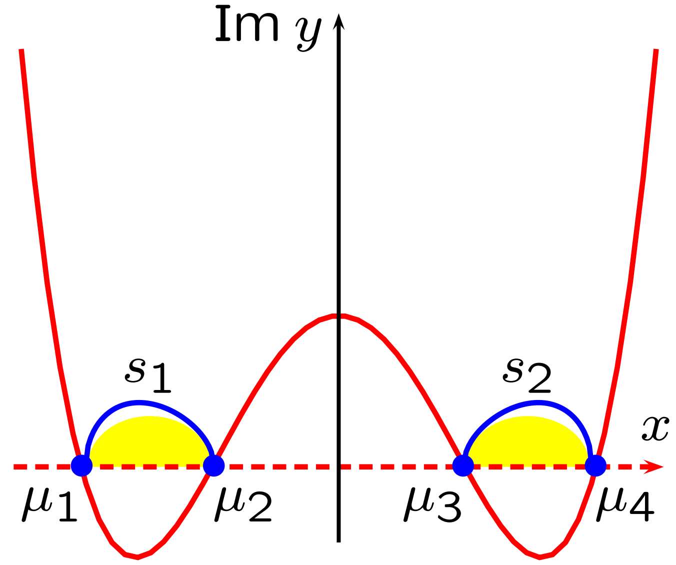
- Asymptotic methods of constructing the genus expansion in matrix models
- The β-eigenvalue model: perturbative approach; the master loop equaiton
- The β-model and Riccati equation: Quantum surfaces as a nonperturbative approach
- "Quantum" algebraic geometry: holomorphic differentials, A- and B-cycles, symmetric forms
- Higher-order corrections for correlation functions and symplectic invariants $\mathcal{F}_g$

't Hooft idea of $1/N$ expansion. In the Matrix integral

$$\int_{N\times N} DH\, e^{-N\,tr\,V(H)} = e^{\sum_{g=0}^{\infty} N^{2-2g}\mathcal{F}_g} := \mathcal{Z}^{(N)}_{1MM}\big(\{t_k\}\big), \quad V(x) = \sum_{k=1}^{p} \frac{1}{k} t_k x^k$$

contributions of different genus enter with different powers of N. (The order is the Euler characteristic of the corresponding triangulated surface.)

The asymptotic distribution of eigenvalues $\rho(x) = \mathrm{Im}\, y(x)$ spans in general n intervals (multigap solutions) and is $y(x) = M(x)\tilde{y}(x)$ where $M(x)$ is a polynomial and $\tilde{y}$ defines the spectral curve—a hyperelliptic Riemann surface $\tilde{y}^2 = \prod_{j=1}^{2n}(x-\mu_j)$



$\mathcal{F}_0$ – satisfies equations of the Whitham–Krichever hierarchy AND WDVV w.r.t. $s_i := \oint_{\mathcal{A}_i} y dx$ and t_k [L.Ch., A.Marshakov, A.Mironov, D.Vassiliev]'03

Asymptotic $(N \to \infty)$ methods for solving matrix integrals

We define the **one-point resolvent** to be a 1-*differential*

$$W_1(\lambda) = \hbar \left\langle \sum_{i=1}^{N} \frac{1}{\lambda - x_i} \right\rangle d\lambda, \quad \hbar = t_0/N, \qquad \left(W_1(\lambda) = \frac{\partial}{\partial V(\lambda)} \mathcal{F} \right)$$

and the t-point resolvents $(t \geq 2)$ to be symmetric t-differentials

$$W_t(\lambda_1, \ldots, \lambda_t) = \hbar^{2-t} \left\langle tr \frac{1}{\lambda_1 - H} \cdots tr \frac{1}{\lambda_t - H} \right\rangle_{\text{conn}} d\lambda_1 \cdots d\lambda_t \qquad \left(W_t = \frac{\partial}{\partial V(\lambda_1)} \cdots \frac{\partial}{\partial V(\lambda_t)} \mathcal{F} \right)$$

("conn"means the connected part of a correlation function). All the W's have the genus expansions $W_t(\lambda_1, \ldots, \lambda_t) = \sum_{h=0}^{\infty} \hbar^{2h} W_t^{(h)}(\lambda_1, \ldots, \lambda_t)$

We introduce spaces $\Omega_t(\Sigma)$ of (meromorphic) symmetric differentials (singularities at branch points only)

$$\frac{\partial}{\partial V(\lambda)} : \Omega_t \mapsto \Omega_{t+1}, \qquad H_\bullet : \Omega_{t+1} \mapsto \Omega_t.$$

- **Loop equation** expresses invariance under the change of integration variables $\delta x_i = \epsilon \frac{1}{x_i - x}$ and is exact:

$$W_1^2(x) - V'(x)W_1(x) + \left\langle tr \frac{V'(x) - V'(H)}{x - H} \right\rangle + \hbar^2 W_2(x,x) = 0.$$

Disregarding the correction term, for $W_1^{(0)}(x) = y(x) + V'(x)/2$ we obtain **algebraic equation** determining the **spectral curve**:

$$y^2(x) = \frac{1}{4}V'(x)^2 + P_{n-1}(x) \equiv U(x)$$

The **"flat" variables** t_k, s_i are: $t_k = \mathrm{res}_\infty x^{-k} y(x)$, $k \geq 0$; $s_i = \oint_{A_i} y(x)dx$.

$$\frac{\partial y(x)dx}{\partial s_i} = \omega_i \text{ —canonical holomorphic differential}, \qquad \oint_{A_i} \omega_j = \delta_{i,j}.$$

$$\frac{\partial y(x)dx}{\partial t_k} = v_k \text{ —Whitham–Krichever meromorphic differential},$$

$$\mathrm{res}_{\,\infty} x^{-p} v_k = \delta_{k,p}, \; k,p = 0,1,\ldots, \qquad \oint_{A_i} v_k \equiv 0.$$

1-, 2- and 3-point correlation functions

For P and Q point on the spectral curve, $B(P,Q)$ is the Bergmann bi-differential symmetric in $P \leftrightarrow Q$, canonically normalized, $\oint_{A_i} B(\cdot,Q) = 0$, and such that

$$B(P,Q)\Big|_{P\to Q} = \left(\frac{1}{(\xi(P)-\xi(Q))^2} + O(1)\right) d\xi(P)d\xi(Q),$$

with no other singularities. $\bar{y}$ denotes the point on the second sheet of the hyper-elliptic curve. Then

$$\frac{dxdz}{(x-z)^2} = B(x,z) + B(x,\bar{z}); \quad \frac{\partial}{\partial V(x)} y(z) = -\frac{1}{2}\left(B(x,z) - B(x,\bar{z})\right)$$

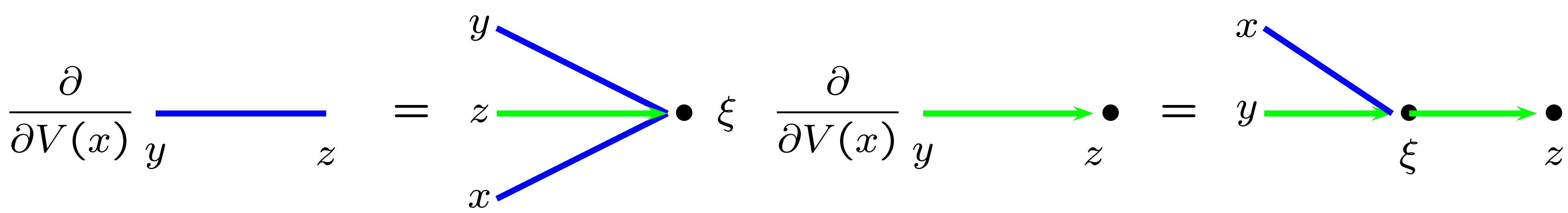
and so

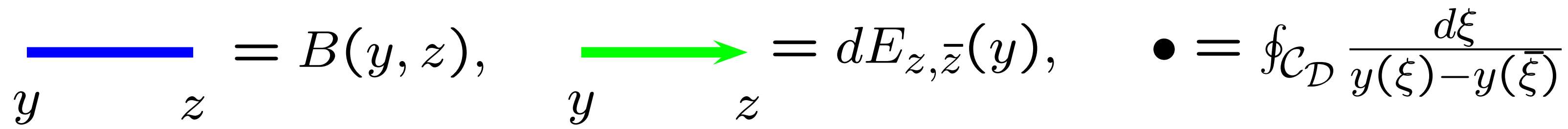
$$\frac{\partial}{\partial V(x)} W_1^{(0)}(y) = W_2^{(0)}(x,y) = B(x,\bar{y}).$$

$$\frac{\partial}{\partial V(x)} W_2^{(0)}(y,z) = W_3^{(0)}(x,y,z) = \oint_{\mathcal{C}_{\mathcal{D}}} \frac{B(x,\xi)B(y,\xi)B(\bar{z},\xi)}{dy(\xi)d\xi}$$

$$= \oint_{\mathcal{C}_{\mathcal{D}}} \frac{dE_{\xi,\bar{\xi}}(x)B(y,\xi)B(\bar{z},\xi)}{(y(\xi)-y(\bar{\xi}))d\xi}.$$

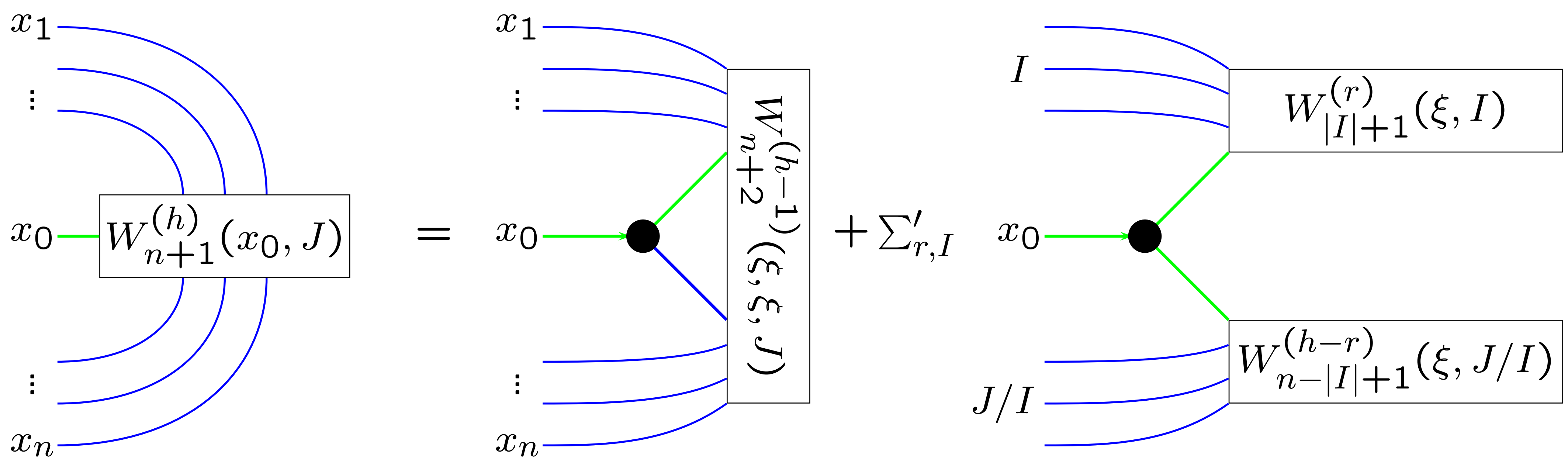
Action of $\partial/\partial V(x)$:





A subgraph of green arrowed propagators indicates the order of taking residues: from branches to the root.

Iterative solution of the loop equation (in the graphic form):



$W^{(h)}_n(J)$ comprises all the diagrams n **external legs and** h **loops** with the corresponding automorphism factors (if any) such that

- we segregate one variable, say, x_1, and take all the **maximum connected rooted subtrees** starting at the vertex x_1 and not going to any other external leg; the directed propagators $dE_{z,\bar{z}}(y)$ are associated with arrowed propagators; this subtree establishes **partial ordering** on the set of vertices;

- all other propagators: h inner propagators and $n-1$ remaining external legs are $B(x, y)$ if the vertices x and y are distinct and $B(x, \bar{x})$ for the tadpoles; *only comparable vertices can be joint by B(x,y)* (a vertex is comparable to itself).

Symplectic invariants $\mathcal{F}_h$ [L.Ch., B.Eynard, N.Orantin]

The above recurrent relation on the resolvents reads

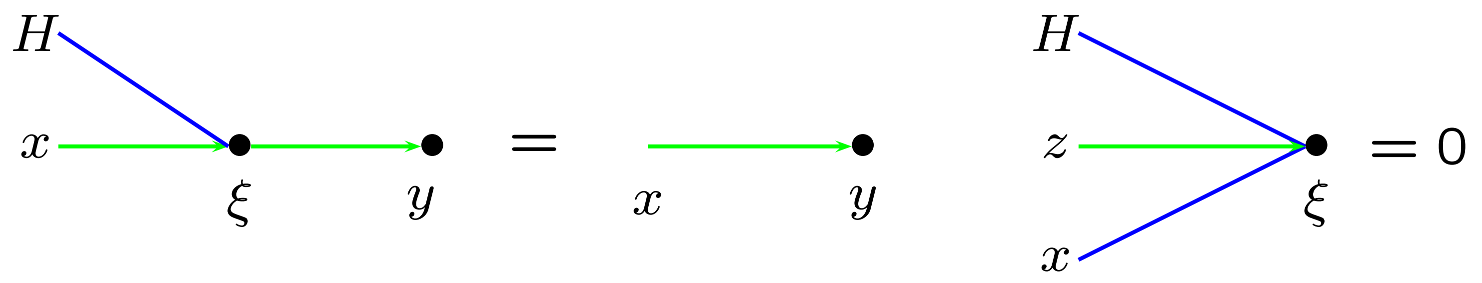
$$W^{(h)}_{n+1}(x_0, J) = \sum_i \mathop{res}_{\xi\to\mu_i} \frac{dE_{\xi,\bar\xi}(x_0)}{2(y(\xi)-y(\bar\xi))d\xi}\left[W^{(h-1)}_{n+2}(\xi,\bar\xi,J) + {\sum_{s,I\subseteq J}}' W^{(s)}_{|I|+1}(\xi, I)W^{(h-s)}_{n-|I|+1}(\bar\xi, J/I)\right],$$

the sum ranges only *stable* correlation functions $W^{(a)}_b$ with $a \geq 0$, $b > 0$, and $2a + b - 2 > 0$.

We use the new operator $H\cdot$ for **inverting** the loop insertion operator:

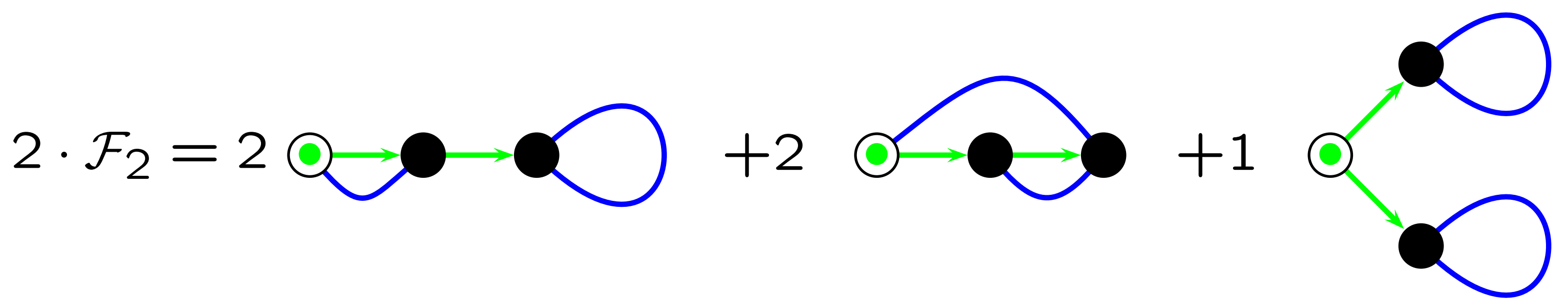
$$H\cdot\varphi := \mathop{res}_{\infty_x} V(x)\varphi(x) - \mathop{res}_{\infty_{\bar x}} V(x)\varphi(x) + t_0\int_{\infty_x}^{\infty_{\bar x}}\varphi(x)dx + \sum_{i=1}^{g} s_i \oint_{B_i}\varphi(x)dx,$$

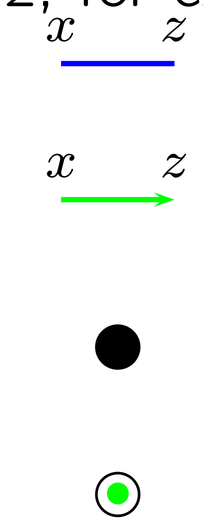
Action of $H_\bullet$:



$$\frac{\partial}{\partial V}\circ H_\bullet - H_\bullet \circ \frac{\partial}{\partial V} = \mathrm{Id}, \quad \text{then} \quad \mathcal{F}_h = \frac{1}{2h-2} H_x \cdot W_1^{(h)}(x).$$

We then have the diagrammatic expression for $\mathcal{F}_h$ with $h \geq 2$; for example





$= B(x,z)dxdz$

$= \int^z d\xi B(x,\xi)dx$

$= \sum_j \mathrm{res}\big|_{\mu_j} \frac{1}{y(x)dx}$

$= \sum_j \mathrm{res}\big|_{\mu_j} \frac{\int^x y(\xi)d\xi}{y(x)dx}$

[green arrows indicate the order of taking residues]

Here $B(x,z)dx\,dz$ is the normalized bi-differential on the Riemann surface Σ with the double pole as $x \to z$, $y(x)dx$ is the 1-differential on Σ, which is a **hyperelliptic** Riemann surface in the one-matrix model case.

Same technique works for

—finding $\mathcal{F}_h$ in one-matrix model with hard edges [L.Ch.]'05

—finding $\mathcal{F}_h$ in the two-matrix model (here Σ is an arbitrary algebraic curve) [B.Eynard, L.Ch., N.Orantin]'06

$$\int DH_1\, DH_2\, e^{-N\, tr(V_1(H_1)+V_2(H_2)+H_1 H_2)}$$

- general procedure of finding $\mathcal{F}_h$ in the β eigenvalue model using Feynman-like diagrams [B.Eynard, L.Ch.]'06

$$\int_N dx_i\, |\Delta(x)|^{2\beta}\, e^{-\frac{N\sqrt{\beta}}{t_0}\sum_{i=1}^N V(x_i)} \qquad \beta = \begin{cases} 1/2 - \text{orthogonal matrices} \\ 1 - \text{Hermitian matrices} \\ 2 - \text{symplectic matrices} \end{cases},$$

for arbitrary β **and any potential for which V' is a rational function** [this includes the AGT-conjecture case], we know the answer for $\mathcal{F}_{g,k}$, where

$$\mathcal{F} = \sum_{g,k=0}^{\infty} N^{2-2g-k}(\sqrt{\beta} - \sqrt{\beta}^{-1})^k \mathcal{F}_{g,k}.$$

Perturbative solutions of the β-eigenvalue model[L.Ch.,B.Eynard]'06

- **Loop equation** expresses invariance under the change of integration variables $\delta x_i = \epsilon \frac{1}{x_i - x}$ and is exact:

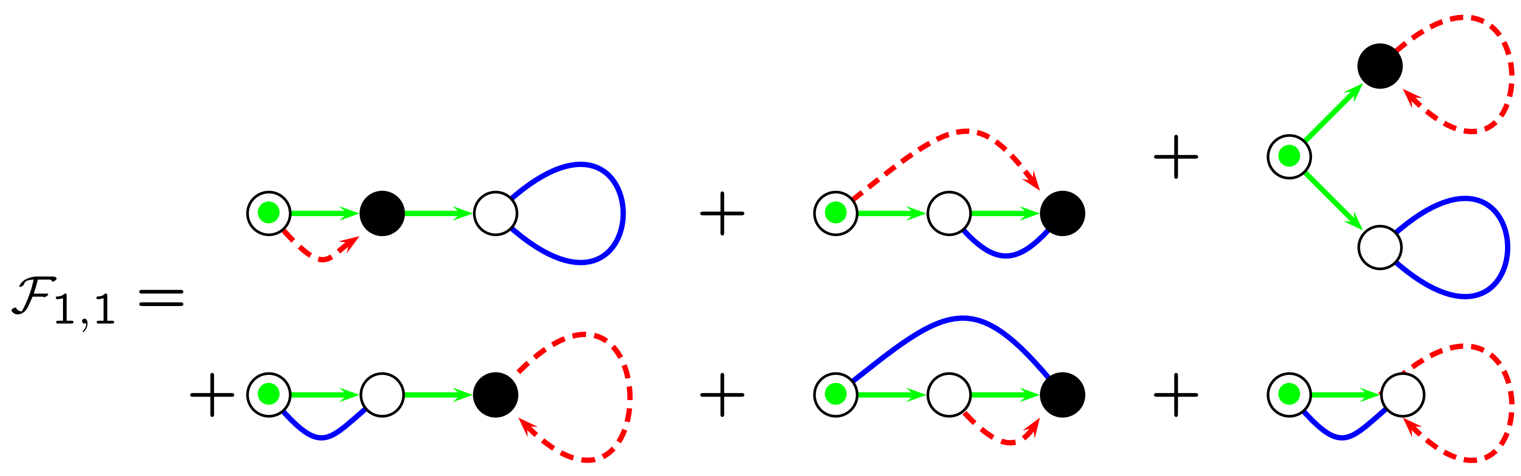
$$W_1^2(x) - V'(x)W_1(x) + \left\langle tr \frac{V'(x) - V'(H)}{x - H} \right\rangle + \frac{1}{N}(\sqrt{\beta} - \sqrt{\beta}^{-1})W_1'(x) + \frac{1}{N^2}W_2(x,x) = 0.$$

In the perturbative approach, we interpret the **both** last two terms as corrections, for $W_1^{(0)}(x) = y(x) + V'(x)/2$ we obtain the equation of the standard hyperelliptic spectral curve $y^2(x) = \frac{1}{4}V'(x)^2 + P_{n-1}(x) \equiv U(x)$. The free energy term is expressed again through the terms $W_1^{(g,k)}(x)$ of one-point resolvent expansion:

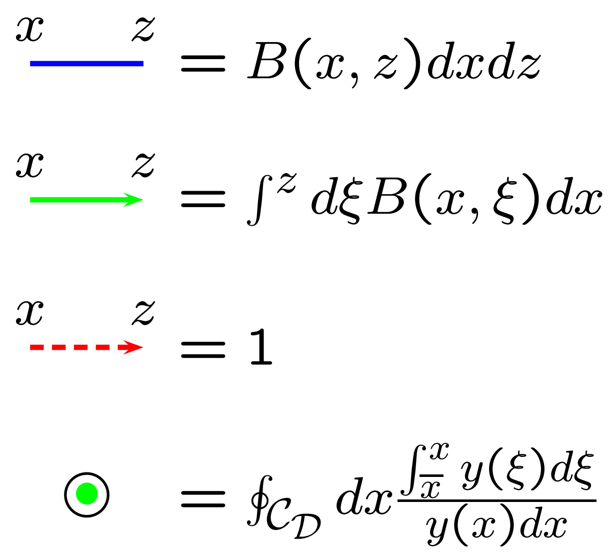
$$\mathcal{F}_{g,k} = \frac{1}{2g + k - 2} H_x \cdot W_1^{(g,k)}(x),$$

with the special expressions for $\mathcal{F}_{1,0}$ [L.Ch.'02] and $\mathcal{F}_{0,2}$ [L.Ch.,B.Eynard'06].

Feynman diagram rules

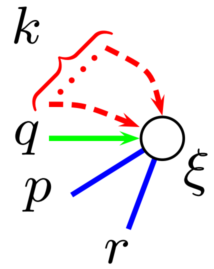


[green arrows indicate the order of doing $\int_{\mathcal{C}_\mathcal{D}}$]

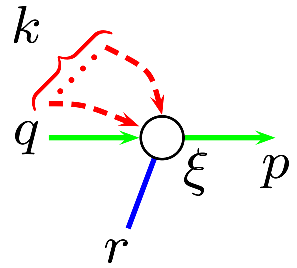


The complete list of vertices is as follows (black vertices are those containing derivatives of $y(\xi)$)

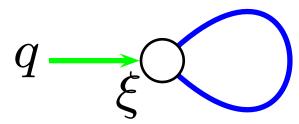
The vertices with three adjacent solid lines



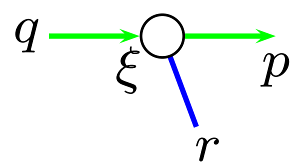
$$\sim \oint_{\mathcal{C}_{\mathcal{D}}^{(\xi)}} dE_{\xi,\bar\xi}(q)\frac{d\xi}{2\pi i\ y(\xi)}\frac{\partial^k}{\partial\xi^k}\Big(B(r,\xi)B(p,\xi)\Big),\quad k\ge 0,\ \begin{cases}\xi<r,p;\\ r,p \text{ can be external}\end{cases}$$



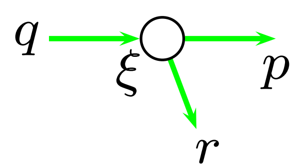
$$\sim \oint_{\mathcal{C}_{\mathcal{D}}^{(\xi)}} dE_{\xi,\bar\xi}(q)\frac{d\xi}{2\pi i\ y(\xi)}\frac{\partial^k}{\partial\xi^k}B(r,\xi),\quad k\ge 0,\ \begin{cases}\xi<r;\\ r \text{ can be external}\end{cases}$$



$$\sim \oint_{\mathcal{C}_{\mathcal{D}}^{(\xi)}} dE_{\xi,\bar\xi}(q)\frac{d\xi}{2\pi i\ y(\xi)}B(\xi,\bar\xi).$$



$$\sim \oint_{\mathcal{C}_{\mathcal{D}}^{(\xi)}} dE_{\xi,\bar\xi}(q)\frac{d\xi}{2\pi i\ y(\xi)},\quad \begin{cases}\xi>r;\\ q \text{ can be external}\end{cases}$$



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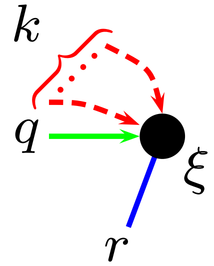
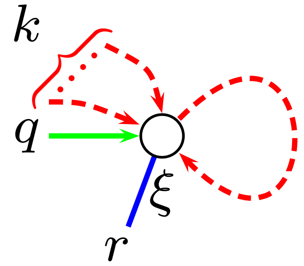
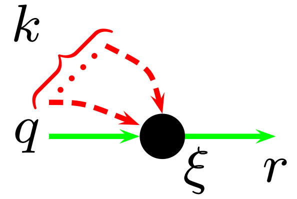
The vertices with two adjacent solid lines



$$\sim \oint_{\mathcal{C}_{\mathcal{D}}^{(\xi)}} dE_{\xi,\bar\xi}(q)\frac{y^{(k)}(\xi)d\xi}{2\pi i\ y(\xi)}B(r,\xi), \quad k\geq 1, \quad \begin{cases} \xi < r; \\ r \text{ can be external} \end{cases}$$



$$\sim \oint_{\mathcal{C}_{\mathcal{D}}^{(\xi)}} dE_{\xi,\bar\xi}(q)\frac{d\xi}{2\pi i\ y(\xi)}\frac{\partial^{k+1}}{\partial\xi^{k+1}}B(r,\xi), \quad k\geq 0, \quad \begin{cases} \xi < r; \\ r \text{ can be external} \end{cases}$$



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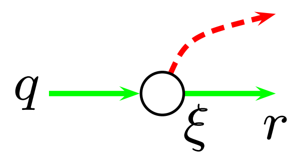
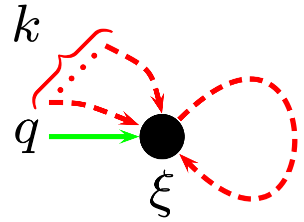


$$\sim \oint_{\mathcal{C}_{\mathcal{D}}^{(\xi)}} dE_{\xi,\bar\xi}(q)\frac{d\xi}{2\pi i\ y(\xi)}, \quad q \text{ can be external.}$$

A vertex with one adjacent solid line



$$\sim \oint_{\mathcal{C}_{\mathcal{D}}^{(\xi)}} dE_{\xi,\bar\xi}(q)\frac{y^{(k+1)}(\xi)d\xi}{2\pi i\ y(\xi)}, \quad k\geq 0,\ q \text{ can be external.}$$

Quantum surfaces = nonperturbative solutions of the β-eigenvalue model[L.Ch.,B.Eynard,O.Marchal]'09-10

- Recall the **loop equation**:

$$W_1^2(x) - V'(x)W_1(x) + \left\langle tr\,\frac{V'(x)-V'(H)}{x-H}\right\rangle + \frac{1}{N}(\sqrt{\beta}-\sqrt{\beta}^{-1})W_1'(x) + \frac{1}{N^2}W_2(x,x) = 0.$$

We now incorporate the term with $W_1'(x)$ into the leading order. This results in **resummation** of the asymptotic series for $\mathcal{F}_{g,k}$ in k.

For $W_1^{(0)}(x) = y(x) + V'(x)/2$ we obtain **Riccati equation** determining the **spectral curve**:

$$y^2(x) + \hbar y'(x) = \frac{1}{4}V'(x)^2 + P_{n-1}(x) \equiv U(x),$$ where we identify $\hbar = (\sqrt{\beta} - \sqrt{\beta}^{-1})/N$.

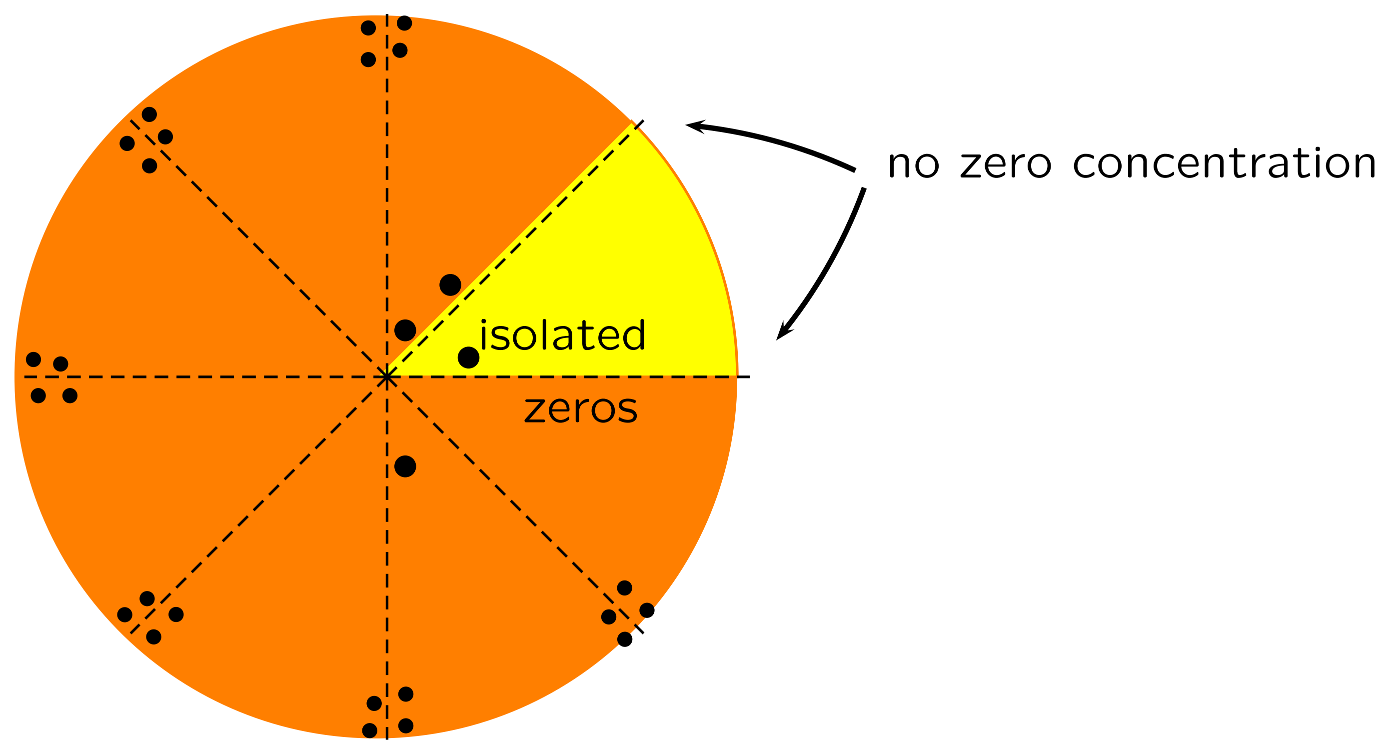
Solution is $y(x) = \hbar\psi'(x)/\psi(x)$, where $\psi(x)$ solves the **Schrödinger equation**

$$\hbar^2\psi''(x) = U(x)\psi(x).$$

with $V'(x) = 2\sqrt{U(x)}_+$.

- **Stokes Sectors** We choose the function $\psi_\alpha(x)$ to be a unique solution of the Schrödinger equation that decreases at the αth sector

$$S_k = \left\{ \mathrm{Arg}(x) \in] -\frac{\theta_0}{d+1} + \pi\frac{k-\frac{1}{2}}{d+1}, -\frac{\theta_0}{d+1} + \pi\frac{k+\frac{1}{2}}{d+1}[\right\}$$



- We **cannot** satisfy asymptotic conditions $W_1^{(0)}(x) \sim t_0/x + O(x^{-2})$ in all directions if we take just one solution $\psi(x)$, so define $W_1^{(0)}(x)$ **sectorwise**:

$$\omega(\overset{\alpha}{x}) := W_1^{(0)}(\overset{\alpha}{x}) = \hbar\frac{\psi'_\alpha(x)}{\psi_\alpha(x)} + \frac{V'(x)}{2}, \text{ for } x \in S_\alpha.$$

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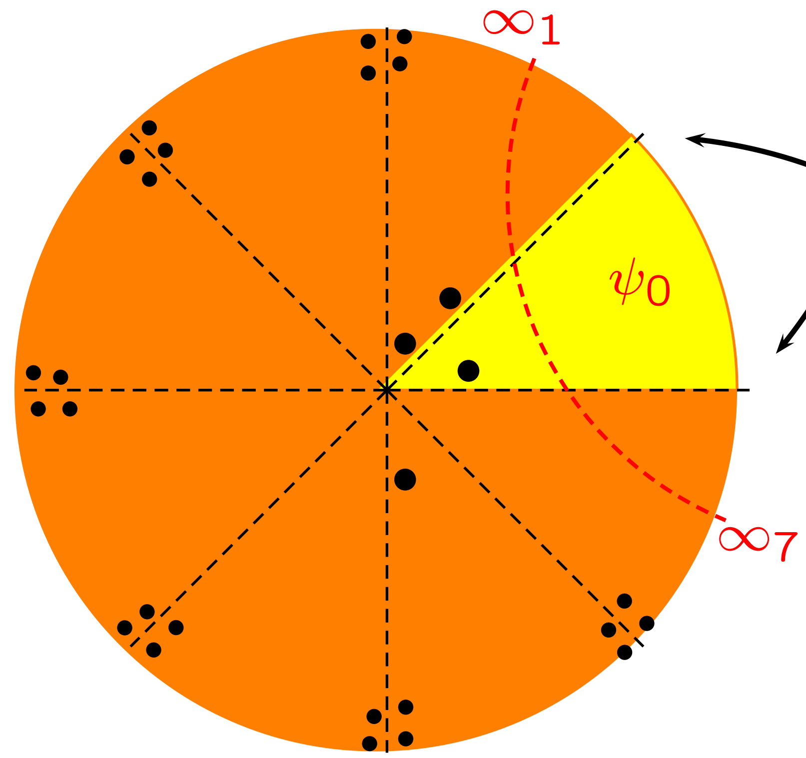

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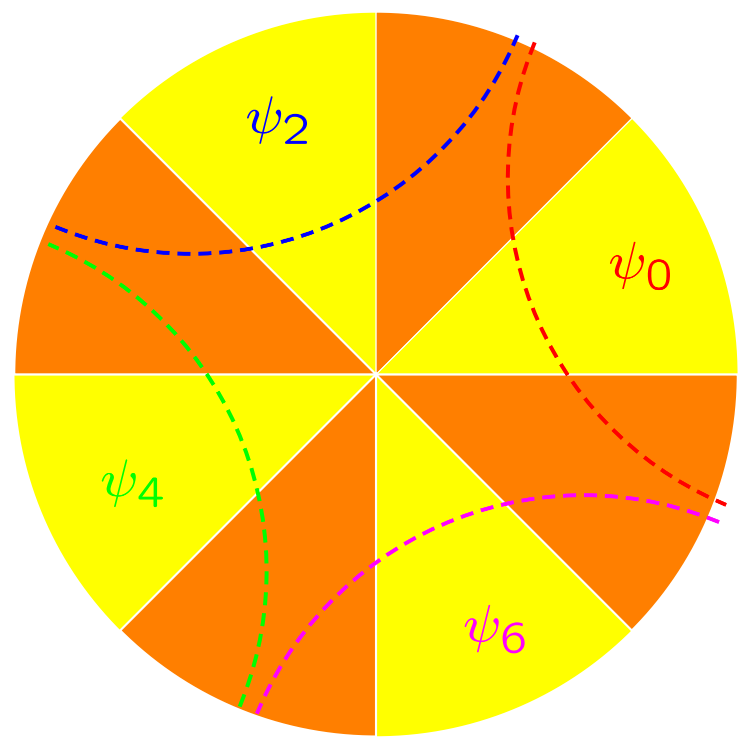
$$\omega(\overset{\alpha}{x}) := W_1^{(0)}(\overset{\alpha}{x}) = \hbar \frac{\psi'_\alpha(x)}{\psi_\alpha(x)} + \frac{V'(x)}{2}, \quad \text{for } x \in S_\alpha.$$

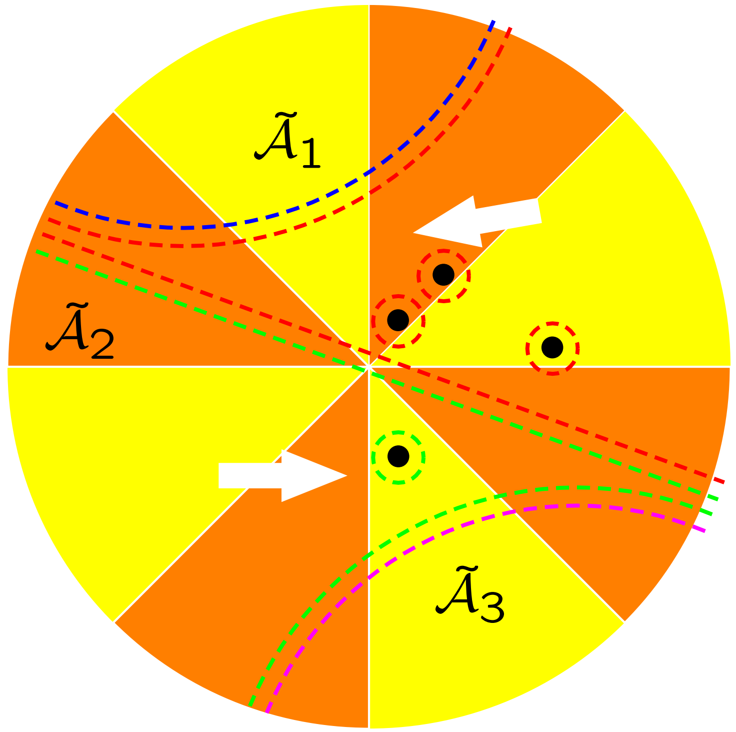
- The contour $\mathcal{C}_D$ and the set of $\mathcal{A}$- and $\mathcal{B}$-cycles

$$\oint_{\mathcal{C}_D} f(x)dx \equiv \sum_{\alpha} \int_{\infty_{\alpha-1}}^{\infty_{\alpha+1}} f(\overset{\alpha}{x})dx$$

The integration contour $\mathcal{C}_D$: $\oint_{\mathcal{C}_D}$ — the analogue of $\text{res}|_{\infty}$

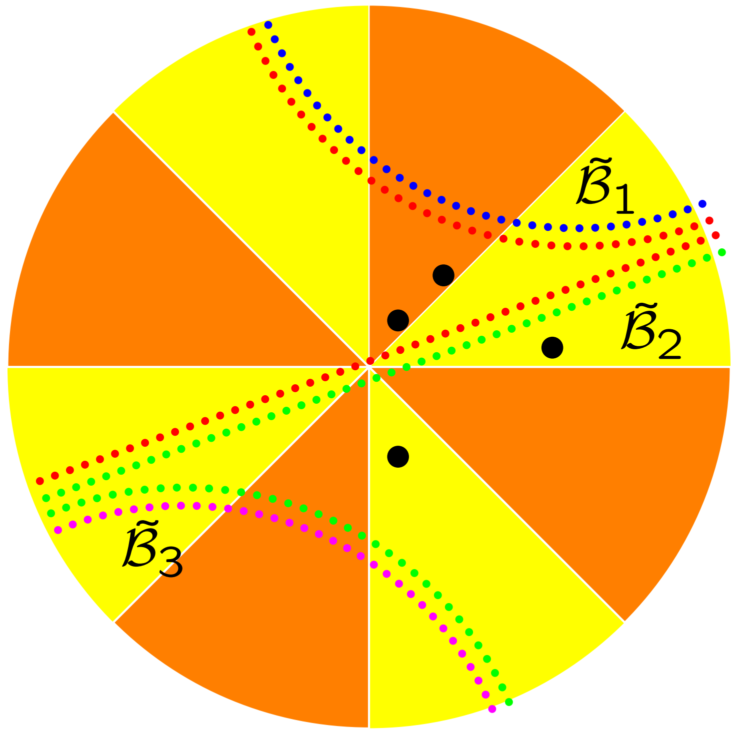


To obtain A-cycles, we "protrude" integration contours to make them running between infinities "in pairs":

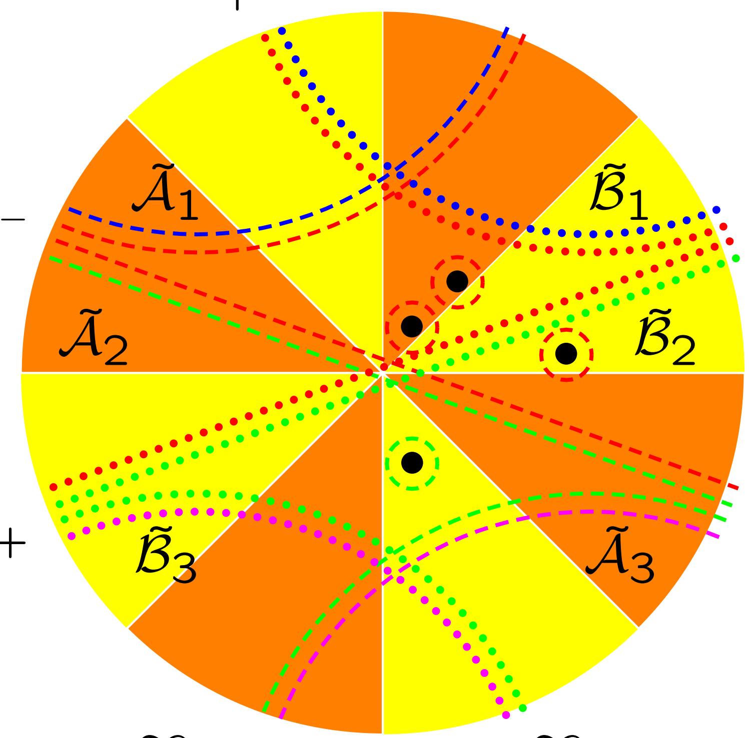


$$\oint_{\tilde{\mathcal{A}}_\alpha} f(x)dx \stackrel{\text{def}}{=} \int_{\infty_{\tilde{\alpha}_-}}^{\infty_{\tilde{\alpha}_+}} \left(f(\overset{\alpha_+}{x}) - f(\overset{\alpha_-}{x})\right) dx + \sum_{s_i^{(\alpha_\pm)}(\alpha)} \underset{}{res}\, f(\overset{\alpha_\pm}{x})$$

B-cycles are, as usual, “dual” to A-cycles:



$$\oint_{\tilde{\mathcal{B}}_\alpha} f(x)dx \stackrel{\text{def}}{=} \int_{\infty_{\alpha_-}}^{\infty_{\alpha_+}} (f(\overset{\alpha_+}{x}) - f(\overset{\alpha_-}{x}))\, dx,$$



- Filling fractions

$$s_\alpha = \frac{1}{2i\pi} \oint_{\tilde{\mathcal{A}}_\alpha} \omega(x)dx \stackrel{\text{def}}{=} \int_{\infty_{\tilde{\alpha}_-}}^{\infty_{\tilde{\alpha}_+}} \Big(\omega(\overset{\alpha_+}{x}) - \omega(\overset{\alpha_-}{x})\Big)dx, \quad \alpha = 1, \ldots, d.$$

The difference

$$\omega(\overset{\alpha_+}{x}) - \omega(\overset{\alpha_-}{x}) = \frac{\mathsf{Wron}_{\alpha_+,\alpha_+}}{\psi_{\alpha_+}(x)\psi_{\alpha_-}(x)},$$

decreases exponentially in sectors where the both solutions ψ_{α_+} and ψ_{α_-} increase.

Variations of the resolvent w.r.t. "flat" coordinates

Flat coordinates: $\epsilon_i = \oint_{\mathcal{A}_i} \omega(x)dx$, $t_k = \oint_{\mathcal{C}_D} \omega(x)x^{-k}dx$, $k = 0, 1, \ldots$.

We consider an infinitesimal polynomial variation $U \to U + \delta U$. We have

$$\hbar\delta y' + 2y\delta y = \delta U; \delta y(\overset{\alpha}{x}) = \frac{1}{\hbar\psi_\alpha^2(x)} \int_{\infty_\alpha}^{x} \psi_\alpha^2(x')\delta U(x')dx'.$$

Here $\delta_{\epsilon_i} U(x) = h_i(x)$, $i = 1, \ldots, d-1$; $\delta_{t_k} U(x) = h_{d+k}(x)$, $k = 0, 1, \ldots$. Every $h_{d+k}(x)$ is a polynomial of degree $d + k - 1$.

Expressions $\frac{1}{\hbar\psi_\alpha^2(x)} \int_{\infty_\alpha}^{x} \psi_\alpha^2(x')h_s(x')dx' = v_s(\overset{\alpha}{x})$ are analogues of holomorphic differentials and Whitham–Krichever meromorphic differentials. Those are **holomorphic differentials** if degree of h_s less or equal $d - 2$, otherwise normalization:

$$\frac{\partial t_k}{\partial t_p} = \delta_{k,p} = \oint_{\mathcal{C}_\mathcal{D}} v_p \cdot x^{-k}dx; \qquad 0 = \frac{\partial \epsilon_i}{\partial t_p} = \oint_{\mathcal{A}_i} v_p(x)dx.$$

- First kind functions $v_k(\overset{\alpha}{x})$. Let h_k, $k=1,\dots,d-1$, be a basis of polynomials of degree $\le d-2$. Then

$$v_k(\overset{\alpha}{x}) = \frac{1}{\hbar\,\psi_\alpha^2(x)} \int_{\infty_\alpha}^{x} h_k(x')\,\psi_\alpha^2(x')\,dx'$$

with **the same** polynomial $h_k(x')$ for all the sheets and such that

$$I_{k,\alpha} = \oint_{\mathcal{A}_\alpha} v_k(x)\,dx = \delta_{k,\alpha} \quad k,\alpha = 1,\dots,d-1;$$

$v_k(\overset{\alpha}{x})$ has double poles with no residue at the zeroes of ψ_α and behaves like $O(1/x^2)$ inside all the sectors including the sector S_α.

- The Riemann matrix of periods

$$\tau_{\alpha,i} \overset{\text{def}}{=} \oint_{\mathcal{B}_\alpha} v_i(x)dx$$

is symmetric [proof is not direct, however...]

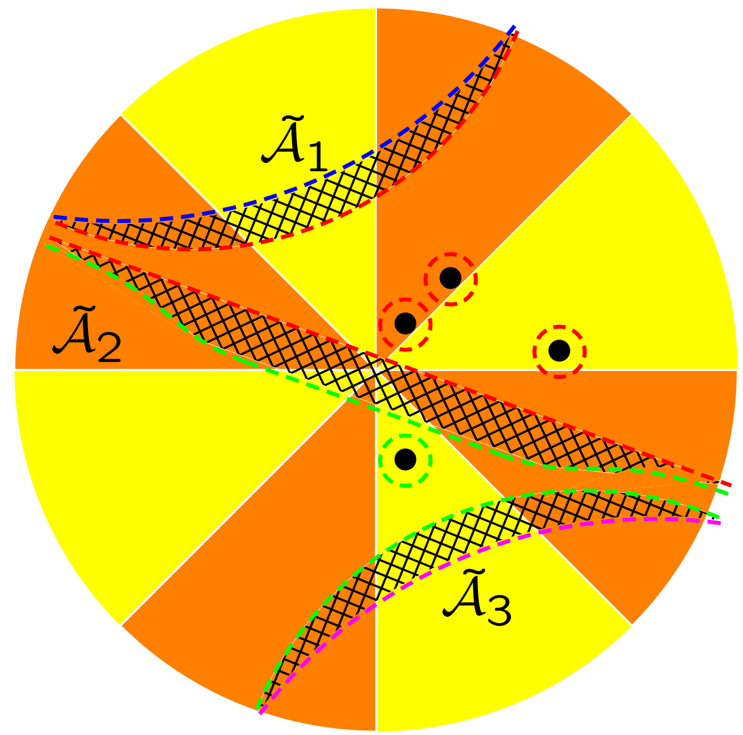
- The recursion kernel K

$$\widehat{K}(\overset{\alpha}{x}, z) = \frac{1}{\hbar}\frac{1}{\psi_\alpha^2(x)} \int_{\infty_\alpha}^{x} \psi_\alpha^2(x')\, \frac{dx'}{x'-z}$$

The recursion kernel $K(\overset{\alpha}{x}, z)$ reads

$$K(\overset{\alpha}{x}, z) = \widehat{K}(\overset{\alpha}{x}, z) - \sum_{j=1}^{d-1} v_j{}'(\overset{\alpha}{x}) C_j(z), \quad \hbar C_\alpha(z) = \oint_{\mathcal{A}_\alpha} \widehat{K}(x,z),\ \alpha = 1, \ldots, g$$

for z in the hatched domain



$K(\overset{\alpha}{x}, z) \Rightarrow$ third kind differential $G(\overset{\alpha}{x}, \overset{\beta}{z}) \Rightarrow$ "quantum" Bergman kernel $B(\overset{\alpha}{x}, \overset{\beta}{z})$

- $K(\overset{\alpha}{x}, z)$ has a discontinuity along $\mathcal{A}$- and $\mathcal{B}$-cycles;

- Third kind differential: kernel $G(\overset{\alpha}{x}, \overset{\beta}{z})$

$$G(\overset{\alpha}{x}, \overset{\beta}{z}) = -\hbar\,\psi_\beta^2(z)\,\partial_z\,\frac{K(\overset{\alpha}{x}, z)}{\psi_\beta^2(z)} = 2\hbar\frac{\psi'_\beta(z)}{\psi_\beta(z)}K(\overset{\alpha}{x}, z) - \hbar\partial_z\,K(\overset{\alpha}{x}, z)$$

  is an analogue of $dE_{Q,\bar{Q}}(P)$ and *has no discontinuity* along the $\mathcal{B}$-cycles and has discontinuity across $\mathcal{A}_\beta$-cycles: $\delta_z G(\overset{\alpha}{x}, \overset{\beta_\pm}{z}) = \mp 2i\pi\, v_{\beta_\pm}(\overset{\alpha}{x})$.

- The "quantum" Bergman kernel $B(\overset{\alpha}{x}, \overset{\beta}{z})$

$$B(\overset{\alpha}{x}, \overset{\beta}{z}) = -\frac{1}{2}\,\partial_z\,G(\overset{\alpha}{x}, \overset{\beta}{z}).$$

  $B(\overset{\alpha}{x}, \overset{\beta}{z})$ *is an analytical function of x and z in the whole complex plane (no cuts) with the double pole with zero residue at $x = z$ for $\alpha = \beta$.*

- Moreover, the kernel B satisfies the loop equations *in the both variables.*

The properties of $B(\overset{\alpha}{x},\overset{\beta}{z})$

- For every $\alpha=1,\dots,g$: $\oint_{\mathcal{A}_i} B(x,\overset{\beta}{z})\,dx=0$, $\quad\oint_{\mathcal{A}_j} B(\overset{\alpha}{x},z)\,dz=0$;
- $\oint_{\mathcal{B}_j} B(\overset{\alpha}{x},z)\,dz=2i\pi v_j(\overset{\alpha}{x})$;
- $B(\overset{\alpha}{x},\overset{\beta}{z})$ is symmetric, $B(\overset{\alpha}{x},\overset{\beta}{z})=B(\overset{\beta}{z},\overset{\alpha}{x})$.

Corollary The period matrix $\tau_{k,\alpha}$ is symmetric: $\tau_{k,\alpha}=\oint_{\mathcal{B}_k}\oint_{\mathcal{B}_\alpha} B(z,x)dz\,dx$.

Since $\frac{\partial}{\partial t_r}\left(\hbar\frac{\psi'_\alpha(x)}{\psi_\alpha(x)}\right)=v_{d+r}(\overset{\alpha}{x})$, we define the loop insertion operator $\frac{\partial}{\partial V(\xi)}:=\sum_{r=1}^{\infty} r\xi^{-r-1}\frac{\partial}{\partial t_r}$

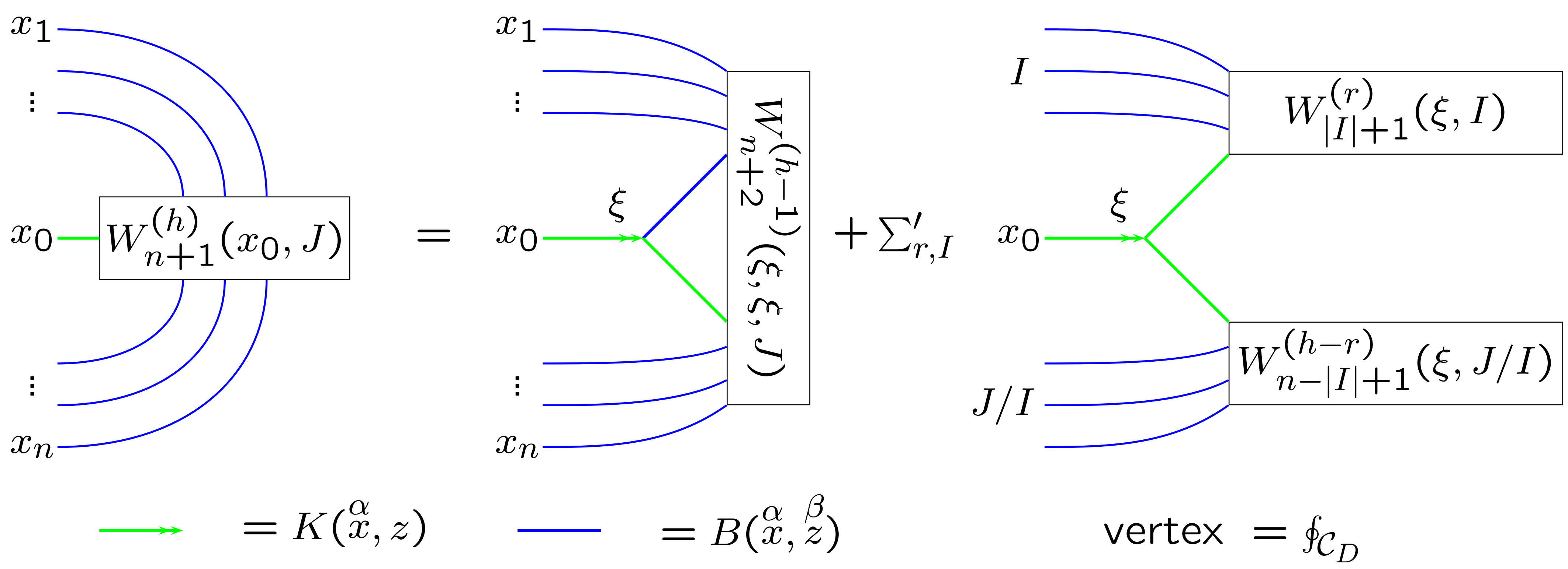
$$\frac{\partial}{\partial V(\xi)}y(\overset{\alpha}{x})=\sum_{r=1}^{\infty}\xi^{-r-1}\oint_{\xi>\mathcal{C}_D} B(\overset{\alpha}{x},z)z^r dz,$$

which sums up to $\oint_{\xi>\mathcal{C}_D} B(\overset{\alpha}{x},z)\frac{1}{\xi-z}dz$ The point ξ lies between some infinity, say, ∞_β, and the integration contour $\mathcal{C}_D$. Pulling the contour of integration through ξ to infinity we obtain zero due to the asymptotic conditions for $B(\overset{\alpha}{x},z)$, so the only nonvanishing contribution comes from the residue at $\xi=z$, $\frac{\partial}{\partial V(\xi)}y(\overset{\alpha}{x})=B(\overset{\alpha}{x},\overset{\beta}{\xi})$.

We identify $B(\overset{\alpha}{x},\overset{\beta}{z})$ with the two-point correlation function.

- Diagrammatic representation for correlation functions (solutions of the loop equation) is formally the same as the one in the original matrix model

Recurrent relation:



Singular behavior of $\mathcal{F}_0$

For the one- and two-matrix models:

$$\text{Sing } \mathcal{F}_0 = \sum_{i=1}^{n} \frac{1}{2} s_i^2 \log s_i; \qquad \frac{\partial^3 \mathcal{F}_0}{\partial s_i \partial s_j \partial s_k} = \frac{\delta_{i,j,k}}{s_i} + \text{reg.}$$

In the quantum surface case (for $\hbar = 1$),

$$\text{Sing } \mathcal{F}_0 = \sum_{i=1}^{n} \int^{s_i} \log \Gamma(\xi) d\xi; \qquad \frac{\partial^3 \mathcal{F}_0}{\partial s_i \partial s_j \partial s_k} = \delta_{i,j,k} \Big(\log \Gamma(s_i) \Big)'' + \text{reg.}$$

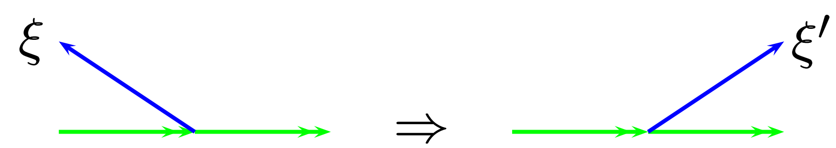
So we have poles of the **second** order at $s_i = 0, -1, -2, \ldots$ (not the **first**-order pole $1/s_i$ as in the matrix-model case), but the **same** singular behavior $\simeq \frac{1}{2} s_i^2 \log s_i$ at large positive s_i.

- Diagrammatic representation for symplectic invariants

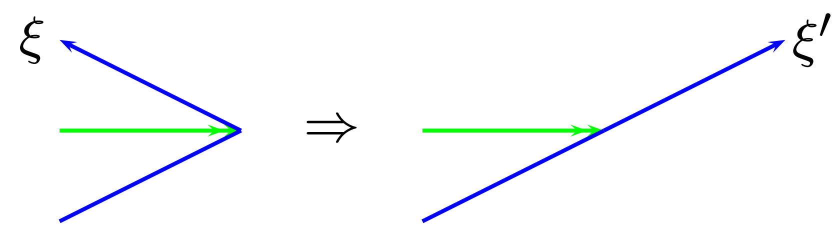
Difficulty: to define $\beta\frac{\partial}{\partial\beta}$ for $|\Delta|^{2\beta}$ in the integral. Solution: instead of one-point function, we need **two-point functions**:

$$\hbar\frac{\partial}{\partial\hbar}W_n^{(h)}(J) = \int_{\mathcal{C}_{D_\xi}} d\xi \left[\int_\infty^\xi W_{n+2}^{(h-1)}(\bar{\xi}', \xi, J)d\xi' + \sum_{r=0}^{h}\sum_{I\subseteq J}\int_\infty^\xi W_{|I|+1}^{(r)}(\bar{\xi}', I)d\xi' \cdot W_{n-|I|+1}^{(h-r)}(\xi, J/I)\right],$$

where $\bar{\xi}$ must be taken to be an "innermost" variable; because $\int^\xi B(\xi', y) = G(\xi, y)$, we replace all the appearances



and

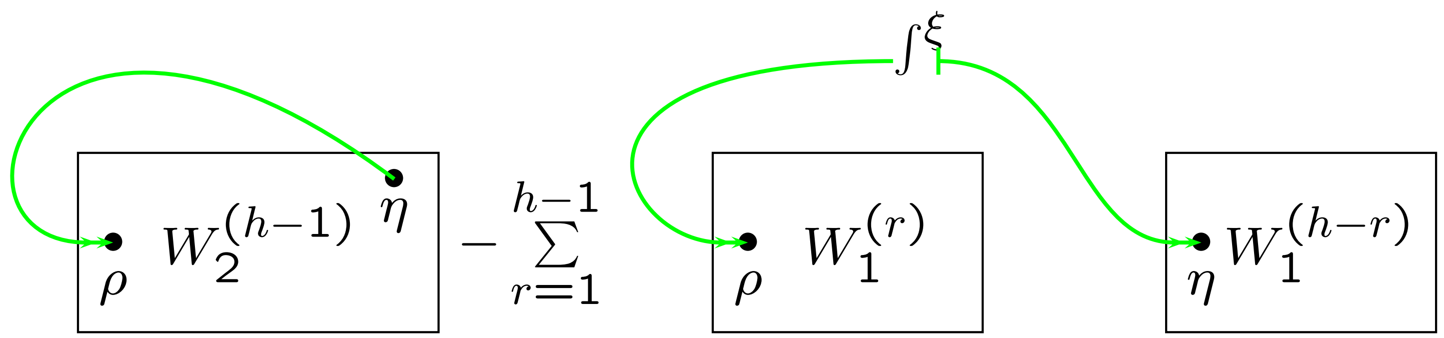


with *no additional factors*.

- The term $\mathcal{F}_h$

For the stable cases ($h \neq 0, 1$), the term $\mathcal{F}_h$ reads: Take all diagrams describing the stable terms $W_1^{(r)}(\bar{\xi})$ and $W_1^{(h-r)}(\xi)$ with $1 \le r \le h-1$ and $W_2^{(h-1)}(\xi, \bar{\xi})$, then

$(2h-2)\mathcal{F}_h =$



Here in the first term the vertices η and ρ are *distinct* (the ρ vertex is always the first three-valent vertex in the rooted tree) and joint by the propagator $K(\eta, \rho)$. In the second term, the integration over ξ survives ($\rho > \mathcal{C}_{D_\xi} > \eta$) and the symbol $\int^\xi$ indicates that we must insert the integration

$$\int_{\rho > \mathcal{C}_{D_\xi} > \eta} d\xi \int_{\infty_\alpha}^{\xi + \delta_\alpha} d\xi' K(\overset{\alpha}{\xi'}, \rho) K(\overset{\alpha}{\xi}, \eta)$$

between the integrations over the variables ρ and η.