Two dimensional gravity in genus-one in Matrix Models, Topological and Liouville approaches.

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Several approaches to 2D quantum geometry

• There exist several different approaches to the 2-D Quantum Gravity. One of them is the continuous approach. In this approach the theory is determined by the functional integral over all metrics. Calculation of this integral in conformal gauge leads to the Liouville field theory. Therefore this approach is called the Liouville Gravity.

• The other way to describe sum over 2d surfaces is the discrete approach. It is based on the idea of approximation of two-dimensional geometry by the ensemble of planar graphs of big size. Technically the ensemble of graphs is usually defined by expansion into a series of perturbation theory of integral over matrixes of size $N \times N$. That is why this approach is called the Matrix Models (further MM).

• There exist the third approach —2d Topological gravity. Witten built axiomatics of this theory by studying intersection theory . It was conjectured and checked (for genus-zero) that correlation numbers in Topological gravity and in Matrix models coincide. It should be mentioned that the coincidence takes place if correlation numbers in OMM are calculated in KdV frame.

The method of orthogonal polynomials

The partition function in One-Matrix Model

$$Z(v_k, N) = \log \int dM e^{-\mathrm{tr} V(v_k, M)}$$

where M is hermitian matrix $N \times N$ and potential

$$V(v_k, M) = N \sum_{k=1}^{p+1} v_k M^{2k}$$

Expansion to Feynman diagrams in respect to the coupling constants v_k can be interpreted as genus expansion

$$Z = \sum_{h=0}^{\infty} N^{2-2h} Z_h,$$

$h -$ genus of surfaces

Now we want to compute the integral over M. The first step is dioganlization the matrix M in the integral giving

$$
\begin{aligned}
&Z(v_k, N) = \log \int \prod_{i=1}^{N} d\lambda_i \Delta^2(\lambda) e^{-\sum_i V(v_k,\lambda_i)} \\
&\{\lambda_i\} - \text{eigenvalues of } M \\
&\Delta(\lambda) = \prod_{i<j} (\lambda_i - \lambda_j) - \text{Vandermonde determinant}
\end{aligned}
\tag{1}
$$

Introducing the set of orthogonal polynomials $P_n(\lambda) = \lambda^n + ...$,

$$\int_{-\infty}^{\infty} d\lambda e^{-V(\lambda)} P_n(\lambda) P_m(\lambda) = s_n \delta_{nm}.$$

one obtains for the partition function

$$Z = N \sum_{k=1}^{N-1} (1 - k/N) \log(s_k/s_{k-1}).$$

Using the relation

$$\lambda P_k(\lambda) = P_{k+1}(\lambda) + R_k P_{k-1}(\lambda)$$

One gets

$$\int e^{-V} P_k \lambda P_{k-1} d\lambda = R_k s_{k-1} = s_k \tag{2}$$

$$R_k = s_k/s_{k-1}$$

Therefore

$$Z = N \sum_{k=1}^{N-1} (1 - k/N) \log R_k$$

We obtain the relation for R_k using

$$ks_{k-1} = \int e^{-V} P_k' P_{k-1} = \int e^{-V} V' P_k P_{k-1}.$$

$$V'(\lambda) = \sum_{k=1}^{p+1} 2kv_k \lambda^{2k-1}$$

and applying $2n-1$ times the previous relation for $\lambda P_k(\lambda)$

$$\lambda^{2n-1} P_k = \lambda^{2n-2}(P_{k+1} + R_k P_{k-1}) =$$
$$= \lambda^{2n-3}(P_{k+2} + (R_{k+1})P_k + (R_k)P_k + (R_k R_{k-1})P_{k-2}) = ...$$

Thus we arrive to the following formula for R_k

$$\boxed{\frac{k}{N} = \tilde{W}(R_k, R_{k\pm1}, ..., R_{k\pm p})}$$

where

$$\tilde{W}(R_k, R_{k\pm1}, ..., R_{k\pm p}) =$$

$$= \sum_{n=1}^{p+1} 2nv_n \sum_{\{\sigma_{2n-1}\}} R_{k+m_1} \cdot ... \cdot R_{k+m_n}$$

$\{\sigma_{2n-1}\}$ denotes all "walks" which consist of $2n-1$ steps, starting in k and finishing in $k-1$.

Evaluation of Z_0 and Z_1

Propose existence of smooth function $R(\xi, N)$ of variable $\xi \in [0, 1]$, and $R(\frac{k}{N}, N) = R_k$, and Taylor expansion for $R(\xi + m/N, N)$

$$R(\xi + m/N, N) = R(\xi, N) + \frac{m}{N} R_\xi(\xi, N) + \frac{m^2}{2N^2} R_{\xi\xi}(\xi, N) + O\Big(\frac{1}{N^3}\Big),$$

Thus

$$\tilde{W}(R(\xi, N)) = W(R(\xi, N)) + \frac{1}{N} W_1(R(\xi, N)) + \frac{1}{N^2} W_2(R(\xi, N)) + O\Big(\frac{1}{N^3}\Big),$$

After calculation

$$W(R(\xi, N)) = \sum_{n=1}^{p+1} \frac{(2n)!}{n!(n-1)!} v_n R^n(\xi, N),$$

$$W_1(R(\xi, N)) = 0,$$

$$W_2(R(\xi, N)) = \frac{R R_{\xi\xi}}{6} W''(R(\xi, N)) + \frac{R R_\xi^2}{12} W'''(R(\xi, N))$$

As a result we have

$$Z = N \sum_{k=1}^{N-1} (1 - k/N) \log R(\xi, N)$$

where $R(\xi, N)$ is solution of equation

$$\xi = W(R(\xi, N)) + \frac{RR_{\xi\xi}}{6N^2} W''(R(\xi, N)) + \frac{RR_\xi^2}{12N^2} W'''(R(\xi, N)) + O\Big(\frac{1}{N^4}\Big),$$

Assuming also the expansion

$$R(\xi, N) = R(\xi) + \frac{1}{N} R_1(\xi) + \frac{1}{N^2} R_2(\xi) + ...,$$

thus

$$\xi = W(R(\xi)),$$
$$R_1(\xi) = 0,$$
$$R_2(\xi) = -\frac{R(\xi)}{12W'(R)} \Big(2R_{\xi\xi} W''(R(\xi)) + R_\xi^2 W'''(R(\xi))\Big).$$

Passing from sum to integral in partition function, we use Euler-Maclorein formula up to N^0 terms

$$Z = N^2 \int_0^1 d\xi (1-\xi) \log R(\xi, N) - \frac{N}{2}(F(1) - F(1/N)) + \\ + \frac{1}{12}(F'(1) - F'(1/N)) + O(1/N),$$

where $F(\xi) = (1-\xi)\log R(\xi, N)$.

Then for partition function in genus-zero and genus-one we obtain

$$Z_0 = \int_0^1 d\xi (1-\xi) \log R,$$

$$Z_1 = -\frac{1}{12}\int_0^1 d\xi (1-\xi) \frac{2R_{\xi\xi}W''(R) + R_\xi^2 W'''(R)}{W'(R)},$$

where $R = R(\xi)$.

The vicinity of p-critical point

The p-critical point are defined by the system of equations

$$W(R_c) = 1, \quad W'(R_c) = 0, \quad \ldots \quad W^{(p)}(R_c) = 0.$$

This system of equations, which determine coefficients v_k^c, $k = 1, \ldots, p$, and define the R_c.

Consider small deviations $\delta v_k = v_k - v_k^c$, and new coordinates t_k in vicinity of the critical point

$$W(R_c) = 1 + t_{p+1}, \quad W'(R_c) = t_p, \quad \ldots \\ W^{(p-1)}(R_c) = t_0, \quad W^{(p)}(R_c) = 0.$$

Denoting $u = R - R_c$ one can obtain

$$\xi = W(u) = u^{p+1} + t_0 u^{p-1} + \sum_{k=1}^{p-1} t_k u^{p-k-1} + 1.$$

Making a substitution $\xi = 1 - y$, one can get

$$\mathcal{P}(u) + y = 0,$$

and the string polynomial $\mathcal{P}(u)$ defined as

$$\mathcal{P}(u) = u^{p+1} + t_0 u^{p-1} + \sum_{k=1}^{p-1} t_k u^{p-k-1}$$

and $u(y)$ is its solution.
Therefore for the partiton functions we obtain

$$Z_0 = \frac{1}{R_c} \int_0^1 dy\, y\, u(y),$$

$$Z_1 = -\frac{1}{12} \int_0^1 dy\, y \left(\frac{2\mathcal{P}''(u) u_{yy} + \mathcal{P}'''(u) u_y^2}{\mathcal{P}'(u)} \right)$$

These expressions can be efficiently simplified and we arrive to the final answer

$$Z_0 = \frac{1}{2}\int_0^{u^*} \mathcal{P}^2(u)du$$

$$Z_1 = -\frac{\log \mathcal{P}'(u^*)}{12}$$

where

$$u^* = u^*(t_0, t_1, ..., t_{p-1})$$

is the "'maximal" root of the polynomial $\mathcal{P}(u)$.

These formalae are indeed the explict expressions of the generating finctions for the correlation numbers in genus zero and genus one.

The same expressions can be obtained from **the double scaling limit** and **Douglas string equation**. The double scaling limit arises when N goes to ∞, while μ and t_k lead to 0 proportionally $(N^{-2}\varepsilon^2)^{\frac{2}{2p+3}}$ and $(N^{-2}\varepsilon^2)^{\frac{k+2}{2p+3}}$ correspondingly,
and ε is some finite parameter. Making suitable replacement of variables, using the rescaling and performing the substitution $Z/N^2 \to Z$ for simplicity, we arrive to the expression for the partition function in the double scaling limit $Z[\mu, t_k, \varepsilon]$

$$Z[\mu, t_k, \varepsilon] = \sum_{h=0}^{\infty} \varepsilon^{2h} Z_h[\mu, t_k],$$

where ε is the parameter, which is responsible for genus expansion.

String equation

We can compute the partition functions Z_h using the String equation which is the equation for function $u(x, \varepsilon, \mu, t_k)$, connected with the partition function $Z[\mu, t_k, \varepsilon]$ as

$$u(x, \varepsilon) = \frac{d^2 Z}{dx^2}$$

It looks as

$$[\hat{P}, \hat{Q}] = 1$$

where

$$\hat{Q} = \varepsilon^2 d^2 + u(x), \quad d \equiv \frac{d}{dx}$$

$$\hat{P} = -\sum_{k=1}^{p+1} t_{p-1-k} \hat{Q}_+^{k-1/2}$$

are two differential operators

and $\hat{Q}_+^{k-1/2}$ stands for the non-negative part of the pseudo-differential operator $\hat{Q}^{k-1/2}$

We look for $u(x)$ in the form

$$u(x,\varepsilon) = \sum_{h=0}^{\infty} \varepsilon^{2h} u_h(x)$$

where, obviously, u_h

$$u_h(x) = \frac{d^2 Z_h}{dx^2}.$$

It is known, that

$$[\hat{Q}_+^{k-1/2}, \hat{Q}] = \frac{dS_k}{dx},$$

where the coefficients $S_k(u)$ obey the recursion relation

$$\frac{dS_{k+1}}{dx} = u\frac{dS_k}{dx} + \frac{1}{2}u_x S_k + \frac{\varepsilon^2}{4}\frac{d^3 S_k}{dx^3},$$

with the boundary conditions $S_0 = \frac{1}{2}$ and $S_k(k \neq 0)$ vanish at $u = 0$.

Use the equations above one can obtain

$$[\hat{P}, \hat{Q}] = 1 \quad \Longrightarrow \quad \sum_{k=1}^{p+1} t_{p-1-k} S_k(u) = -x$$

The solution of the recursion relations, including the first three terms is

$$S_k(u) = \frac{C_{2k}^k}{2^{2k+1}} \left(u^k + \frac{\varepsilon^2 k(k-1)}{6} u^{k-2} u_{xx} + \frac{\varepsilon^2 k(k-1)(k-2)}{12} u^{k-3} u_x^2 \right).$$

where $\quad C_{2k}^k = \dfrac{(2k)!}{k!k!}$

Thus after rescaling the parameter $t_k \to \frac{2^{2k+1}}{C_{2k}^k} t_k$, we can obtain that

$$\mathcal{P}(u) + \varepsilon^2 \left(\frac{1}{6} \mathcal{P}''(u) u_{xx} + \frac{1}{12} \mathcal{P}'''(u) u_x^2 \right) = O(\varepsilon^4),$$

where $\mathcal{P}(u)$ is the string polynomial and $x = t_{p-1}$, $t_{-2} = 1$, $t_{-1} = 0$.

Using the expansion for $u(x, \varepsilon)$, we get to the zeroth order in the ε, that $u_0(x)$ obeys

$$\mathcal{P}(u_0) = 0,$$

therefore

$$u_0 = u^*(t_1, ..., t_{p-2}, x),$$

where u^* is the real maximal root of polynomial $\mathcal{P}(u)$. To the second order in the ε gives for the $u_1(t_1, ..., t_{p-2}, x)$ the following expression

$$u_1 = -\frac{\mathcal{P}'''(u^*)(u_x^*)^2 + 2\mathcal{P}''(u^*)u_{xx}^*}{12\mathcal{P}'(u^*)}.$$

Knowing u_0 and u_1 we can find corresponding the partition functions Z_0 and Z_1, using the fact that if Z and u^* are connected by relation

$$\frac{\partial^2 Z}{\partial x^2} = f(u^*),$$

then

$$Z = -\int_0^{u^*} \mathcal{P}(u)\mathcal{P}'(u)f(u)du.$$

This formula can be checked by straightforward calculation.

Integrating by parts and omitting the regular terms, we arrived to the expressions obtained above

$$Z_0 = \frac{1}{2}\int_0^{u^*} \mathcal{P}^2(u)du$$

and

$$Z_1 = -\frac{\log \mathcal{P}'(u^*)}{12}.$$

Evaluation of correlation numbers in genus-one in KdV frame

The singular part of the partition function on torus $Z_1(t_0, t_1, ...t_{p-1})$ is

$$Z_1 = -\frac{\log \mathcal{P}'(u^*)}{12},$$

where $\mathcal{P}(u)$ is the polynomial of degree $p+1$ (p is natural number)

$$\mathcal{P}(u) = u^{p+1} + t_0 u^{p-1} + \sum_{k=1}^{p-1} t_k u^{p-k-1},$$

Formula for correlation numbers is

$$\langle O_{k_1} ... O_{k_n} \rangle_1 = \left. \frac{\partial^n Z_1}{\partial t_{k_1} ... \partial t_{k_n}} \right|_{t_1 = ... = t_{p-1} = 0}$$

$$\langle O_k \rangle_1 = \frac{p+k}{24} u_c^{-k-2},$$

$$\langle O_{k_1} O_{k_2} \rangle_1 = \frac{(p+2+k_1+k_2)(k_1+k_2) + 2p - k_1 k_2}{48} u_c^{-k_1-k_2-4},$$

Comparison with Topological Gravity

E.Witten recursion relation

$$\langle\sigma_{k_1}\sigma_{k_2}...\sigma_{k_s}\rangle_0 = k_1 \sum_{S=X\cup Y}\langle\sigma_{k_1-1}\prod_{i\in X}\sigma_{k_i}\sigma_0\rangle_0\langle\sigma_0\prod_{j\in Y}\sigma_{k_j}\sigma_{k_{s-1}}\sigma_{k_s}\rangle_0,$$

$$\langle\sigma_{k_1}\sigma_{k_2}...\sigma_{k_s}\rangle_1 = \frac{1}{12}k_1\langle\sigma_{k_1-1}\sigma_{k_2}...\sigma_{k_s}\sigma_0\sigma_0\rangle_0 + \\ + k_1 \sum_{S=X\cup Y}\langle\sigma_{k_1-1}\prod_{i\in X}\sigma_{k_i}\sigma_0\rangle_0\langle\sigma_0\prod_{j\in Y}\sigma_{k_j}\rangle_1,$$

It follows from basis recursion relation

$$\langle\sigma_{k_1}\sigma_{k_2}\sigma_{k_3}\rangle_0 = k_1\langle\sigma_{k_1-1}\sigma_0\rangle_0\langle\sigma_0\sigma_{k_2}\sigma_{k_3}\rangle_0,$$
$$\langle\sigma_k\rangle_1 = \frac{1}{12}k\langle\sigma_{k-1}\sigma_0\sigma_0\rangle_0 + k\langle\sigma_{k-1}\sigma_0\rangle_0\langle\sigma_0\rangle_1$$

and

$$\frac{\partial}{\partial a_k}\langle N\rangle = \langle\sigma_k N\rangle,$$

In One-Matrix Model

$$\sigma_k \leftrightarrow O_{p-k-1}, \qquad a_k \leftrightarrow t_{p-k-1}$$

We need to check

$$\langle O_{p-k_1-1} O_{p-k_2-1} O_{p-k_3-1}\rangle_0 = k_1 \langle O_{p-k_1} O_{p-1}\rangle_0 \langle O_{p-1} O_{p-k_2-1} O_{p-k_3-1}\rangle_0,$$

$$\langle O_{p-k-1}\rangle_1 = \frac{1}{12} k \langle O_{p-k} O_{p-1} O_{p-1}\rangle_0 + k \langle O_{p-k} O_{p-1}\rangle_0 \langle O_{p-1}\rangle_1$$

At arbitrary $\{t_k\}$ one can get

$$\langle O_{k_1} O_{k_2}\rangle_0 = \frac{\partial^2 Z_0}{\partial t_{k_1} \partial t_{k_2}} = \frac{(u^*)^{2p-k_1-k_2-1}}{2p-k_1-k_2-1},$$

$$\langle O_{k_1} O_{k_2} O_{k_3}\rangle_0 = \frac{\partial^3 Z_0}{\partial t_{k_1} \partial t_{k_2} \partial t_{k_3}} = -\frac{(u^*)^{3p-k_1-k_2-k_3-3}}{\mathcal{P}'(u^*)}.$$

$$\langle O_k\rangle_1 = \frac{\partial Z_1}{\partial t_k} = -\frac{p-k-1}{12\mathcal{P}'(u^*)}(u^*)^{p-k-2} + \frac{\mathcal{P}''(u^*)}{12(\mathcal{P}'(u^*))^2}(u^*)^{p-k-1}$$

Use this expressions, we see that recursion relation are fulfilled.

Evaluation of correlation numbers in CFT frame

KdV frame $\longrightarrow$ **CFT frame**

$\updownarrow$ $\updownarrow$

$\{t_k\} \longrightarrow$ "resonanse" transformation $t_k = t_k(\{\lambda_k\}) \longrightarrow \{\lambda_k\}$

As a result

$$\mathcal{P}(u, \{t_k\}) = u^{p+1} + t_0 u^{p-1} + \sum_{k=1}^{p-1} t_k u^{p-k-1}$$

$$\downarrow$$

$$Q(x, \{\lambda_k\}) = \sum_{n=0}^{\infty} \sum_{k_1 \ldots k_n = 1}^{p-1} \frac{\lambda_{k_1} \ldots \lambda_{k_n}}{n!} \frac{d^{n-1}}{dx^{n-1}} L_{p-\sum k_i - n}(x),$$

where

$$x = u/u_c, \quad u_c = u^*(\{\lambda_k\} = 0)$$

$L_n(x)$ – Legendre polynomials.

Formula for correlation numbers is

$$\langle O_{k_1}...O_{k_n}\rangle_1 = \left.\frac{\partial^n Z_1}{\partial\lambda_{k_1}...\partial\lambda_{k_n}}\right|_{\lambda_1=...=\lambda_{p-1}=0}$$

First two correlation numbers in CFT frame

$$\langle O_k\rangle_1 = \frac{(2p-k)(k+1)}{24},$$

$$\langle O_{k_1}O_{k_2}\rangle_1 = -\frac{(1+k_1)(1+k_2)\Big((k_1+k_2-2p+2)(k_1+k_2)-k_1k_2-4p\Big)}{24}.$$

Conclusion

- We have derived the torus partition function Z_1 in p-critical One-Matrix Model. Using the explicit expression for the partition function in genus-one we compute the correlation numbers in KdV, as well as in CFT frames.

- The results in CFT frame should be compared against the correlation numbers in the Minimal Liouville gravity, which have not been computed yet. We expect the coincidence in genus-one similarly one observed in genus-zero.

- The results in KdV frame have been compared with Witten's results for the correlation numbers of the 2d topological gravity and found to coincide.