Brane Universe: Global Geometry

Victor Berezin[1]

[1]Institute for Nuclear Research, RAS, Moscow

- Aim:
  Investigate possible global geometries of "brane universe" scenarios.
  Our Universe - thin shell (membrane) = "brane" - embedded into the space-time of larger number of dimensions, "bulk".
- Strategy:
  Simplify everything as much as possible and construct some exactly solvable model - the source of our physical intuition.

$(N+1)$-dimensional space-time containing a N-dimensional brane (= thin shell). Metric

$$ds^2 = g_{\mu\nu}(y) dy^\mu dy^\nu \; ; \quad \mu, \nu = 0, 1, ...N \tag{1}$$

Brane is time-like and has the "cosmological symmetry" = homogeneity + isotropy.

- Local approach

Outside the shell - bulk geometry possesses same symmetry: bulk does not depend on the brane position $\longrightarrow$ everywhere in the normal Gaussian coordinate system

$$\begin{aligned} ds^2 &= -dn^2 + \gamma_{i,j} dx^i dx^j = (i,j = 0,2,...N) \\ &= -dn^2 + B^2(n,t)dt^2 - A^2(n,t)dl^2_{N-1}\,, \\ dl^2_{N-1} &= \frac{dr^2}{1-kr^2} + r^2 d\Omega^2_{N-2}\,, \end{aligned} \qquad (2)$$

dl^2_{N-1} - Robertson-Walker
$k = +1 - (N-1) - dim$ sphere
$k = -1$ - rotational hyperboloid
$k + 0$ - flat space

Only one brane at $n = 0$.
Energy-momentum tensor

$$T_{\mu\nu} = S_{\mu\nu}\delta(n) + [T_{\mu\nu}]\Theta(n) + T_{-\mu\nu} \tag{3}$$

$S_{\mu\nu}$ - surface energy-momentum tensor
$[X] = (X_+ - X_-)$ - jump, $n > 0 - ('' + ''),\ n < 0 - ('' - '')$
$S_{\mu\nu} \neq 0$ - singular shell (brane)
Extrinsic curvature tensor

$$K_{ij} = -\frac{1}{2}\frac{\partial \gamma_{ij}}{\partial n} = -\frac{1}{2}\gamma_{ij,n} \quad n = const \tag{4}$$

Decomposition of the Einstein equations (not all)

$$\begin{aligned} -[K_{ik}] \quad &+ \quad \gamma_{ik}\,[K_-] = 8\pi G\, S_{ik}\,, \\ S^l_{i|l} \quad &+ \quad [T^n_i] = 0\,, \end{aligned} \tag{5}$$

$$-\left(K_{-ik,n} - \gamma_{ik} K_{-,n} + 2K_{-il} K^{l}_{-k} - K_{-ik} K_{-}\right) \tag{6}$$

$$-\frac{1}{2}\gamma_{ik}\left(K^{l}_{-j} K^{j}_{-l} + K_{-}\right) + {}^{(N)}G_{ik} = 8\pi G\, T_{-ik} \tag{7}$$

$${}^{(N)}G_{ik} = 8\pi G\, T_{-ik} + T^{ind}_{-ik} \tag{8}$$

Cosmological time on the surface $n = const$:

$$d\tau_n = B(n,t)dt \tag{9}$$

Scale factor

$$\begin{aligned} a(\tau_n) &= A(n,t)\,,\ a_\tau = \frac{da}{d\tau}\,,\ a_{\tau\tau} = \frac{d^2a}{d\tau^2}\,, \\ {}^{(N)}G^0_0 &= \frac{(N-1)(N-2)}{2}\frac{a_\tau^2 + k}{a^2}\,, \\ {}^{(N)}G^2_2 &= \frac{N-2}{2}\left(\frac{2a_{\tau\tau}}{a} + (N-3)\frac{a_\tau^2 + k}{a^2}\right) = G^3_3 = \ldots = G^N_N\,. \end{aligned} \tag{10}$$

- Global approach

Cosmological principle $\longrightarrow$ allows to use another approach: to deal with the invariants of bulk geometry only. $(d+2)$ -decomposition $(d+2=N+1)$

$$ds^2 = \gamma_{AB}(x)dx^A dx^B - R^2(x)dl_d^2\,, \quad A,B = 0,1 \qquad (11)$$

dl_d^2 - Robertson-Walker, curvature $= d(d-1)k,\ k = \pm 1, 0.$
$R(t,q)$ - radius = scale factor.
Two invariants $R(t,q)$ and $\Delta(t,q)$:

$$\Delta = \gamma^{AB} R_{,A} R_{,B} \qquad (12)$$

Δ - squared normal to the surfaces $R = const$.
Einstein equations in vectorial form

$$\left(R^{d-1}(\Delta + k)\right)_{,A} = \frac{16\pi G}{d} R^d \left(T\,R_{,A} - T_A^B\,R_{,B}\right)\,, \qquad (13)$$

$$\gamma^{AC}\,R_{||CB} = -\frac{8\pi G}{d} R\,T_B^A\,, \quad A \neq B\,, \qquad (14)$$

Δ - brings important geometrical information. Minkowski $\rightarrow \Delta = -1$ Curved space-times - Δ may have both signs

$\Delta < 0, \to R = const$ - time-like, R can be chosen as a spatial coordinate ($\dot{R} = 0$, $R'^2 = 1 \to \Delta = \gamma^{11} < 0) \longrightarrow R$-regions.
Sign $R_{,q}$ cannot be changed $\to R_+$-regions ($R_{,q} > 0$), R_--regions ($R_{,q} < 0$).
$\Delta > 0 \ \to R = const$ - space-like, R can be chosen a time coordinate ($R' = 0$, $\dot{R}^2 = 1 \to \Delta = \gamma^{00} > 0) \longrightarrow T$-region. Sign $R_{,t}$ cannot be changed $\longrightarrow$
T_+-regions ($R_{,t} > 0$) - inevitable expansion
T_--regions ($R_{,t} > 0$) - inevitable contraction
$R-$ and $T-$regions are separated by surfaces $\Delta = 0$ - apparent horizons.
Global geometry = set of $R-$ and $T-$regions and apparent horizons.
Physical selection rule: geodesic completeness = time-like and null geodesics start and end either at infinities or at singularities.

Simplification: step II

$n \neq 0$ - vacuum $\rightarrow$

$$8\pi\, G\, T^{\nu}_{\mu} = \Lambda\, \delta^{\nu}_{\mu} \quad \longrightarrow 8\pi\, G\, T^{A}_{B} = \Lambda\, \delta^{A}_{B} \tag{15}$$

$$\Delta = -k + \frac{2\, G\, m}{R^{N-2}} + \frac{2}{N(N-1)} \Lambda\, R^2 \tag{16}$$

m - dimension of mass. Different k $\rightarrow$ different bulks $\rightarrow$ global feature. Two-dimensional metric:
R-regions, $\Delta < 0$, $|q| = R$ $\rightarrow$

$$\begin{aligned} ds_2^2 = (-\Delta)\, dt^2 - \frac{dR^2}{(-\Delta)} &= (-\Delta)(dT^2 - dR^{\star 2}) \\ dR\star &= \pm \frac{dR}{|\Delta|}. \end{aligned} \tag{17}$$

T-regions, $\Delta > 0$, $|t| = R$ $\rightarrow$

$$ds_2^2 = \frac{dR^2}{\Delta} - \Delta dq^2 = \Delta \left(dR^{\star 2} - dq^2 \right) , \tag{18}$$

Simplification: step III

Single brane: absence of singularity at $R = 0 \ \rightarrow m = 0$. In Gaussian coordinates

$$\begin{aligned} \Delta &= \frac{1}{B^2(n,t)} R^2(n,t)_{,t} - R^2(n,t)_{,n} = f^2(t) - R^2_{,n}\,, \\ R_{,n} &= \pm\sqrt{f^2(t) - \Delta} = \sigma\sqrt{f^2(t) - \Delta}\,. \end{aligned} \tag{19}$$

$\sigma = \pm 1$, its sign coincides with the sign of R-region, but could be arbitrary in T-region. On different sides of the brane $R_{,n}(+) = -R_{,n}(-) \ \ \rightarrow \sigma = \sigma_- = -\sigma_+ \ \rightarrow Z_2$-symmetry and $sign S^0_0 = \sigma$. Choice of time coordinate: on the brane $B(0,t) = 1, \ t = \tau$ -cosmological time, $R(0,t) = a(\tau)\, f(t) = a_\tau$. Vacuum shell (brane)

$$S^0_0 = S^2_2 \ \ \rightarrow S^0_0 = const \tag{20}$$

The constraint equation gives complete solution to the problem

$$\frac{(N-1)(N-2)}{2} \frac{a^2_\tau + k}{a^2} = \frac{N-2}{N} \left(\Lambda + \frac{N}{2(N-1)} (4\pi G)^2 \left(S^0_0\right)^2 \right)$$

"Cosmological radius" $R_0 = \sqrt{\frac{N(N-1)}{2\Lambda}}$

$$R = R_0 \sqrt{f^2(t) + k} \sin \left(\frac{\sigma n}{R_0} + \varphi(t) \right) \tag{22}$$

On the brane: $n = 0$

$$\frac{\sigma}{R_0} \cot \varphi = \frac{4\pi G}{N-1} S_0^0 , \rightarrow \varphi = \varphi_0 = const$$

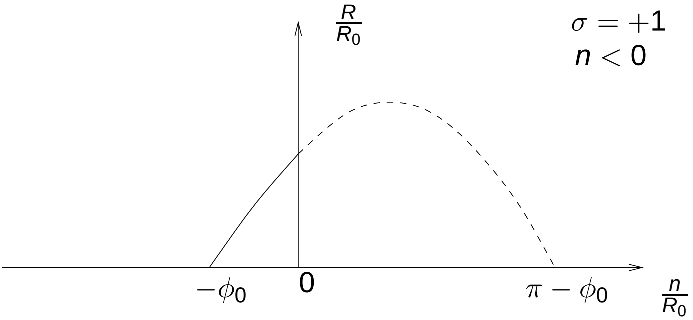
$$\frac{a_\tau^2 + k}{a^2} = \frac{2\Lambda}{N(N-1)} + \left(\frac{4\pi G}{N-1} \right)^2 \left(S_0^0 \right)^2 = \frac{1}{R_0^2 \sin^2 \varphi_0} \tag{23}$$

$$R = R_0 \sin \left(\frac{\sigma n}{R_0} + \varphi_0 \right) \begin{cases} \cosh \frac{t}{a_0} , & \text{for} \quad k = +1 \\ e^{\frac{t}{a_0}} , & \text{for} \quad k = 0 \\ \left| \sinh \frac{t}{a_0} \right| , & \text{for} \quad k = -1 \end{cases}$$

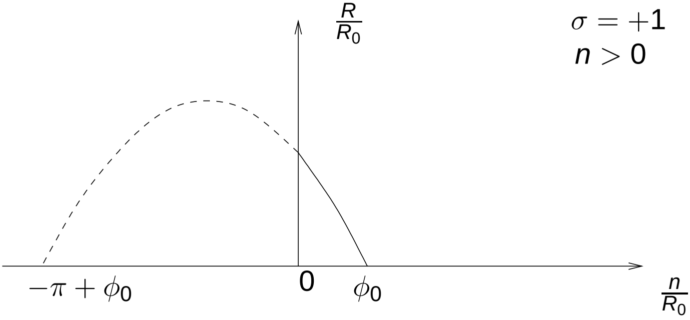
$$a_0 = R_0 \sin \varphi_0 \tag{24}$$

Different $\sigma = \pm 1 \rightarrow$ different matching.

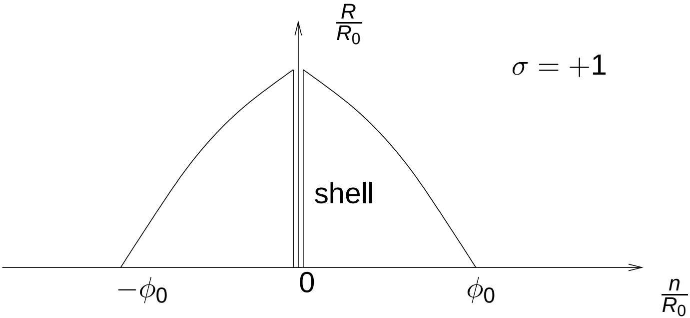
- $R(n)$: $t = const$
- $\Lambda > 0$ $S_0^0 > 0$ $(\sigma = \sigma_- = -\sigma_+ = +1)$



Dashed curve shows a continuation of the function $R(n)$ beyond the shell

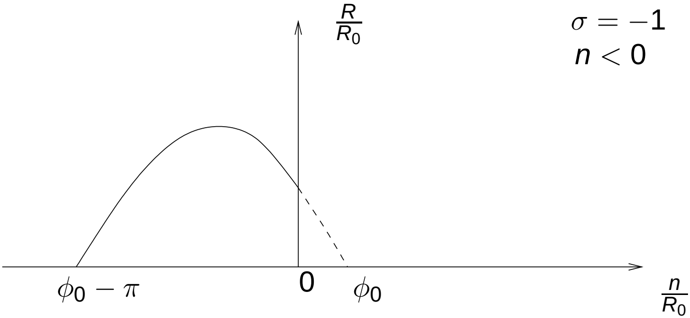


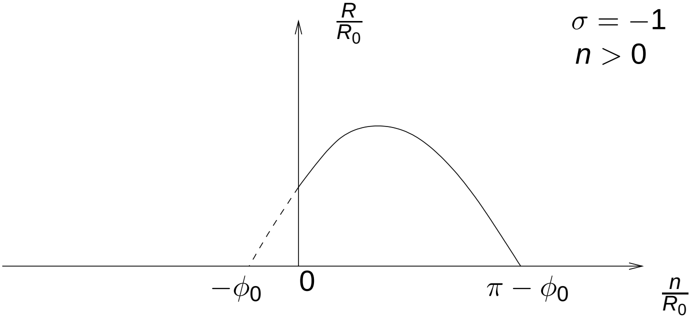
$\frac{R}{R_0}$
$\sigma = +1$
$n > 0$
$-\pi + \phi_0$
0
ϕ_0
$\frac{n}{R_0}$

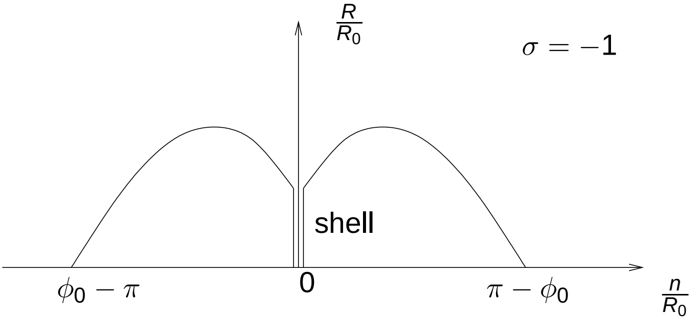


$\frac{R}{R_0}$
$\sigma = +1$
shell
$-\phi_0$
0
ϕ_0
$\frac{n}{R_0}$

- $R(n) \quad t = const$
- $\Lambda > 0 \; S_0^0 < 0 \; (\sigma = \sigma_- = -\sigma_+ = -1)$

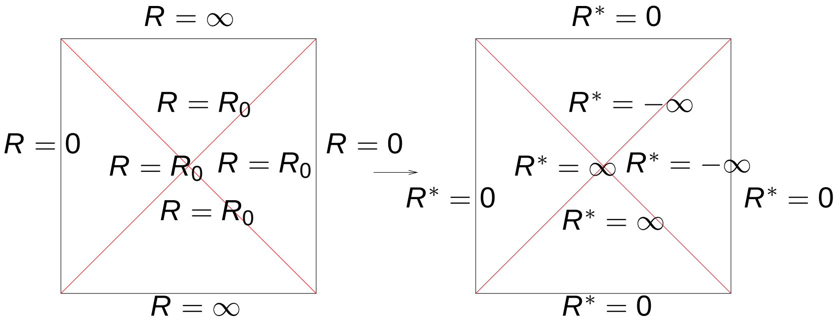



$\frac{R}{R_0}$
$\sigma = -1$
$n > 0$
$-\phi_0$
0
$\pi - \phi_0$
$\frac{n}{R_0}$

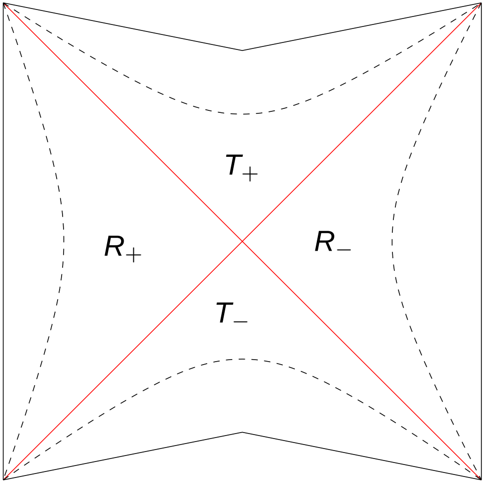

$\frac{R}{R_0}$
$\sigma = -1$
shell
$\phi_0 - \pi$
0
$\pi - \phi_0$
$\frac{n}{R_0}$

- Bulk: Carter-Penrose diagrams
- $\Lambda > 0 \quad k = +1$

$$dR^{\star} = \pm \frac{dR}{1 - \frac{R^2}{R_0^2}} \rightarrow R^{\star} = \pm \log \frac{1 + \frac{R}{R_0}}{1 - \frac{R}{R_0}} \tag{25}$$

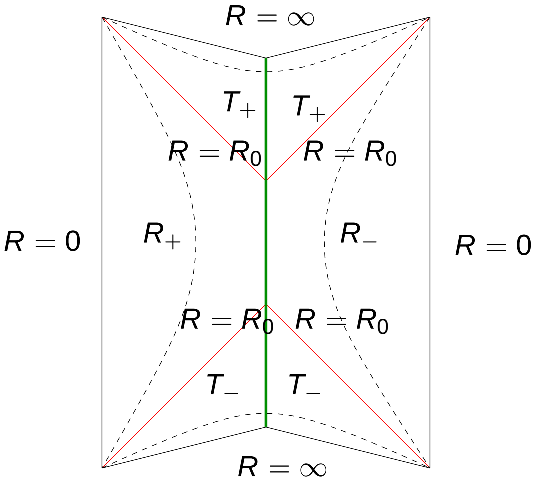


Time coordinate points up, radial coordinate goes from left to right, null curves are straight lines with slope $\pm 45°$

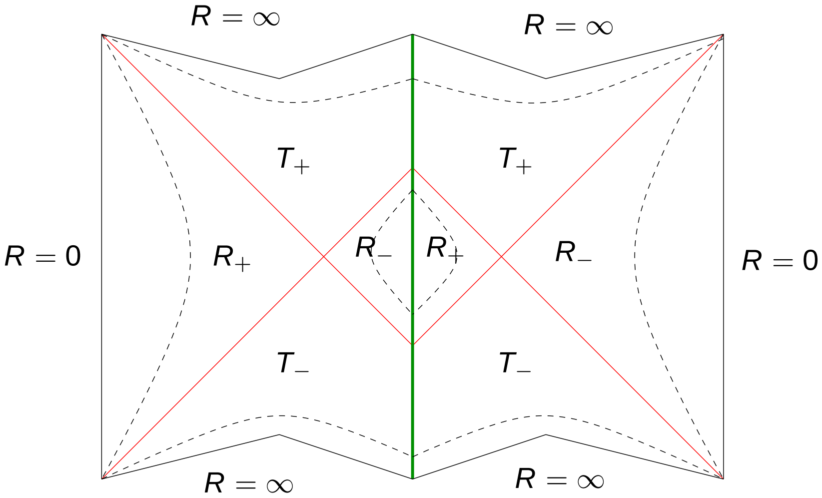


Dashed lines represent surfaces $R = const$, time-like in R-regions and space-like in T-regions

- Complete geometry: Carter-Penrose diagrams
- $\Lambda > 0 \quad k = +1$
- $S_0^0 > 0$



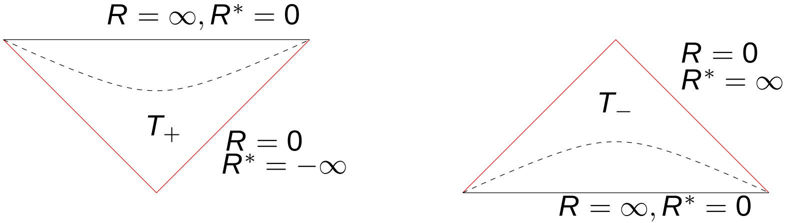
- $S_0^0 < 0$



Non-traversable wormholes on both sides.
Several branes $\to$ asymmetric energy level splitting

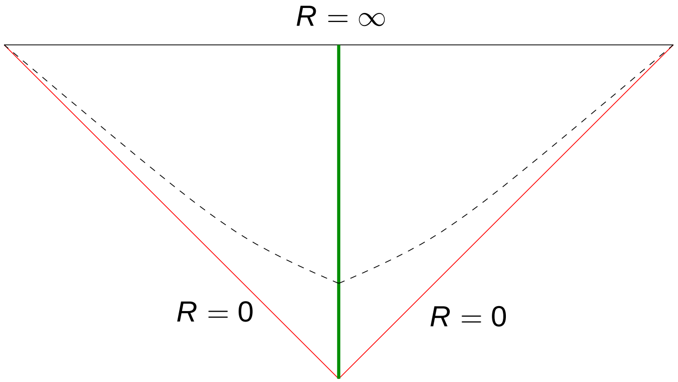
- $\Lambda > 0 \quad k = 0$
- Bulk

$R^{\star} = \mp \frac{1}{R}$

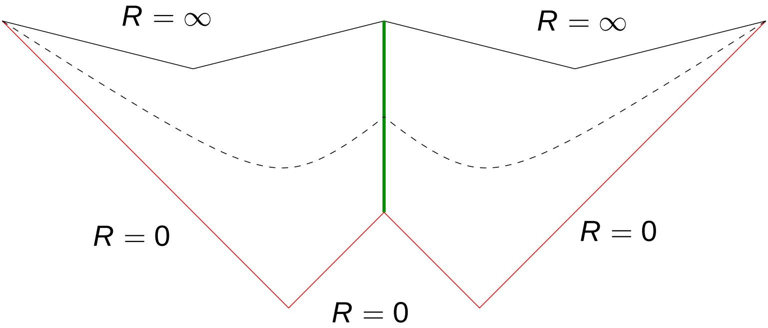


$R = 0$ - apparent horizons

- Complete geometry
- $S_0^0 > 0$



- $S_0^0 < 0$



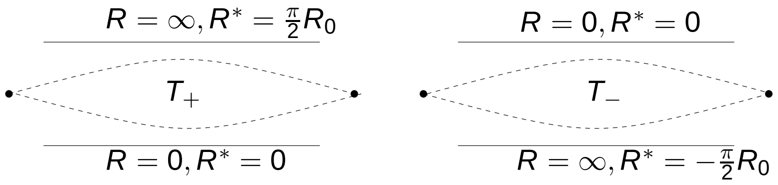
- $\Lambda > 0 \quad k = -1$
- Bulk

$T_{\pm}$-regions everywhere

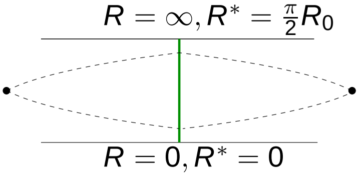
$$R^{\star} = \pm \arctan \frac{R}{R_0}\,, \implies R = \pm R_0 \tan \frac{R^{\star}}{R_0}\,, \tag{26}$$

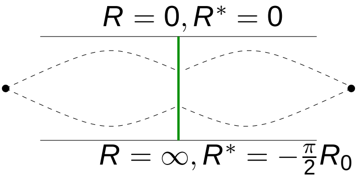
$\Lambda > 0,\, k = -1 \;\; (+-) \leftrightarrow (-+0 \;\; \Lambda < 0,\, k = +1$

$$R_{\pm} \;\leftrightarrow\; T_{\pm}$$



- $\Lambda > 0 \quad k = -1$
- Complete geometry





Left figure is for $S_0^0 > 0$. Right figure is for $S_0^0 < 0$.

$$R_{,n} = \sigma\sqrt{f^2(t) + k + \frac{R^2}{R_0^2}}\,, \quad R_0 = \sqrt{\frac{N(N-1)}{2|\Lambda|}} \qquad (27)$$

- (1): $f^2(t) + k > 0, \quad T_0^{0ind} > 0$
- (2): $f^2(t) + k < 0, \quad T_0^{0ind} < 0$

Case (1):

$$R = R_0\sqrt{f^2(t) + k}\,\sinh\left(\frac{\sigma n}{R_0} + \varphi(t)\right). \qquad (28)$$

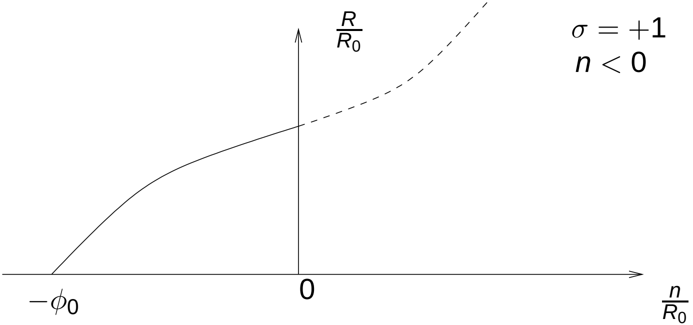
On the brane: $n = 0$

$$\frac{\sigma n}{R_0}\coth\varphi(t) = \frac{4\pi G}{N-1}S_0^0\,, \rightarrow \phi(t) = \phi_0 = const$$

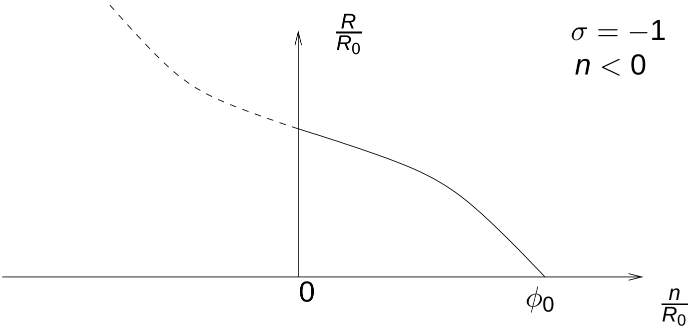
$$\frac{a_\tau^2 + k}{a^2} = \frac{2\Lambda}{N(N-1)} + \left(\frac{4\pi G}{N-1}\right)^2 S_0^{0\,2} = \frac{1}{R_0^2\sinh^2\varphi_0} \qquad (29)$$

Since $|S_0^0| > \sqrt{\frac{(N-1)\,|\Lambda|}{2\pi G N}} \longrightarrow$ "Heavy Shells"

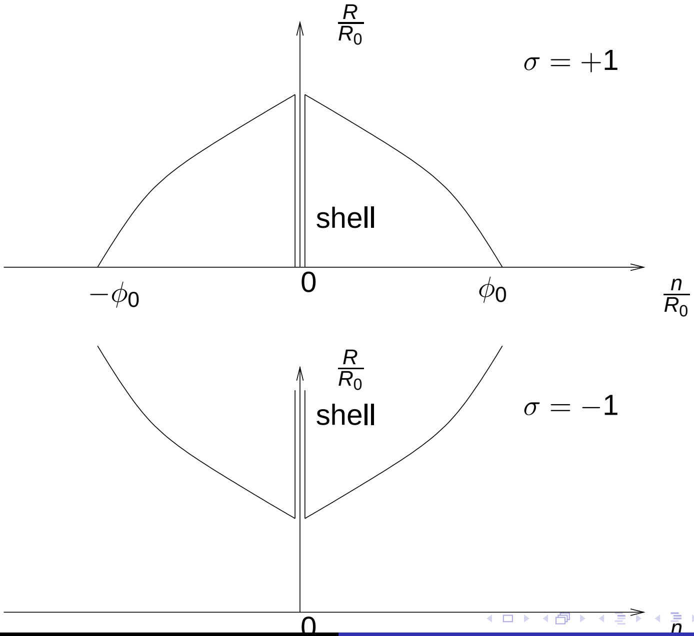
- $\Lambda < 0$ "Heavy Shells"
- $R(n)$ $t = const$



Dashed curve shows $R(n)$ beyond the shell

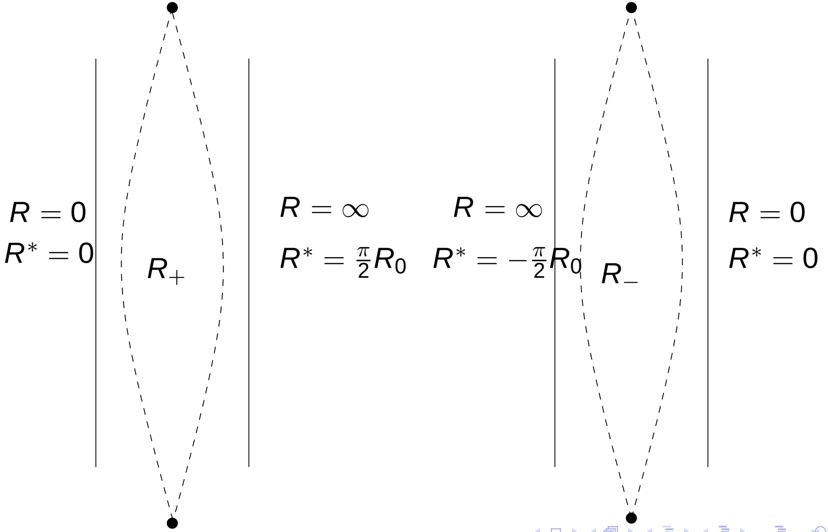

$\frac{R}{R_0}$
$\sigma = -1$
$n < 0$
0
ϕ_0
$\frac{n}{R_0}$

- Complete geometry

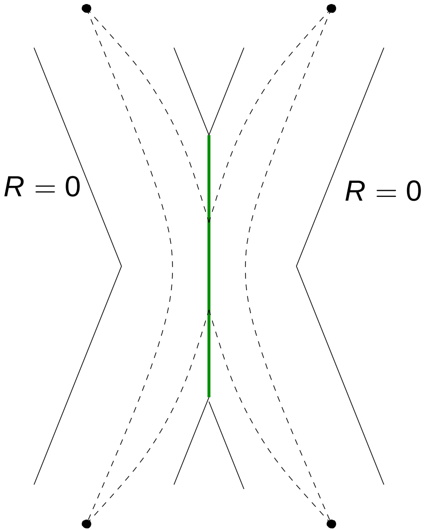


- $\Lambda < 0 \quad k = +1 \quad$ (AdS)
- Bulk: Carter-Penrose diagrams

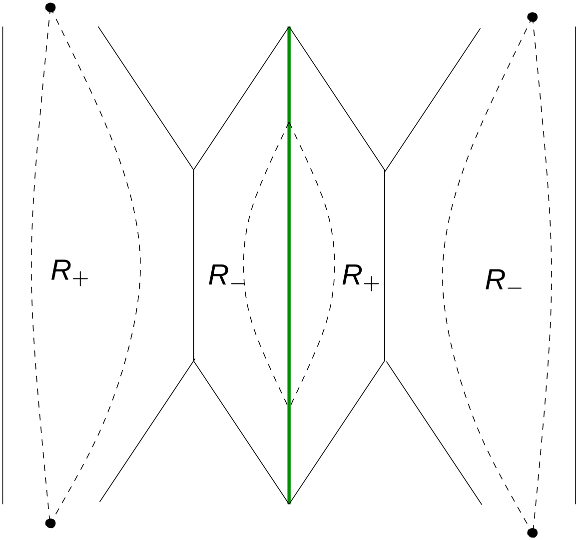
R-regions everywhere



- Complete geometry: Carter-Penrose diagrams

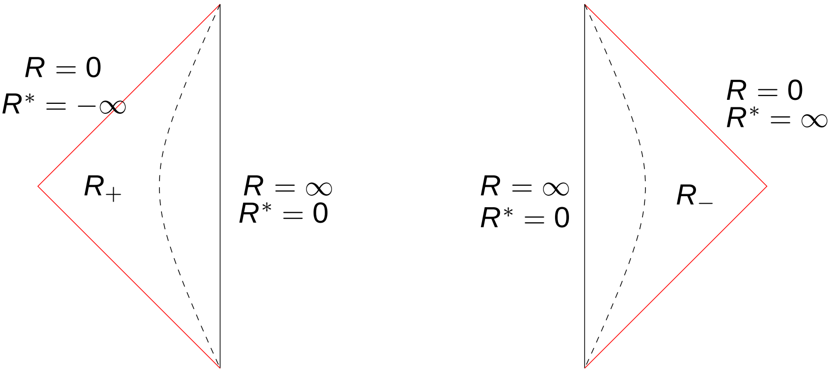


$S^0_0 > 0$

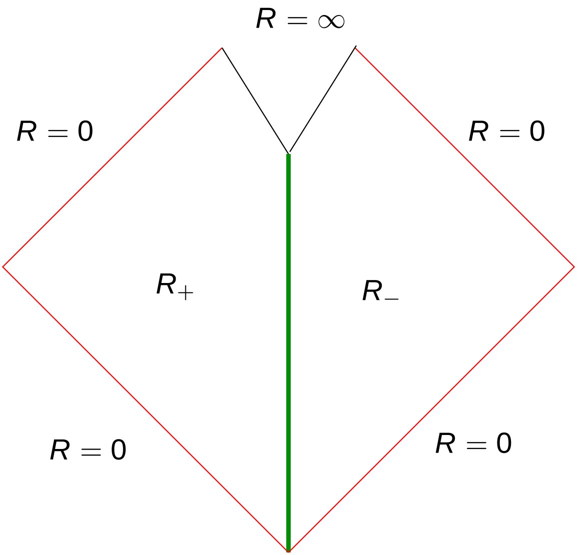


$$S_0^0 < 0$$

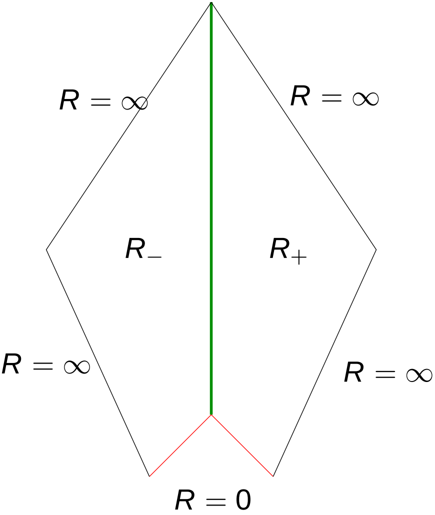
- $\Lambda < 0 \quad k = 0$
- Bulk - only R-regions



- Complete geometry



$S_0^0 > 0$



$$S_0^0 < 0$$

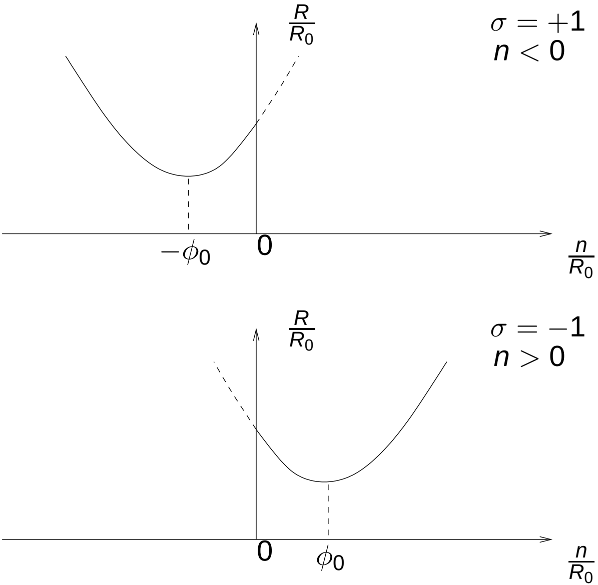
- $\Lambda < 0 \quad k = -1$
  Most interesting case. Admits both "heavy" and "light" shells
- "Heavy": $\left|S_0^0\right| > \sqrt{\frac{(N-1)|\Lambda|}{2\pi GN}}$

$$R = R_0 \sinh\left(\frac{t}{R_0 \sinh\varphi_0}\right) \sinh\left(\frac{\sigma n}{R_0} + \varphi_0\right),$$

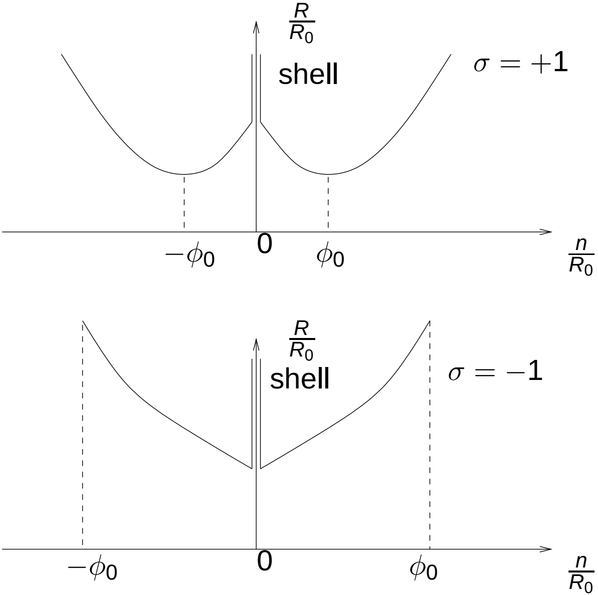
- "Light": $\left|S_0^0\right| < \sqrt{\frac{(N-1)\,|\Lambda|}{2\pi GN}}$

$$\begin{aligned}
R &= R_0 \sin\left(\frac{t}{R_0 \cosh\varphi_0}\right) \cosh\left(\frac{\sigma n}{R_0} + \varphi_0\right), \\
R &= \pm R_0 \tanh\frac{R^\star}{R_0}, \qquad , 0 \le R \le R_0 \\
R &= R_0 \coth\frac{R^\star}{R_0}, \qquad R_0 \le R < \infty
\end{aligned}$$

- "Light shells": $R(n) \quad t = const$

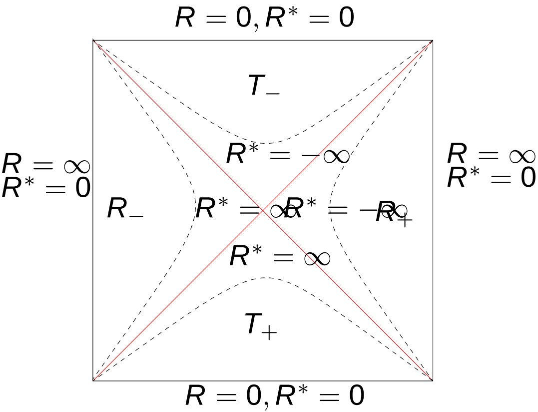


- Complete geometry for $R(n)$



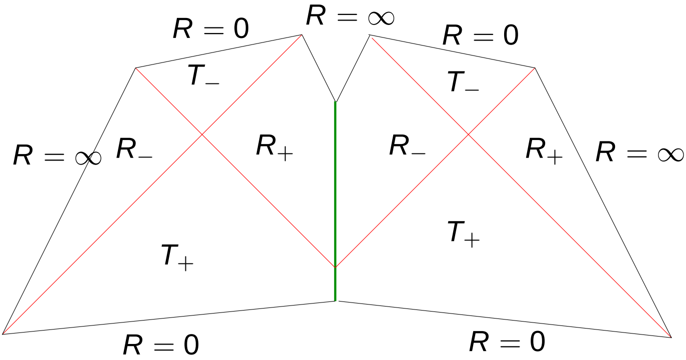
- Bulk: Carter-Penrose diagram

$$\Lambda > 0,\ k = +1 \quad (+-) \leftrightarrow (-+) \quad \Lambda < 0,\ k = -1$$

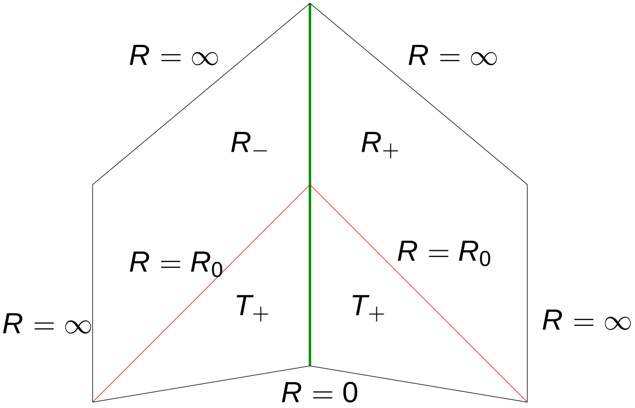


Unusual: Einstein-Rosen bridge $\rightarrow$ wormhole geometry.
"Black hole with zero mass". No singularities at $R = 0$

- Complete geometry: Carter-Penrose diagrams
- "Heavy Shells"

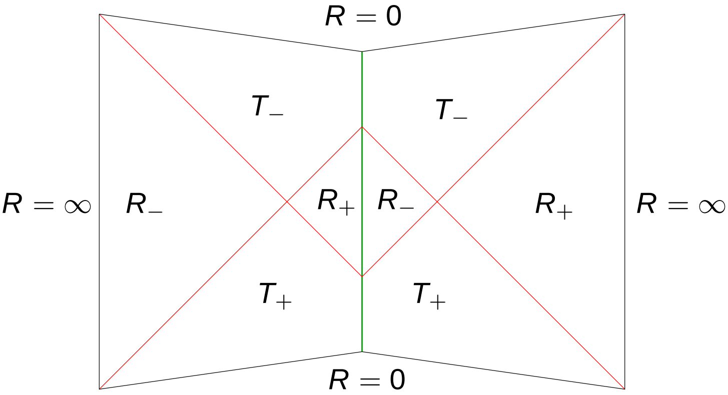


$$S_0^0 > 0$$

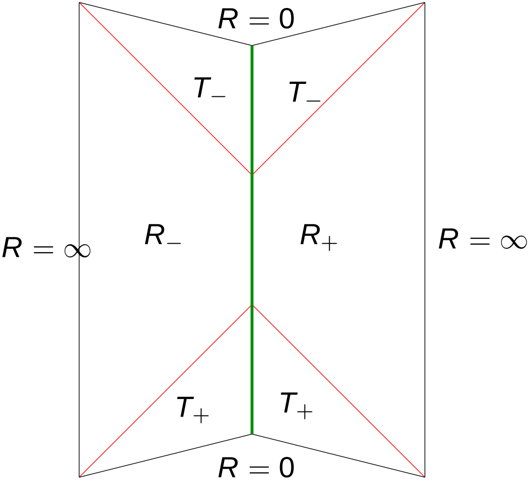


$$S_0^0 < 0$$

- "Light Shells"



$$S^0_0 > 0$$



$$S^0_0 < 0$$

Bound motion. Quantization $\rightarrow$ bound states.
Unfolded extension both to the past and to the future.

Thank You All

THE END