Two-loop resummation in (F)APT

A. P. Bakulev

Bogoliubov Lab. Theor. Phys., JINR (Dubna, Russia)

OUTLINE

- **Intro**: Analytic Perturbation Theory (**APT**) in QCD
- **Problems of APT** and their resolution in **FAPT**:
- **Technical development of FAPT**: thresholds
- **Resummation** in **APT** and **FAPT**
- **Applications**: Higgs decay $H^0 \to b\bar{b}$
- **Conclusions**

Collaborators & Publications

Collaborators:

S. Mikhailov (Dubna) and **N. Stefanis (Bochum)**

Publications:

- A. B., Mikhailov, Stefanis — **PRD 72 (2005) 074014**
- A. B., Mikhailov, Stefanis — **PRD 75 (2007) 056005**
- A. B.&Mikhailov — **arXiv:0803.3013 [hep-ph]**
- A. B. — **Phys. Part. Nucl. 40 (2009) 715**
- A. B., Mikhailov, Stefanis — **arXiv:1004.4125 [hep-ph]**

Analytic Perturbation Theory

in

QCD

History of APT

Euclidean	Minkowskian
$Q^2 = \vec{q}^2 - q_0^2 \geq 0$	$s = q_0^2 - \vec{q}^2 \geq 0$

RG+Analyticity
ghost-free $\overline{\alpha}_{\text{QED}}(Q^2)$
Bogoliubov et al. 1959

pQCD+RG: resum π^2-terms
Arctg(s), UV Non-Power Series
Radyush., Krasn. & Pivov. 1982

DispRel+renormalons
IR finite $\alpha_s^{\text{eff}}(Q^2)$
Dokshitzer et al. 1995

pQCD+renormalons
Arctg(s) at **LE** region
Ball, Beneke & Braun 1994-95

RG+Analyticity
ghost-free $\alpha_{\text{E}}(Q^2)$
Shirkov & Solovtsov 1996

Integral Transformation:
$\mathcal{R}\left[\overline{\alpha}_s\right] \to$ **Arctg**(s)
Jones & Solovtsov 1995

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Jones & Solovtsov 1995

pQCD+RG+Analyticity

Transforms: $\hat{\mathcal{D}} = \hat{\mathcal{R}}^{-1}$

Couplings: $\alpha_{\mathrm{E}}(Q^2) \Leftrightarrow \alpha_{\mathrm{M}}(s)$

Milton & Solovtsov 1996–97

Analytic (global) pQCD+Analyticity

Global couplings: $\mathcal{A}_n(Q^2) \Leftrightarrow \mathfrak{A}_n(s)$

Non-Power perturbative expansions

Shirkov 1999–2001

History of F(ractional)APT

Euclidean

$Q^2 = \vec{q}^2 - q_0^2 \geq 0$

Minkowskian

$s = q_0^2 - \vec{q}^2 \geq 0$

Global Fractional APT (FAPT)

Analytization of α_s^ν: $\mathcal{A}_\nu(Q^2) \Leftrightarrow \mathfrak{A}_\nu(s)$

A. B. & Mikhailov & Stefanis 2005–2006

History of F(ractional)APT

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Analytization of α_s^ν: $\mathcal{A}_\nu(Q^2) \Leftrightarrow \mathfrak{A}_\nu(s)$

Analytization of $\alpha_s^\nu \times \mathbf{Log}^m$: $\mathcal{L}_{\nu,m}(Q^2) \Leftrightarrow \mathfrak{L}_{\nu,m}(s)$

A. B. & Mikhailov & Stefanis 2005–2006

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A. B. & Mikhailov & Stefanis 2005–2006

Resummation in 1-loop APT

S. Mikhailov 2004

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A. B. & Mikhailov & Stefanis 2005–2006

Resummation in 1-loop global FAPT

A. B. & Mikhailov 2008

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A. B. & Mikhailov & Stefanis 2005–2006

Resummation in 1-loop global FAPT

A. B. & Mikhailov 2008

Analytization of $\alpha_s\nu(1 + c_1\alpha_s)^{\nu'}$: $\mathcal{B}_{\nu,\nu'}(Q^2) \Leftrightarrow \mathfrak{B}_{\nu,\nu'}(s)$

A. B. 2008–2009

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A. B. & Mikhailov & Stefanis 2005–2006

Resummation in 1-loop global FAPT

A. B. & Mikhailov 2008

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A. B. 2008–2009

Resummation in 2-loop global FAPT

with 2-loop evolution factors $\mathcal{B}_{\nu,\nu'}(Q^2) \Leftrightarrow \mathfrak{B}_{\nu,\nu'}(s)$

A. B. & Mikhailov & Stefanis 2010

Intro: PT in QCD

- coupling $\alpha_s(\mu^2) = (4\pi/b_0)\, a_s[L]$ with $L = \ln(\mu^2/\Lambda^2)$
- RG equation $\frac{d\, a_s[L]}{d\, L} = -a_s^2 - c_1\, a_s^3 - \ldots$
- 1-loop solution generates Landau pole singularity: $a_s[L] = 1/L$

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- PT series: $D[L] = 1 + d_1 a_s[L] + d_2 a_s^2[L] + \ldots$

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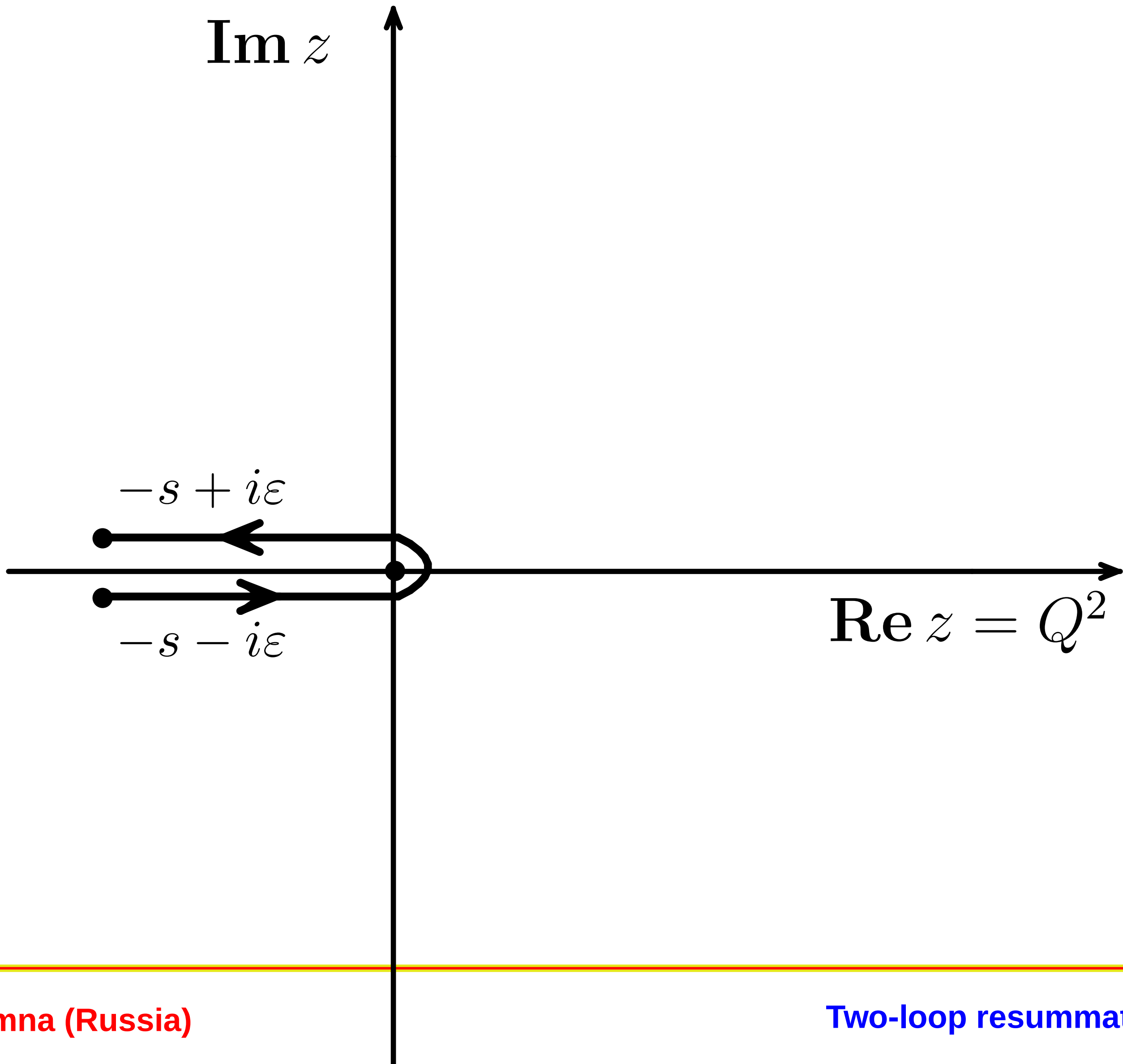
- PT series: $D[L] = 1 + d_1 a_s[L] + d_2 a_s^2[L] + \ldots$
- RG evolution: $B(Q^2) = \left[Z(Q^2)/Z(\mu^2)\right]\, B(\mu^2)$ reduces in 1-loop approximation to

$$Z \sim a^{\nu}[L]\Big|_{\nu\,=\,\nu_0\,\equiv\,\gamma_0/(2b_0)}$$

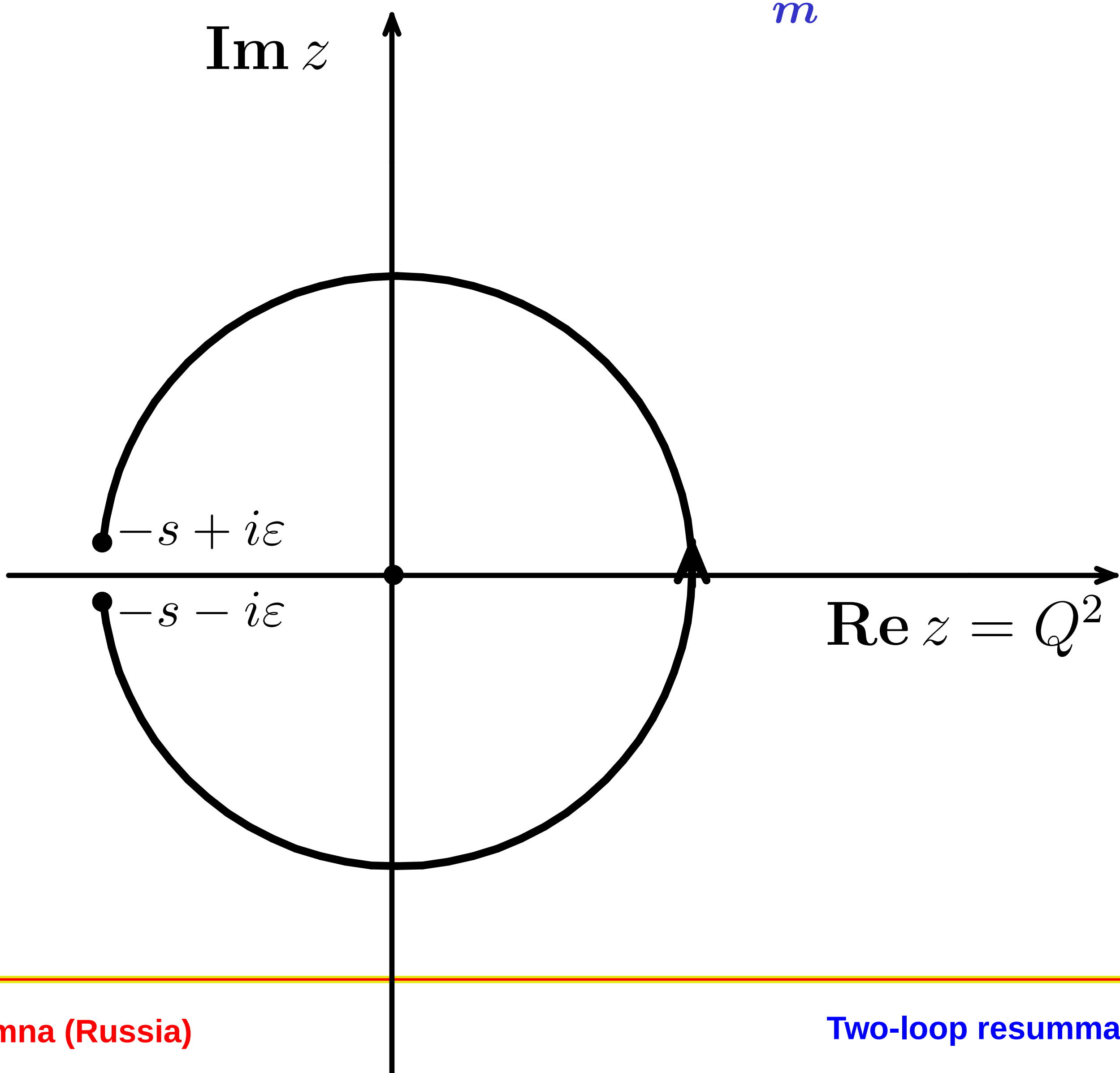
Problem in QCD PT: Minkowski region?

Quantities in Minkowski region $= \oint f(z)D(z)dz$.



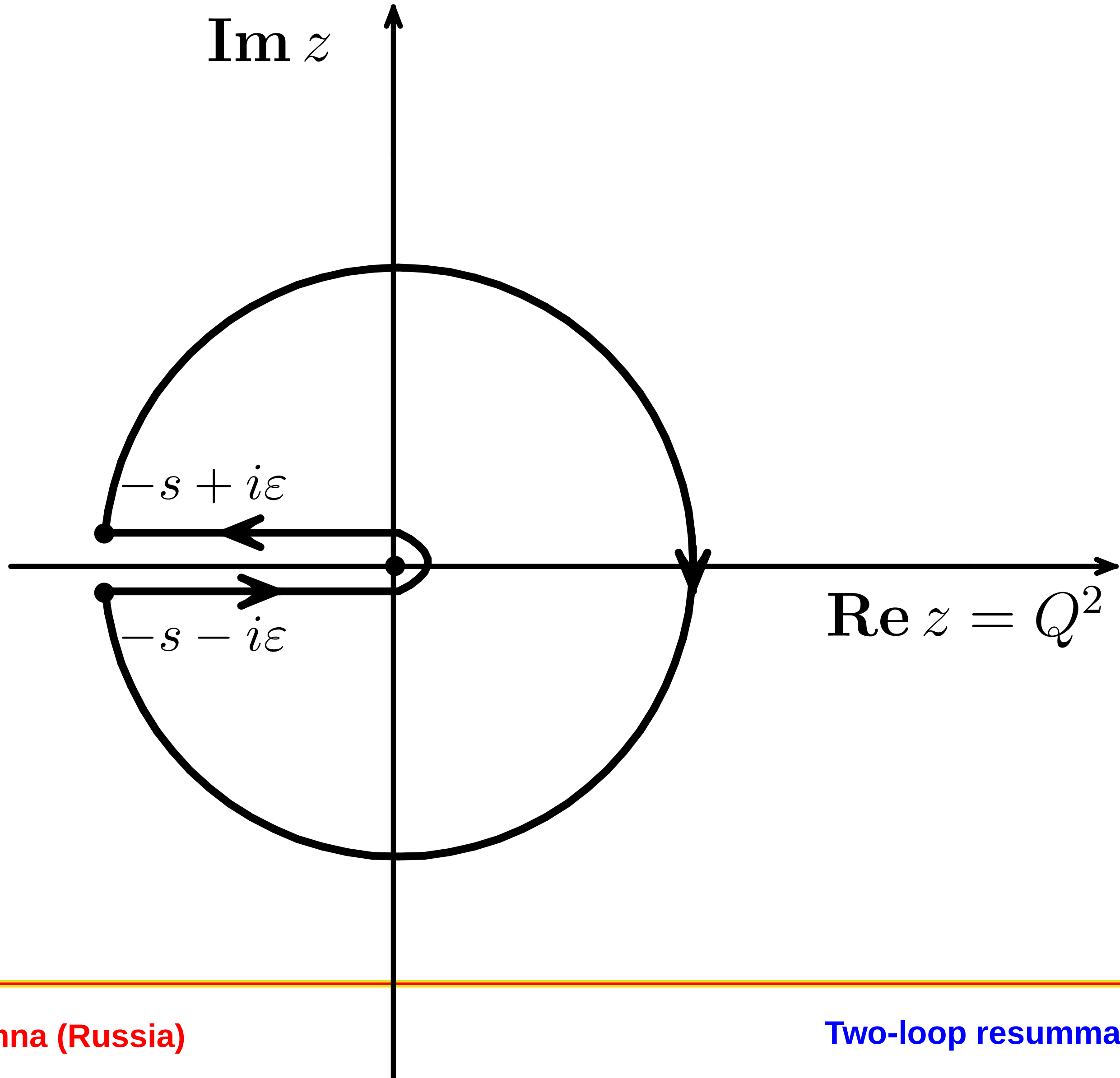
Problem in QCD PT: Minkowski region?

In $\oint f(z)D(z)dz$ one uses $D(z) = \sum_m d_m \alpha_s^m(z)$.



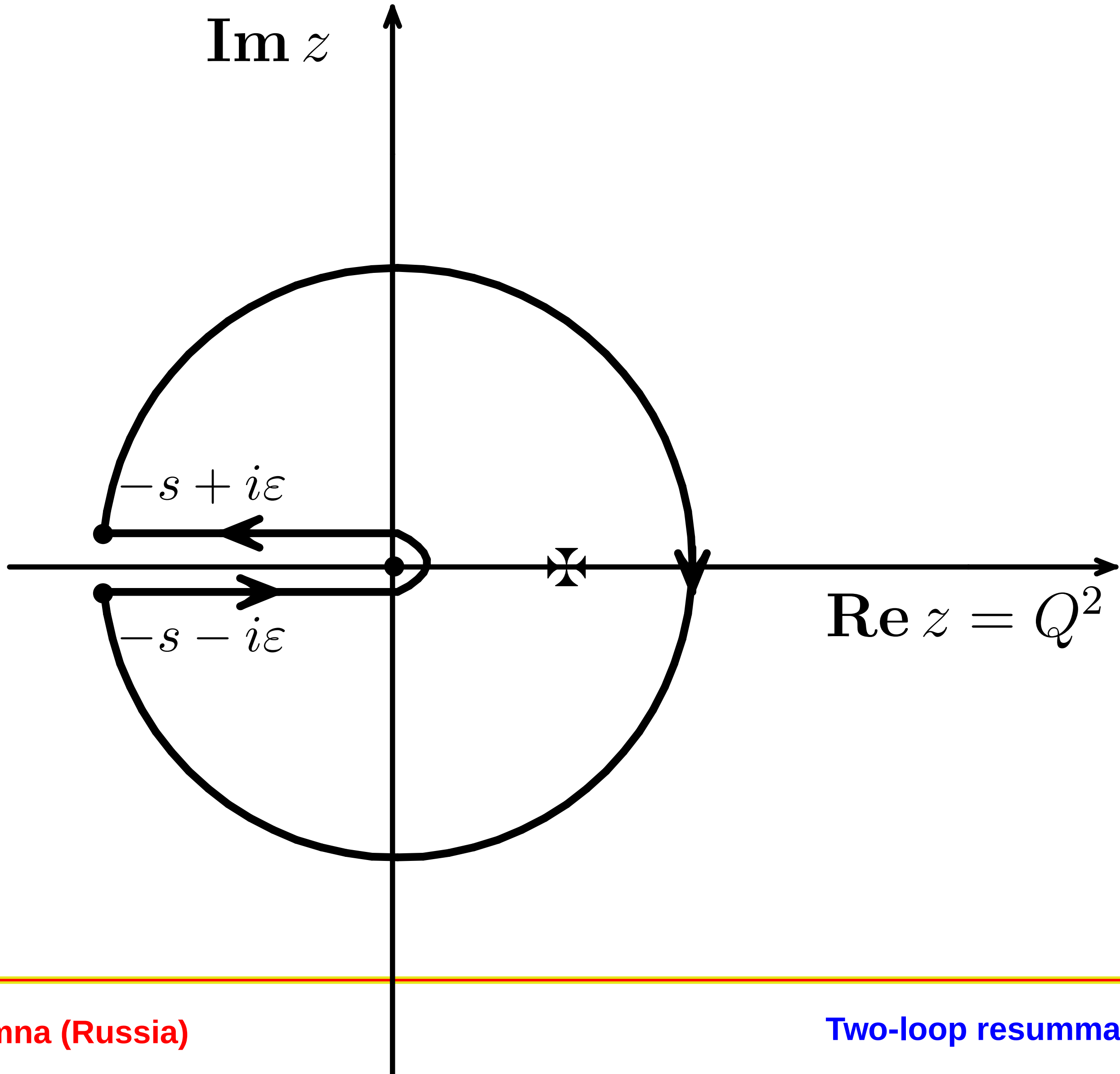
Problem in QCD PT: Minkowski region?

This change of integration contour is legitimate if $D(z)f(z)$ is analytic inside



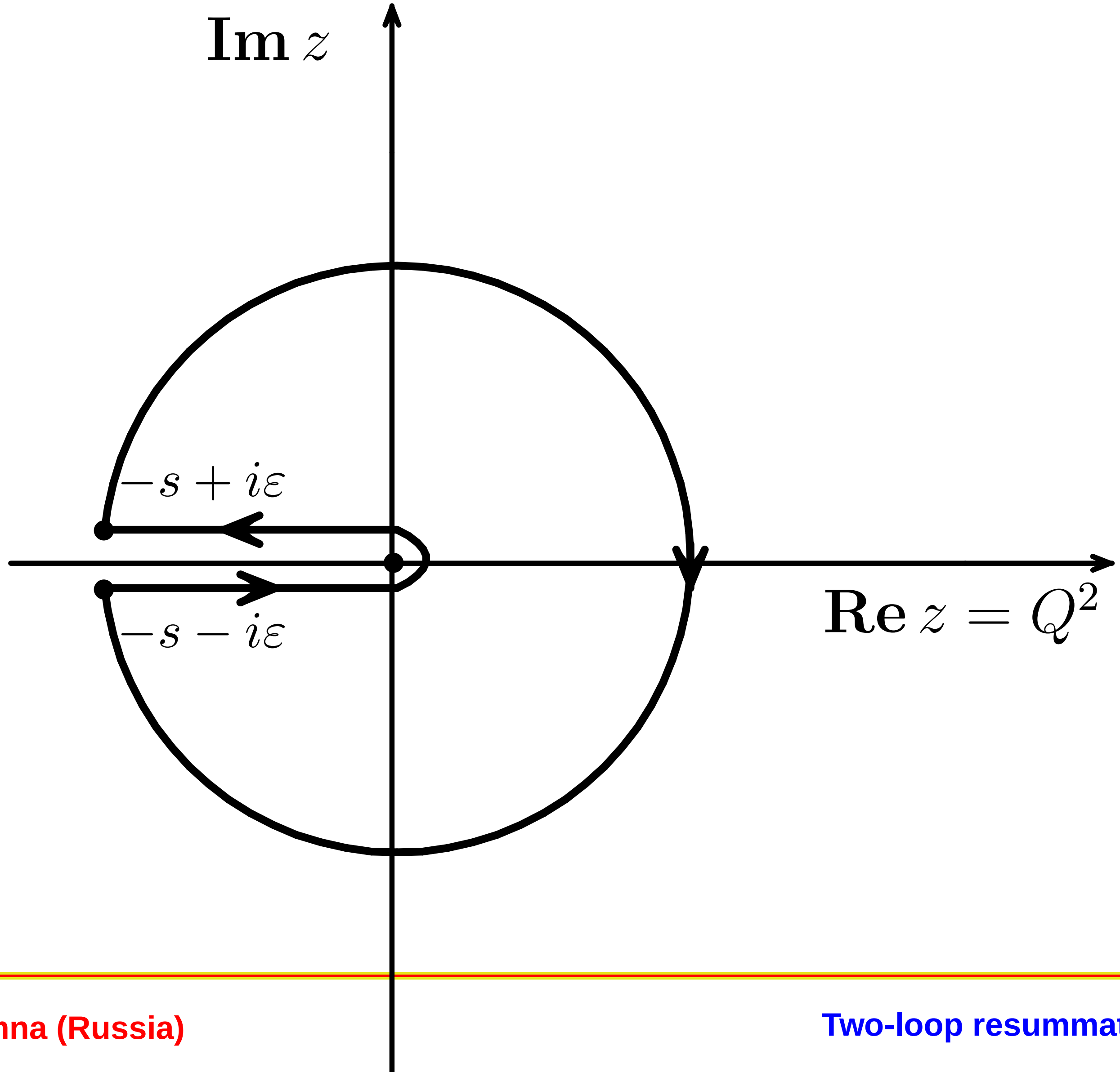
Problem in QCD PT: Minkowski region?

But $\alpha_s(z)$ and hence $D(z)f(z)$ have Landau pole singularity just inside!



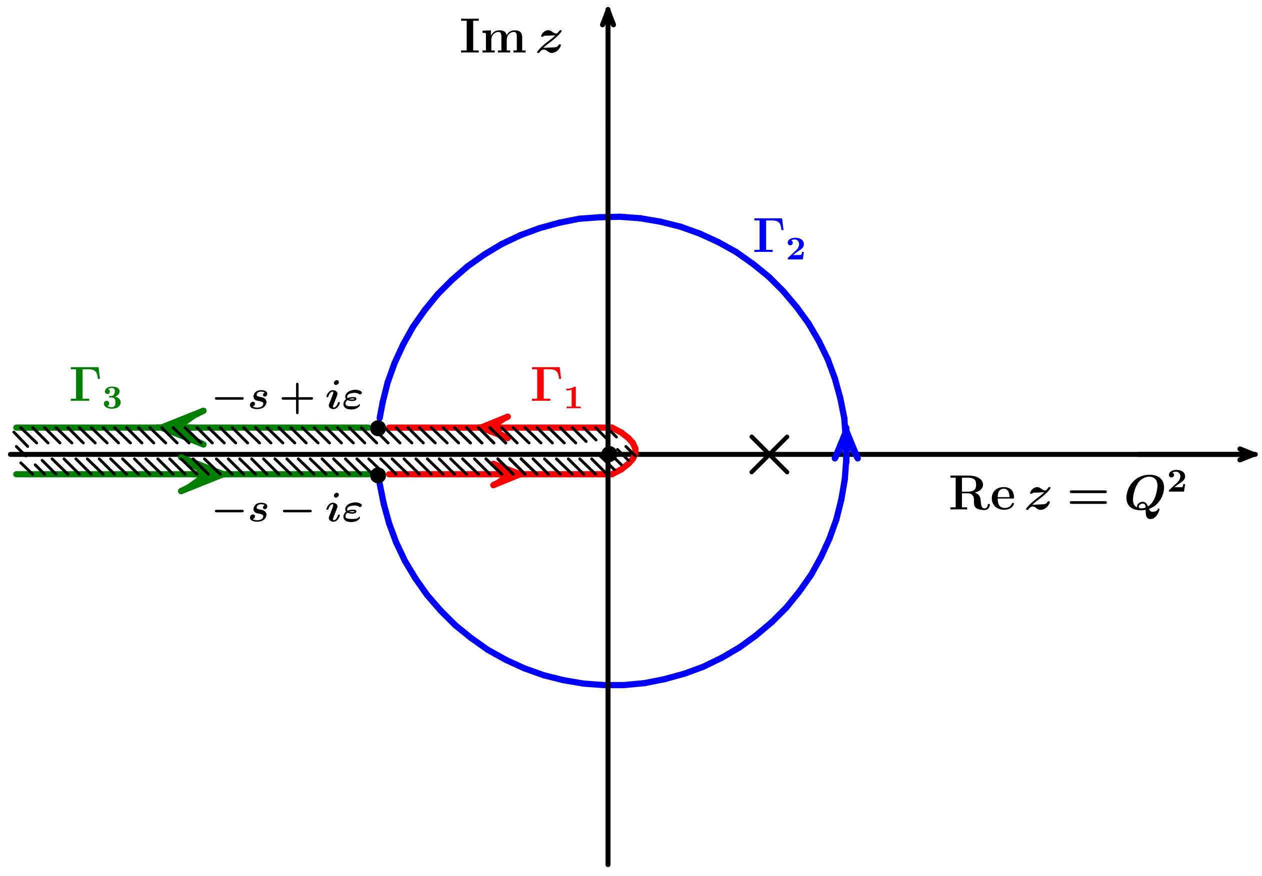
Problem in QCD PT: Minkowski region?

In **APT** effective couplings $\mathcal{A}_n(z)$ are analytic functions $\Rightarrow$ Problem does not appear! Equivalence to CIPT for $\boldsymbol{R(s)}$.



Equivalence CIPT and APT for $R(s)$

$$\text{CIPT}\left\{\oint_{\Gamma_2}\frac{D(z)dz}{z}\right\} = \text{APT}\left\{\oint_{\Gamma_3}\frac{D(z)dz}{z}\right\}$$



Basics of APT

- Different effective couplings in **Euclidean** **(S&S)** and **Minkowskian** **(R&K&P)** regions

Basics of APT

- Different effective couplings in **Euclidean (S&S)** and **Minkowskian (R&K&P)** regions
- Based on **RG** $\oplus$ **Causality**

  $\Downarrow$ UV asymptotics

  $\Downarrow$ Spectrality

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  $\Downarrow$ UV asymptotics $\qquad$ $\Downarrow$ Spectrality

- Euclidean: $-q^2 = Q^2, \;\; L = \ln Q^2/\Lambda^2, \;\; \{\mathcal{A}_n(L)\}_{n\in\mathbb{N}}$

- Minkowskian: $q^2 = s, \;\; L_s = \ln s/\Lambda^2, \;\; \{\mathfrak{A}_n(L_s)\}_{n\in\mathbb{N}}$

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- PT $\quad \sum_m d_m a_s^m(Q^2) \quad \Rightarrow \quad \sum_m d_m \mathcal{A}_m(Q^2)$ $\quad$ **APT**

  m is power $\quad \Rightarrow \quad$ m is **index**

Spectral representation

By **analytization** we mean "Källen–Lehmann" representation

$$\left[f(Q^2)\right]_{\text{an}} = \int_0^\infty \frac{\rho_f(\sigma)}{\sigma + Q^2 - i\epsilon}\, d\sigma$$

Then (note here **pole remover**):

$$\rho(\sigma) = \frac{1}{L_\sigma^2 + \pi^2}$$

$$\mathcal{A}_1[L] = \int_0^\infty \frac{\rho(\sigma)}{\sigma + Q^2}\, d\sigma = \frac{1}{L} - \frac{1}{e^L - 1}$$

$$\mathfrak{A}_1[L_s] = \int_s^\infty \frac{\rho(\sigma)}{\sigma}\, d\sigma = \frac{1}{\pi} \arccos \frac{L_s}{\sqrt{\pi^2 + L_s^2}}$$

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with spectral density $\rho_f(\sigma) = \textbf{Im}\left[f(-\sigma)\right]/\pi$. Then:

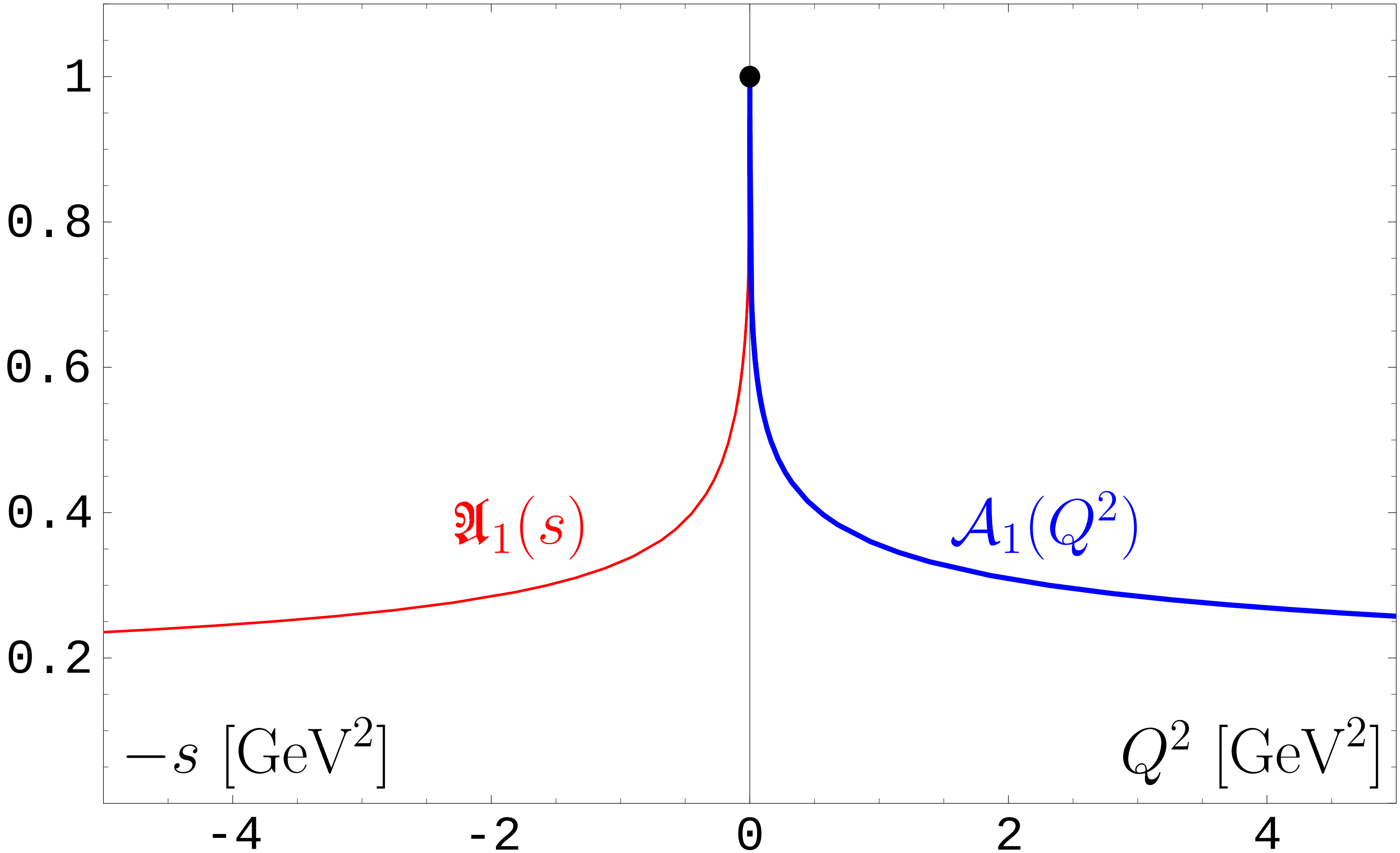
$$\mathcal{A}_n[L] = \int_0^\infty \frac{\rho_n(\sigma)}{\sigma + Q^2}\, d\sigma = \frac{1}{(n-1)!}\left(-\frac{d}{dL}\right)^{n-1} \mathcal{A}_1[L]$$

$$\mathfrak{A}_n[L_s] = \int_s^\infty \frac{\rho_n(\sigma)}{\sigma}\, d\sigma = \frac{1}{(n-1)!}\left(-\frac{d}{dL_s}\right)^{n-1} \mathfrak{A}_1[L_s]$$

$$a_s^n[L] = \frac{1}{(n-1)!}\left(-\frac{d}{dL}\right)^{n-1} a_s[L]$$

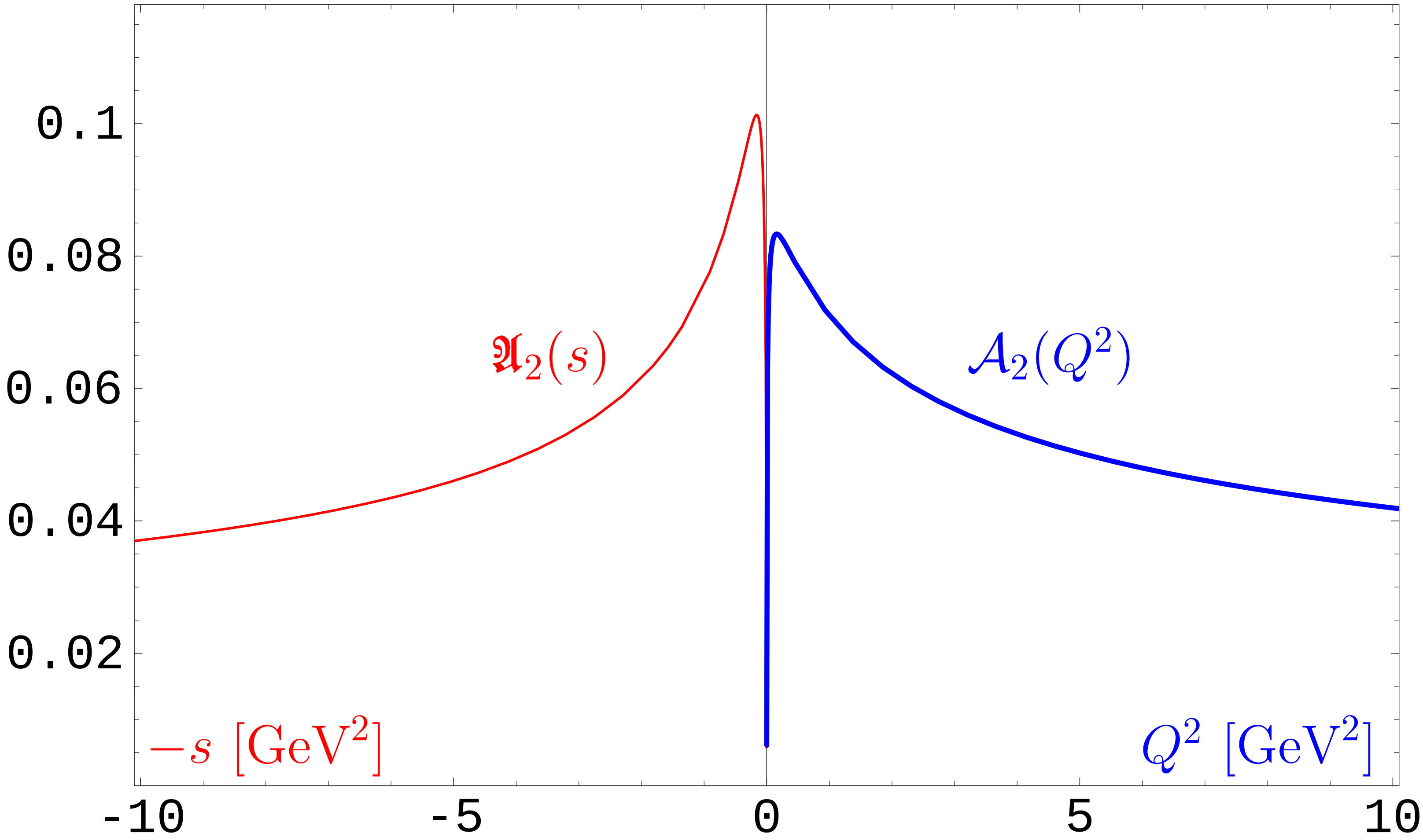
APT graphics: Distorting mirror

First, couplings: $\mathfrak{A}_1(s)$ and $\mathcal{A}_1(Q^2)$



APT graphics: Distorting mirror

Second, square-images: $\mathfrak{A}_2(s)$ and $\mathcal{A}_2(Q^2)$



Problems of APT.
Resolution:
Fractional APT

Problems of APT

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$$Z[L] = \textbf{exp}\left\{\int^{a_s[L]} \frac{\gamma(a)}{\beta(a)}\, da\right\} \xrightarrow{\textbf{1-loop}} [a_s[L]]^{\gamma_0/(2\beta_0)}$$

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- Factorization $\to \left[a_s[L]\right]^n L^m$
- Sudakov resummation $\to \textbf{exp}\left[-a_s[L]\cdot f(x)\right]$

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- Factorization $\to$ $[a_s[L]]^n\, L^m$
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New functions: $(a_s)^\nu\,,\ (a_s)^\nu\,\textbf{ln}(a_s),\ (a_s)^\nu\, L^m,\ e^{-a_s},\ \ldots$

Constructing one-loop FAPT

In one-loop **APT** we have a very nice recurrence relation

$$\mathcal{A}_n[L] = \frac{1}{(n-1)!}\left(-\frac{d}{dL}\right)^{n-1}\mathcal{A}_1[L]$$

and the same in Minkowski domain

$$\mathfrak{A}_n[L] = \frac{1}{(n-1)!}\left(-\frac{d}{dL}\right)^{n-1}\mathfrak{A}_1[L]\,.$$

We can use it to construct **FAPT**.

FAPT(E): Properties of $\mathcal{A}_\nu[L]$

First, Euclidean coupling ($L = L(Q^2)$):

$$\mathcal{A}_\nu[L] = \frac{1}{L^\nu} - \frac{F(e^{-L}, 1-\nu)}{\Gamma(\nu)}$$

Here $F(z,\nu)$ is reduced **Lerch** transcendent. function. It is analytic function in ν.

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- $\mathcal{A}_0[L] = 1$;
- $\mathcal{A}_{-m}[L] = L^m$ for $m \in \mathbb{N}$;
- $\mathcal{A}_m[L] = (-1)^m \mathcal{A}_m[-L]$ for $m \geq 2\,,\ m \in \mathbb{N}$;
- $\mathcal{A}_m[\pm\infty] = 0$ for $m \geq 2\,,\ m \in \mathbb{N}$;

FAPT(M): Properties of $\mathfrak{A}_\nu[L]$

Now, Minkowskian coupling ($L = L(s)$):

$$\mathfrak{A}_\nu[L] = \frac{\sin\left[(\nu-1)\arccos\left(L/\sqrt{\pi^2+L^2}\right)\right]}{\pi(\nu-1)\left(\pi^2+L^2\right)^{(\nu-1)/2}}$$

Here we need only elementary functions.

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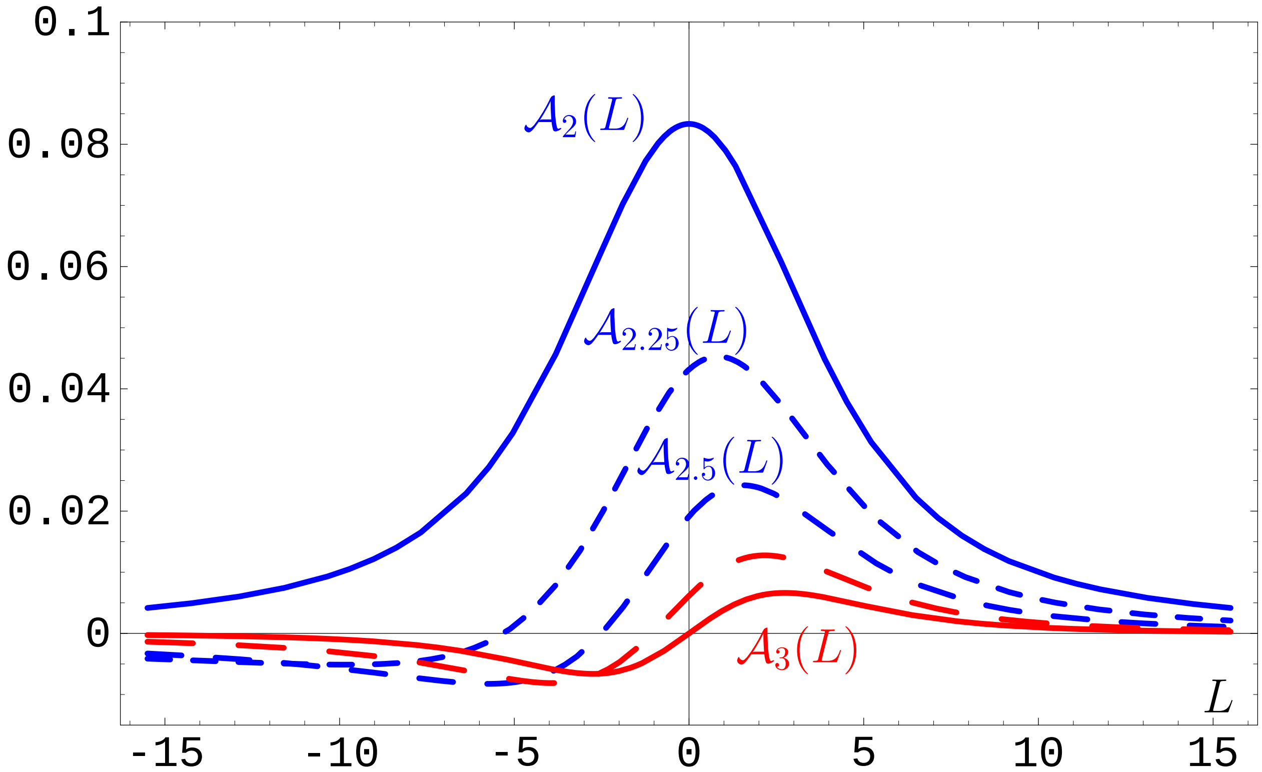
Here we need only elementary functions. Properties:

- $\mathfrak{A}_0[L] = 1$;
- $\mathfrak{A}_{-1}[L] = L$;
- $\mathfrak{A}_{-2}[L] = L^2 - \dfrac{\pi^2}{3}, \quad \mathfrak{A}_{-3}[L] = L\left(L^2 - \pi^2\right), \ \ldots$;
- $\mathfrak{A}_m[L] = (-1)^m \mathfrak{A}_m[-L]$ for $m \geq 2\,,\ m \in \mathbb{N}$;
- $\mathfrak{A}_m[\pm\infty] = 0$ for $m \geq 2\,,\ m \in \mathbb{N}$

FAPT(E): Graphics of $\mathcal{A}_\nu[L]$ vs. L

$$\mathcal{A}_\nu[L] = \frac{1}{L^\nu} - \frac{F(e^{-L}, 1-\nu)}{\Gamma(\nu)}$$

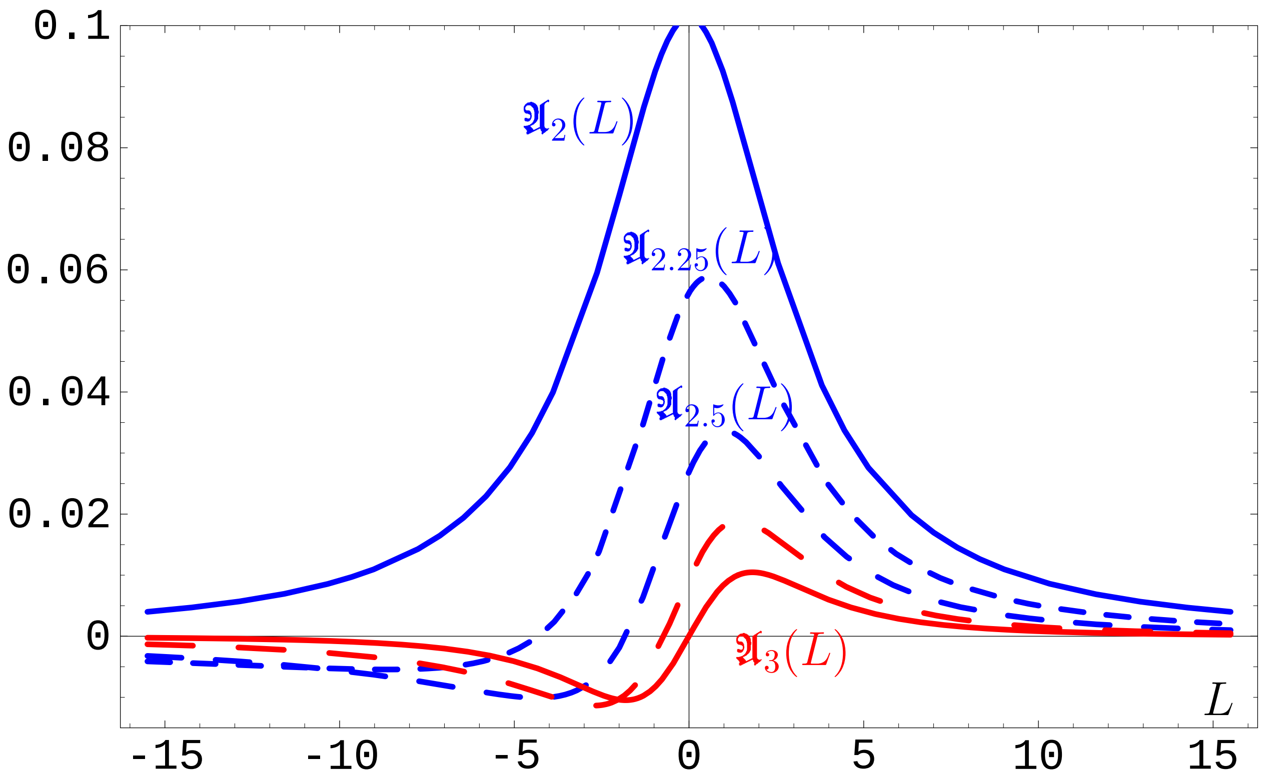
Graphics for fractional $\nu \in [2, 3]$:



FAPT(M): Graphics of $\mathfrak{A}_\nu[L]$ vs. L

$$\mathfrak{A}_\nu[L] = \frac{\sin\left[(\nu-1)\arccos\left(L/\sqrt{\pi^2+L^2}\right)\right]}{\pi(\nu-1)\left(\pi^2+L^2\right)^{(\nu-1)/2}}$$

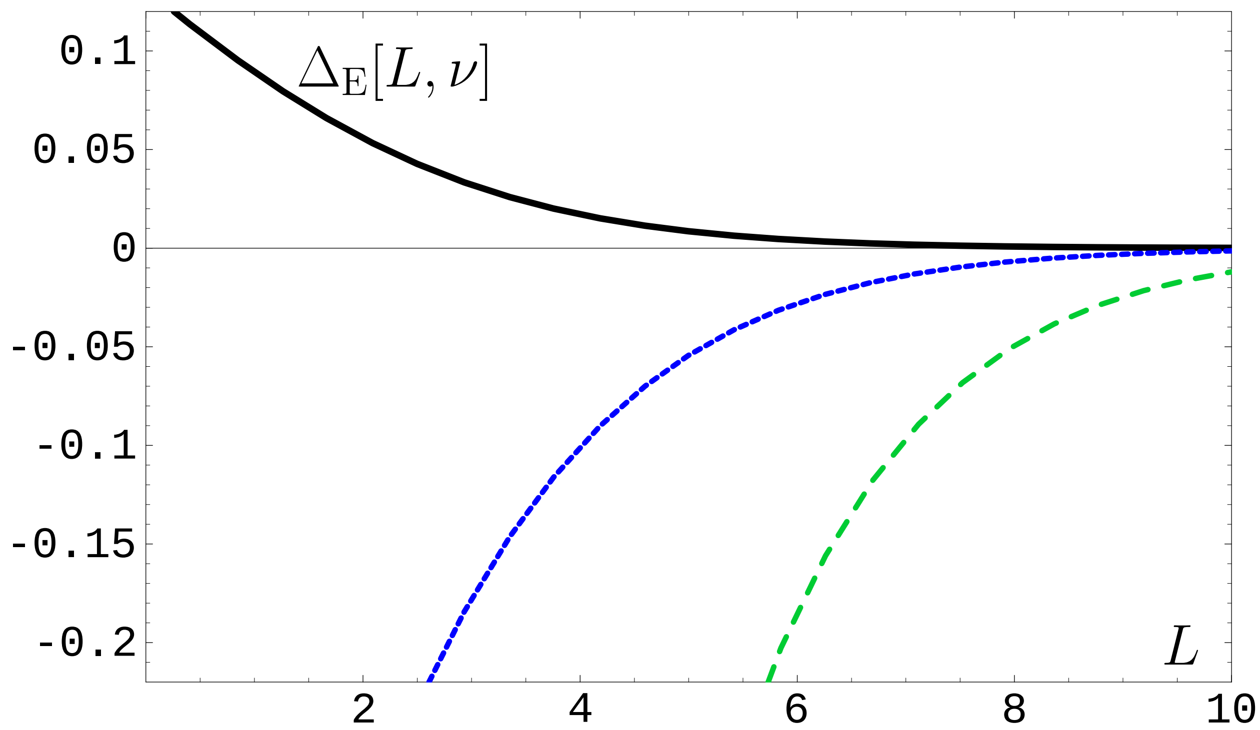
Compare with graphics in Minkowskian region :



FAPT(E): Comparing $\mathcal{A}_\nu$ with $\left(\mathcal{A}_1\right)^\nu$

$$\Delta_{\rm E}(L,\nu) = \frac{\mathcal{A}_\nu[L] - \left(\mathcal{A}_1[L]\right)^\nu}{\mathcal{A}_\nu[L]}$$

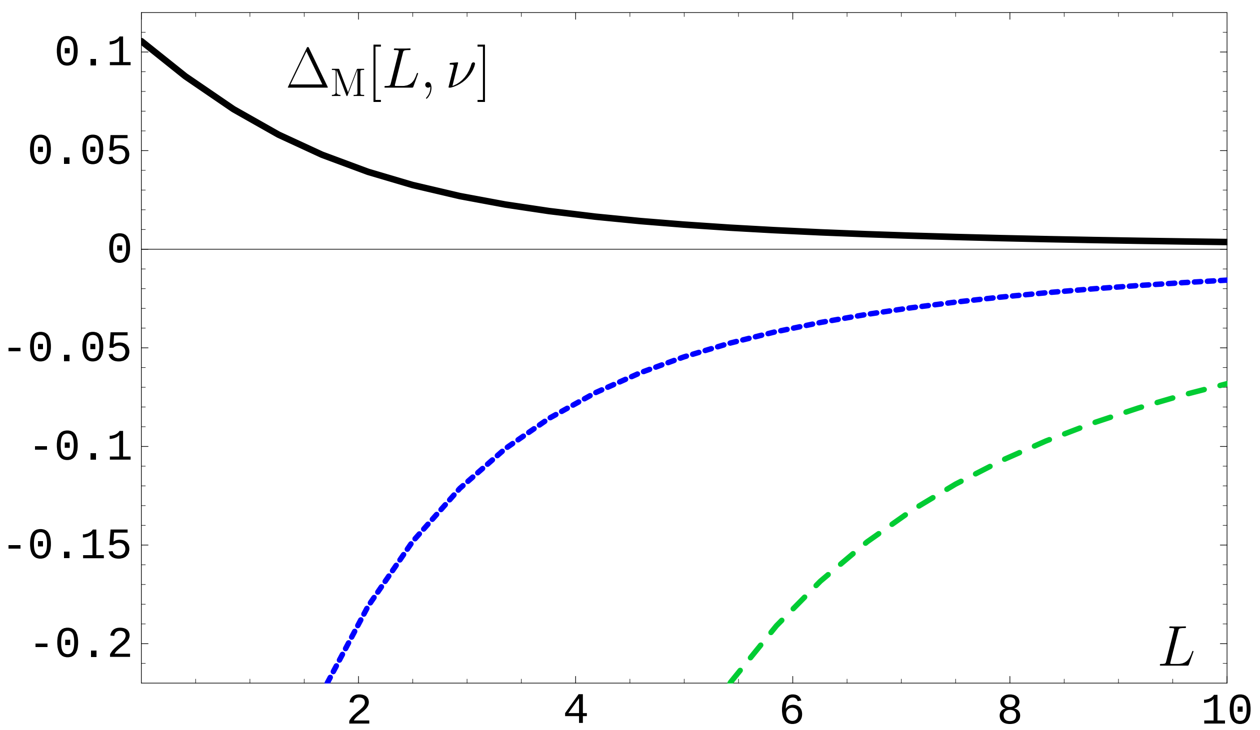
Graphics for fractional $\nu =$**0.62**, **1.62** and **2.62**:



FAPT(M): Comparing $\mathfrak{A}_\nu$ with $(\mathfrak{A}_1)^\nu$

$$\Delta_{\rm M}(L,\nu) = \frac{\mathfrak{A}_\nu[L] - \left(\mathfrak{A}_1[L]\right)^\nu}{\mathfrak{A}_\nu[L]}$$

Minkowskian graphics for $\nu =$**0.62**, **1.62** and **2.62**:



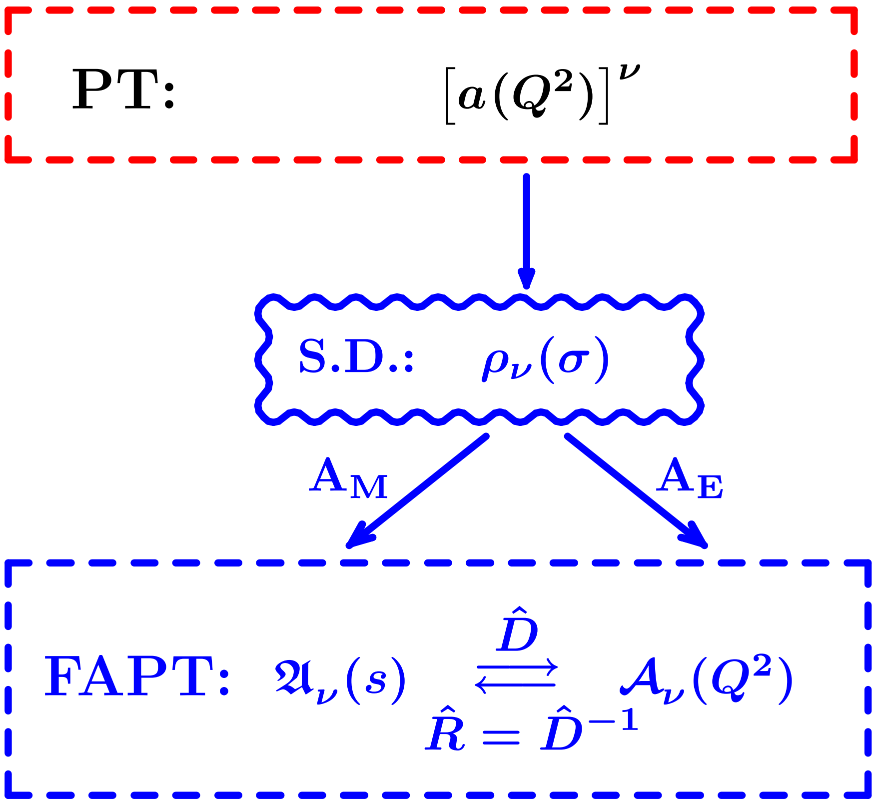
Comparison of PT, APT, and FAPT

Theory	PT	APT	FAPT
Set	$\left\{a^{\nu}\right\}_{\nu\in\mathbb{R}}$	$\left\{\mathcal{A}_m, \mathfrak{A}_m\right\}_{m\in\mathbb{N}}$	$\left\{\mathcal{A}_\nu, \mathfrak{A}_\nu\right\}_{\nu\in\mathbb{R}}$
Series	$\sum\limits_m f_m\, a^m$	$\sum\limits_m f_m\, \mathcal{A}_m$	$\sum\limits_m f_m\, \mathcal{A}_m$
Inv. powers	$(a[L])^{-m}$	—	$\mathcal{A}_{-m}[L] = L^m$
Products	$a^{\mu} a^{\nu} = a^{\mu+\nu}$	—	—
Index deriv.	$a^{\nu} \ln^k a$	—	$\mathcal{D}^k \mathcal{A}_\nu$
Logarithms	$a^{\nu} L^k$	—	$\mathcal{A}_{\nu-k}$

Development of FAPT:

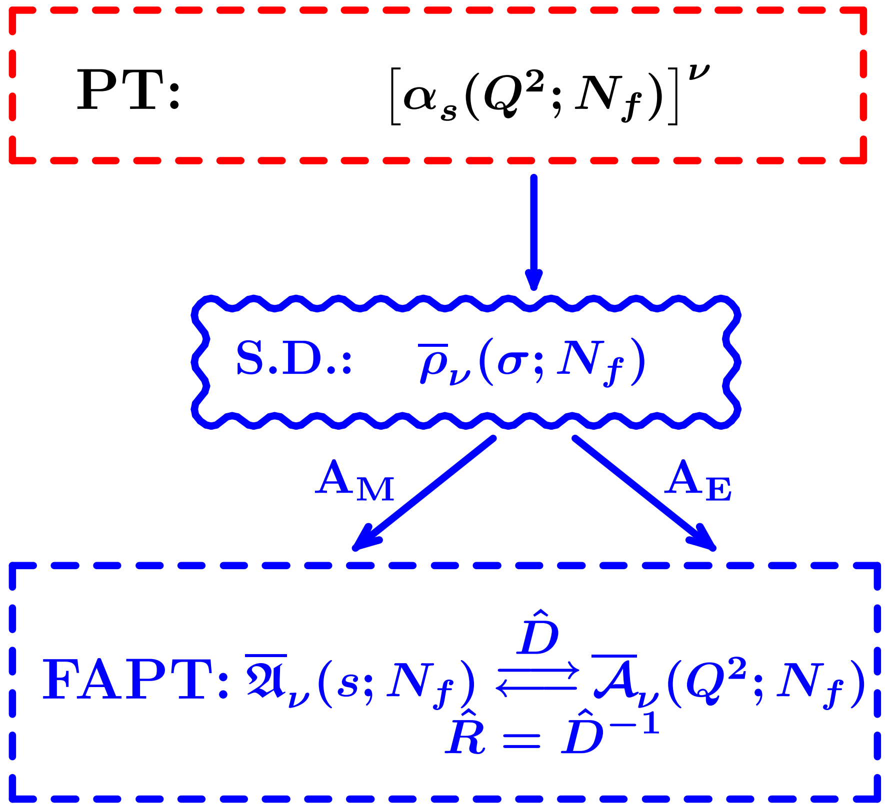
Heavy-Quark Thresholds

Conceptual scheme of FAPT



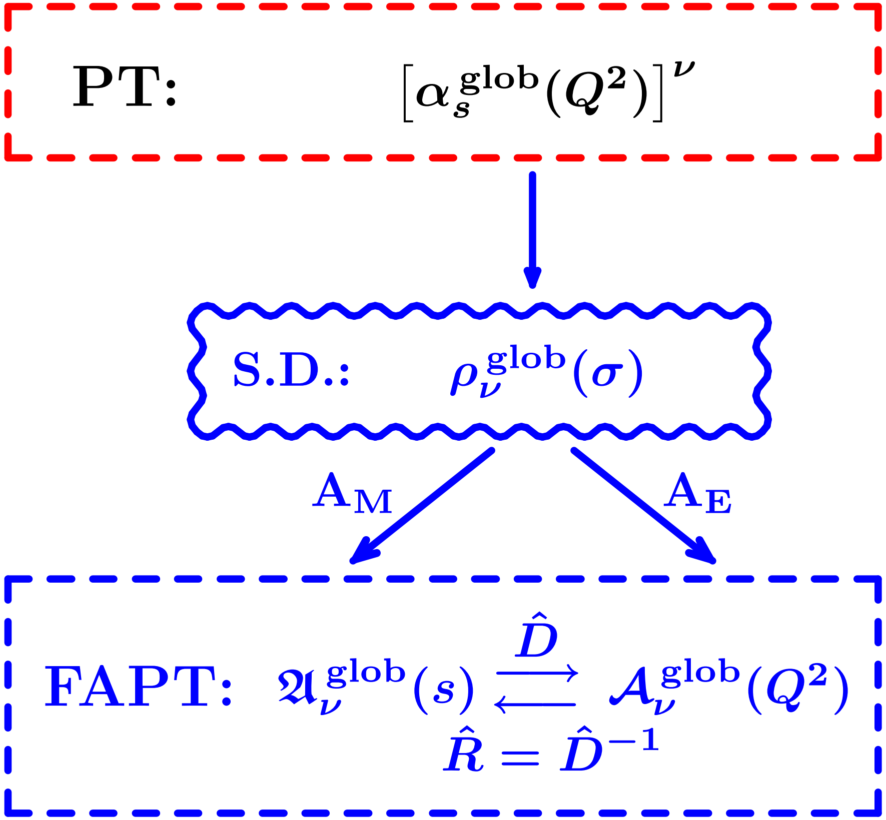
Here N_f is fixed and factorized out.

Conceptual scheme of *FAPT*



Here N_f is fixed, but not factorized out.

Conceptual scheme of *FAPT*



Here we see how “analytization” takes into account N_f-dependence.

Global FAPT: Single threshold case

- Consider for simplicity only one threshold at $s = m_c^2$ with transition $N_f = 3 \to N_f = 4$.
- Denote: $L_4 = \ln\left(m_c^2/\Lambda_3^2\right)$ and $\lambda_4 = \ln\left(\Lambda_3^2/\Lambda_4^2\right)$.

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Then:

$$\mathfrak{A}_\nu^{\text{glob}}[L] = \theta\left(L < L_4\right)\left[\overline{\mathfrak{A}}_\nu[L;3] - \overline{\mathfrak{A}}_\nu[L_4;3] + \overline{\mathfrak{A}}_\nu[L_4+\lambda_4;4]\right]$$
$$+\,\theta\left(L \geq L_4\right)\overline{\mathfrak{A}}_\nu[L+\lambda_4;4]$$

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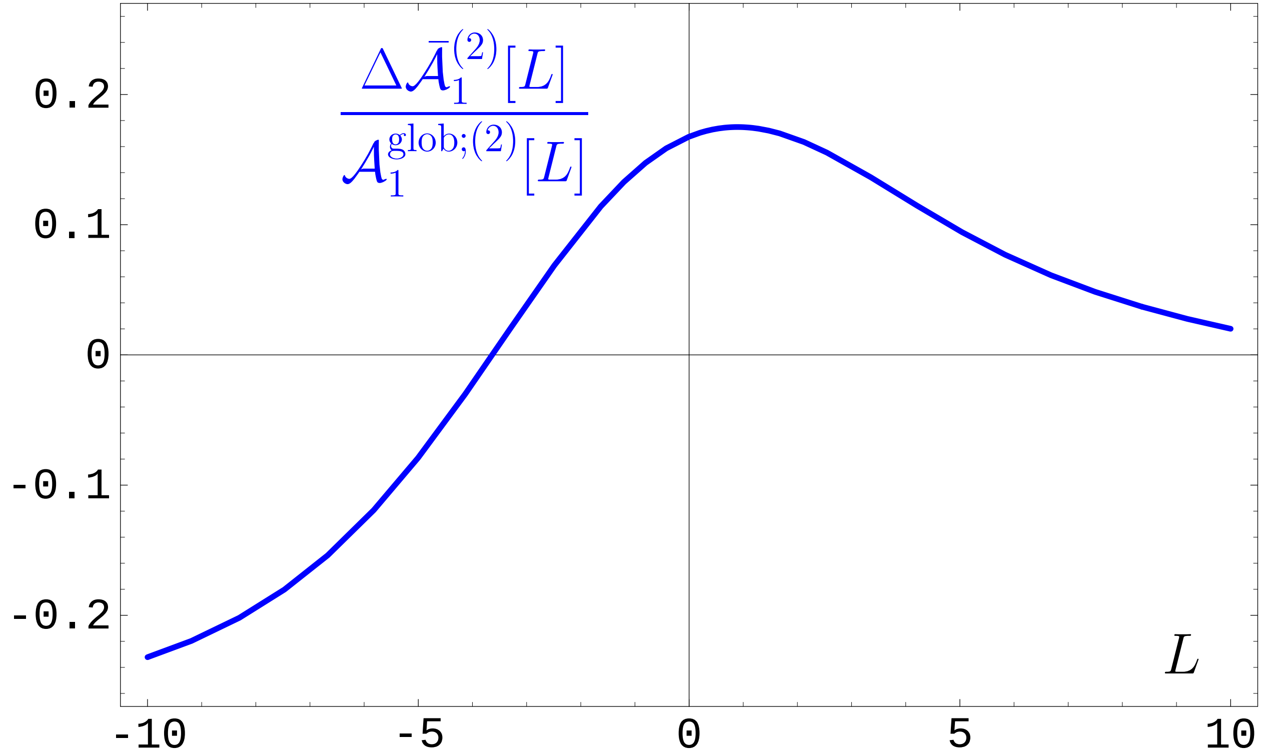
and

$$\mathcal{A}_\nu^{\text{glob}}[L] = \overline{\mathcal{A}}_\nu[L+\lambda_4;4] + \int_{-\infty}^{L_4} \frac{\overline{\rho}_\nu[L_\sigma;3] - \overline{\rho}_\nu[L_\sigma+\lambda_4;4]}{1+e^{L-L_\sigma}}\,dL_\sigma$$

Graphical comparison: Fixed-N_f—Global

$$\mathcal{A}_\nu^{\text{glob}}[L] = \overline{\mathcal{A}}_\nu[L+\lambda_4;4] + \Delta\overline{\mathcal{A}}_\nu[L]\,;$$

$\Delta\overline{\mathcal{A}}_1[L]/\mathcal{A}_1^{\text{glob}}[L]$ — **solid**:



Resummation
in
one-loop APT and FAPT

Resummation in one-loop APT

Consider series $$\mathcal{D}[L] = d_0 + \sum_{n=1}^{\infty} d_n\, \mathcal{A}_n[L]$$

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Let exist the generating function $P(t)$ for coefficients:

$$d_n = d_1 \int_0^{\infty} P(t)\, t^{n-1} dt \quad \text{with} \quad \int_0^{\infty} P(t)\, dt = 1\,.$$

We define a shorthand notation

$$\langle\langle f(t) \rangle\rangle_{P(t)} \equiv \int_0^{\infty} f(t)\, P(t)\, dt\,.$$

Then coefficients $d_n = d_1 \,\langle\langle t^{n-1} \rangle\rangle_{P(t)}$.

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We have one-loop recurrence relation:

$$\mathcal{A}_{n+1}[L] = \frac{1}{\Gamma(n+1)} \left(-\frac{d}{dL}\right)^n \mathcal{A}_1[L]\,.$$

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$$\mathcal{D}[L] = d_0 + d_1 \, \langle\langle \mathcal{A}_1[L-t] \rangle\rangle_{P(t)}$$

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Result:

$$\mathcal{D}[L] = d_0 + d_1 \, \langle\langle \mathcal{A}_1[L-t] \rangle\rangle_{P(t)}$$

and for Minkowski region:

$$\mathcal{R}[L] = d_0 + d_1 \, \langle\langle \mathfrak{A}_1[L-t] \rangle\rangle_{P(t)}$$

Resummation in Global Minkowskian APT

Consider series $\mathcal{R}[L] = d_0 + \sum_{n=1}^{\infty} d_n\, \mathfrak{A}_n^{\text{glob}}[L]$

with coefficients $d_n = d_1 \langle\langle t^{n-1} \rangle\rangle_{P(t)}$.

Result:

$$\mathcal{R}[L] = d_0 \;+\; d_1 \langle\langle \theta\,(L < L_4) \left[\Delta_4 \overline{\mathfrak{A}}_1[t] + \overline{\mathfrak{A}}_1\left[L - \frac{t}{\beta_3}; 3\right]\right] \rangle\rangle_{P(t)}$$

$$+\; d_1 \langle\langle \theta\,(L \geq L_4) \overline{\mathfrak{A}}_1\left[L + \lambda_4 - \frac{t}{\beta_4}; 4\right] \rangle\rangle_{P(t)}\,.$$

where

$$\Delta_4 \overline{\mathfrak{A}}_1[t] = \overline{\mathfrak{A}}_1\left[L_4 + \lambda_4 - \frac{t}{\beta_4}; 4\right] - \overline{\mathfrak{A}}_1\left[L_3 - \frac{t}{\beta_3}; 3\right].$$

Resummation in Global Euclidean APT

In Euclidean domain the result is more complicated:

$$\mathcal{D}[L] = d_0 + d_1 \langle\langle \int\limits_{-\infty}^{L_4} \frac{\bar{\rho}_1\left[L_\sigma; 3\right] dL_\sigma}{1 + e^{L - L_\sigma - t/\beta_3}} \rangle\rangle_{P(t)}$$

$$+ \langle\langle \Delta_4[L, t] \rangle\rangle_{P(t)} + d_1 \langle\langle \int\limits_{L_4}^{\infty} \frac{\bar{\rho}_1\left[L_\sigma + \lambda_4; 4\right] dL_\sigma}{1 + e^{L - L_\sigma - t/\beta_4}} \rangle\rangle_{P(t)} \,.$$

where

$$\Delta_4[L, t] = \int\limits_0^1 \frac{\bar{\rho}_1\left[L_4 + \lambda_4 - tx/\beta_4; 4\right] t}{\beta_4 \left[1 + e^{L - L_4 - t\bar{x}/\beta_4}\right]} \, dx$$

$$- \int\limits_0^1 \frac{\bar{\rho}_1\left[L_3 - tx/\beta_3; 3\right] t}{\beta_3 \left[1 + e^{L - L_4 - t\bar{x}/\beta_3}\right]} \, dx.$$

Resummation in FAPT

Consider seria $\mathcal{R}_\nu[L] = d_0\,\mathfrak{A}_\nu[L] + \sum_{n=1}^{\infty} d_n\,\mathfrak{A}_{n+\nu}[L]$

and $\mathcal{D}_\nu[L] = d_0\,\mathcal{A}_\nu[L] + \sum_{n=1}^{\infty} d_n\,\mathcal{A}_{n+\nu}[L]$

with coefficients $d_n = d_1\,\langle\langle t^{n-1}\rangle\rangle_{P(t)}$.

Result:

$$\begin{aligned} \mathcal{R}_\nu[L] &= d_0\,\mathfrak{A}_\nu[L] + d_1\,\langle\langle \mathfrak{A}_{1+\nu}[L-t]\rangle\rangle_{P_\nu(t)}\,; \\ \mathcal{D}_\nu[L] &= d_0\,\mathcal{A}_\nu[L] + d_1\,\langle\langle \mathcal{A}_{1+\nu}[L-t]\rangle\rangle_{P_\nu(t)}\,. \end{aligned}$$

where $P_\nu(t) = \int_0^1 P\left(\frac{t}{1-z}\right)\nu\,z^{\nu-1}\frac{dz}{1-z}\,.$

Resummation in Global Minkowskian FAPT

Consider series $\mathcal{R}_\nu[L] = d_0\,\mathfrak{A}_\nu^{\textbf{glob}} + \sum_{n=1}^{\infty} d_n\,\mathfrak{A}_{n+\nu}^{\textbf{glob}}[L]$

with coefficients $d_n = d_1\,\langle\langle t^{n-1}\rangle\rangle_{P(t)}$.

Then result is complete analog of the Global APT(M) result with natural substitutions:

$$\overline{\mathfrak{A}}_1[L] \to \overline{\mathfrak{A}}_{1+\nu}[L] \quad \text{and} \quad P(t) \to P_\nu(t)$$

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Resummation in Global Euclidean FAPT

Consider series $\mathcal{D}_\nu[L] = d_0\,\mathcal{A}_\nu^{\textbf{glob}} + \sum_{n=1}^{\infty} d_n\,\mathcal{A}_{n+\nu}^{\textbf{glob}}[L]$

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Then result is complete analog of the Global APT(E) result with natural substitutions:

$$\overline{\rho}_1[L] \to \overline{\rho}_{1+\nu}[L] \quad \text{and} \quad P(t) \to P_\nu(t)$$

with $P_\nu(t) = \int_0^1 P\left(\frac{t}{1-z}\right)\nu\,z^{\nu-1}\frac{dz}{1-z}$.

Resummation
in
two-loop APT and FAPT

Resummation in two-loop APT

Consider series $\mathcal{S}[L] = \sum_{n=1}^{\infty} \langle\langle t^{n-1} \rangle\rangle_{P(t)}\, \mathcal{F}_n[L]$.

Here $\mathcal{F}_n[L] = \mathcal{A}_n^{(2)}[L]$ or $\mathfrak{A}_n^{(2)}[L]$.

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Here $\mathcal{F}_n[L] = \mathcal{A}_n^{(2)}[L]$ or $\mathfrak{A}_n^{(2)}[L]$.

We have two-loop recurrence relation ($c_1 = b_1/b_0^2$):

$$-\frac{1}{n}\frac{d}{dL}\mathcal{F}_n[L] = \mathcal{F}_{n+1}[L] + c_1\,\mathcal{F}_{n+2}[L]$$

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Result ($\tau(t) = t - c_1 \ln(1 + t/c_1)$):

$$\mathcal{S}[L] = \left\langle\!\!\left\langle \frac{c_1\,\mathcal{F}_1[L] + t\,\mathcal{F}_1[L-\tau(t)]}{c_1+t} + \frac{c_1\,t}{c_1+t}\,\mathcal{F}_2[L-\tau(t)] \right\rangle\!\!\right\rangle_{P(t)} - \left\langle\!\!\left\langle \frac{c_1\,t}{c_1+t}\int_0^t \frac{dt'}{c_1+t'}\,\frac{d\mathcal{F}_1[L+\tau(t')-\tau(t)]}{dL} \right\rangle\!\!\right\rangle_{P(t)}.$$

Resummation in two-loop global APT

Consider series $\rho_{\Sigma}^{(2)}[L, N_f] =$

$$\beta_f \sum_{n=1}^{\infty} \langle\langle t^{n-1} \rangle\rangle_{P(t)} \, \overline{\rho}_n^{(2)}[L, N_f] = \sum_{n=1}^{\infty} \langle\langle \left[\frac{t}{\beta_f}\right]^{n-1} \rangle\rangle_{P(t)} \, \rho_n^{(2)}[L]$$

Resummation in two-loop global APT

Thus ($t_f = t/\beta_f$): $\rho_{\Sigma}^{(2)}[L, N_f] = \sum_{n=1}^{\infty} \langle\langle t_f^{n-1} \rangle\rangle_{P(t)}\, \rho_n^{(2)}[L]$

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Result of summation is ($t_f = t/\beta_f$):

$$\rho_{\Sigma}^{(2)}[L, N_f] = \left\langle\!\!\left\langle \frac{c_1\,\rho_1^{(2)}[L]+t_f\,\rho_1^{(2)}[L-\tau(t_f)]}{c_1+t_f} + \frac{c_1\,t_f}{c_1+t_f}\,\rho_2^{(2)}[L-\tau(t_f)] \right.\right.$$
$$\left.\left. - \frac{c_1\,t_f}{c_1+t_f}\int_0^{t_f}\frac{dt'}{c_1+t'}\,\frac{d\rho_1^{(2)}[L+\tau(t')-\tau(t_f)]}{dL} \right\rangle\!\!\right\rangle_{P(t)}\,.$$

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Consider series $\mathcal{S}_\nu[L] = \sum_{n=1}^{\infty} \langle\langle t^{n-1} \rangle\rangle_{P(t)} \, \mathcal{F}_{n+\nu}[L]$.

Here $\mathcal{F}_\nu[L] = \mathcal{A}_\nu^{(2)}[L]$ or $\mathfrak{A}_\nu^{(2)}[L]$ (or $\rho_\nu^{(2)}[L]$ — for global).

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$$\mathcal{S}[L] = \Big\langle\Big\langle \mathcal{F}_{1+\nu}[L] - \tfrac{t^2}{c_1+t} \int_0^1 z^\nu dz\, \dot{\mathcal{F}}_{1+\nu}[L + \tau(t\,z) - \tau(t)]$$
$$+ \tfrac{c_1\,t}{c_1+t} \left\{ \mathcal{F}_{2+\nu}[L] - \int_0^1 dz\, \tfrac{t^2\,z^{\nu+1}}{c_1+t\,z}\, \dot{\mathcal{F}}_{2+\nu}[L + \tau(t\,z) - \tau(t)] \right\} \Big\rangle\Big\rangle_{P(t)}$$

Resummation in two-loop (global) FAPT

Consider series $\mathcal{S}_{\nu_0,\nu_1}[L] = \sum_{n=1}^{\infty} \langle\langle t^{n-1} \rangle\rangle_{P(t)} \mathcal{F}_{n+\nu_0,\nu_1}[L]$.

Here $\mathcal{F}_{n+\nu_0,\nu_1}[L] = \mathcal{B}^{(2)}_{n+\nu_0,\nu_1}[L]$ or $\mathfrak{B}^{(2)}_{n+\nu_0,\nu_1}[L]$
(or $\rho^{(2)}_{n+\nu_0,\nu_1}[L]$ — for global),
where

$$\mathcal{B}_{\nu;\nu_1}[L] = \mathbf{A}_{\mathbf{E},\mathbf{M}} \left[a^{\nu}_{(2)}[L] \left(1 + c_1\, a_{(2)}\right)^{\nu_1} [L] \right]$$

is the analytic image of the two-loop evolution factor.

We have constructed formulas of resummation for $\mathcal{S}_{\nu_0,\nu_1}[L]$ as well.

Higgs boson decay

$$H^0 \to b\bar{b}$$

Higgs boson decay into $b\bar{b}$-pair

This decay can be expressed in QCD by means of the correlator of quark scalar currents $J_{\text{S}}(x) = :\bar{b}(x)b(x):$

$$\Pi(Q^2) = (4\pi)^2 i \int dx e^{iqx} \langle 0|\, T[\, J_{\text{S}}(x) J_{\text{S}}(0)\,]\, |0\rangle$$

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in terms of discontinuity of its imaginary part

$$R_{\mathsf{S}}(s) = \mathbf{Im}\, \Pi(-s - i\epsilon)/(2\pi\, s)\,,$$

so that

$$\Gamma_{\mathsf{H}\to b\bar{b}}(M_{\mathsf{H}}) = \frac{G_F}{4\sqrt{2}\pi} M_{\mathsf{H}}\, m_b^2(M_{\mathsf{H}})\, R_{\mathsf{S}}(s = M_{\mathsf{H}}^2)\,.$$

FAPT(M) analysis of R_S

Running mass $m(Q^2)$ is described by the RG equation

$$m^2(Q^2) = \hat{m}^2 \alpha_s^{\nu_0}(Q^2) \left[1 + \frac{c_1 b_0 \alpha_s(Q^2)}{4\pi^2}\right]^{\nu_1} .$$

with RG-invariant mass $\hat{m}^2$ (for b-quark $\hat{m}_b \approx 8.53$ **GeV**) and $\nu_0 = 1.04$, $\nu_1 = 1.86$.

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$$\left[3\,\hat{m}_b^2\right]^{-1} \widetilde{D}_{\mathbf{S}}(Q^2) = \alpha_s^{\nu_0}(Q^2) + \sum_{m>0} \frac{d_m}{\pi^m}\, \alpha_s^{m+\nu_0}(Q^2)\,.$$

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In 1-loop FAPT(M) we obtain

$$\widetilde{\mathcal{R}}_{\mathsf{S}}^{(1);N}[L] = 3\hat{m}^2 \left[\mathfrak{A}_{\nu_0}^{(1);\text{glob}}[L] + \sum_{m>0}^{N} \frac{d_m}{\pi^m}\, \mathfrak{A}_{m+\nu_0}^{(1);\text{glob}}[L]\right]$$

FAPT(M) analysis of $R_{\mathbf{S}}$

Running mass $m(Q^2)$ is described by the RG equation

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In 2-loop FAPT(M) we obtain

$$\widetilde{\mathcal{R}}_{\mathbf{S}}^{(2);N}[L] = 3\hat{m}^2 \left[\mathfrak{B}_{\nu_0,\nu_1}^{(2);\text{glob}}[L] + \sum_{m>0}^{N} \frac{d_m}{\pi^m}\, \mathfrak{B}_{m+\nu_0,\nu_1}^{(2);\text{glob}}[L]\right]$$

Model for perturbative coefficients

Coefficients of our series, $\tilde{d}_m = d_m/d_1$, with $d_1 = 17/3$:

Model	$\tilde{d}_1$	$\tilde{d}_2$	$\tilde{d}_3$	$\tilde{d}_4$	$\tilde{d}_5$
pQCD	1	7.42	62.3		—

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$c = 2.5,\ \beta = -0.48$	1	7.42	62.3		

We use model $\tilde{d}_n^{\text{mod}} = \dfrac{c^{n-1}(\beta\,\Gamma(n) + \Gamma(n+1))}{\beta + 1}$

with parameters β and c estimated by known $\tilde{d}_n$ and with use of **Lipatov** asymptotics.

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$c = 2.5,\ \beta = -0.48$	1	7.42	62.3	$\mathbf{662}$	—
$c = 2.4,\ \beta = -0.52$	1	7.50	61.1	$\mathbf{625}$	$\mathbf{7826}$
"PMS" model	—	—	64.8	$\mathbf{547}$	$\mathbf{7782}$

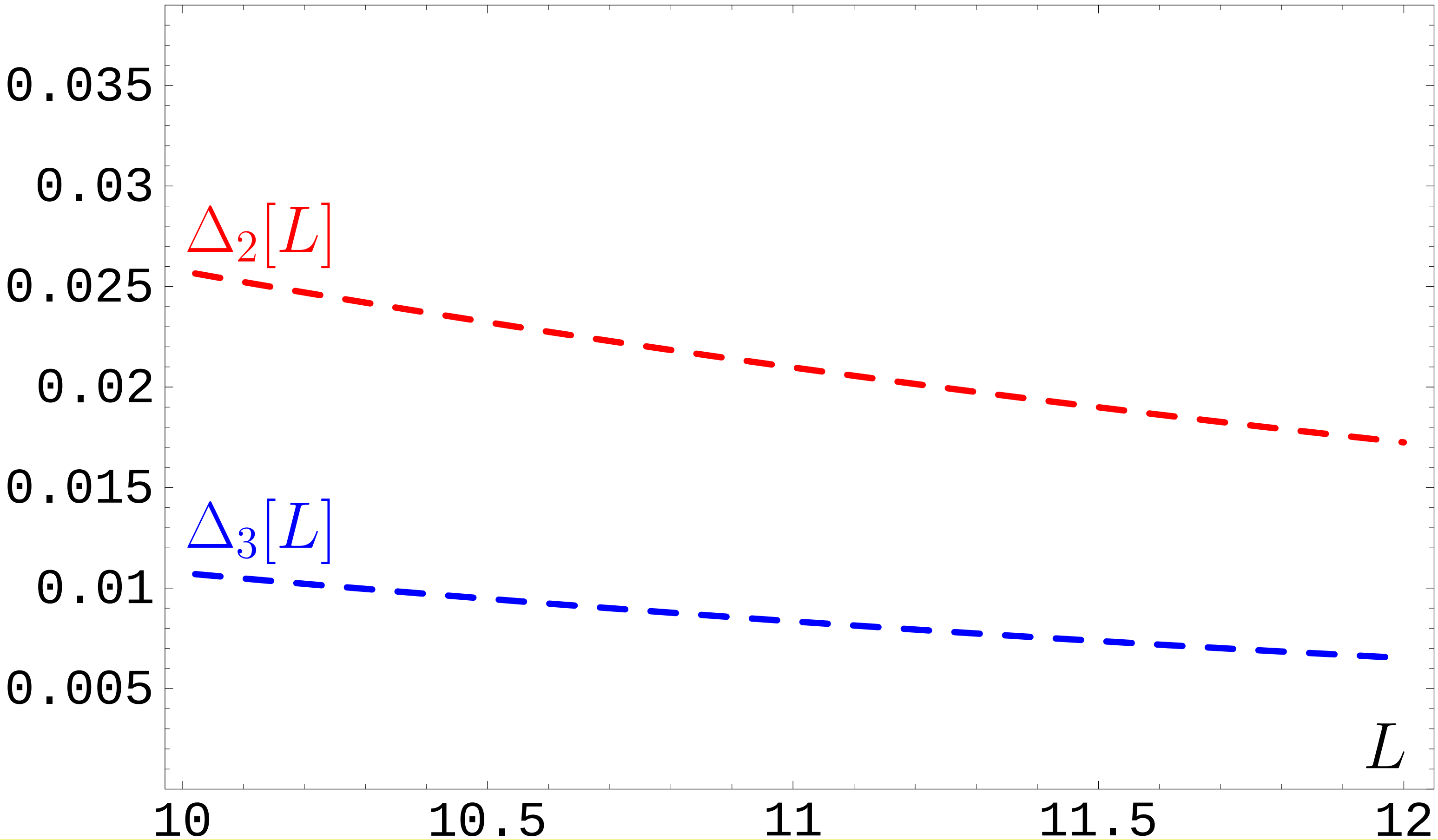
We use model $\tilde{d}_n^{\text{mod}} = \dfrac{c^{n-1}(\beta\,\Gamma(n) + \Gamma(n+1))}{\beta + 1}$

with parameters β and c estimated by known $\tilde{d}_n$ and with use of **Lipatov** asymptotics.

FAPT(M) for $\Gamma_{H\to\bar{b}b}(m_H)$: Truncation errors

We define relative errors of series truncation at Nth term:

$$\Delta_N[L] = 1 - \widetilde{\mathcal{R}}_{\mathbf{S}}^{(2;N)}[L]/\widetilde{\mathcal{R}}_{\mathbf{S}}^{(2;\infty)}[L]$$



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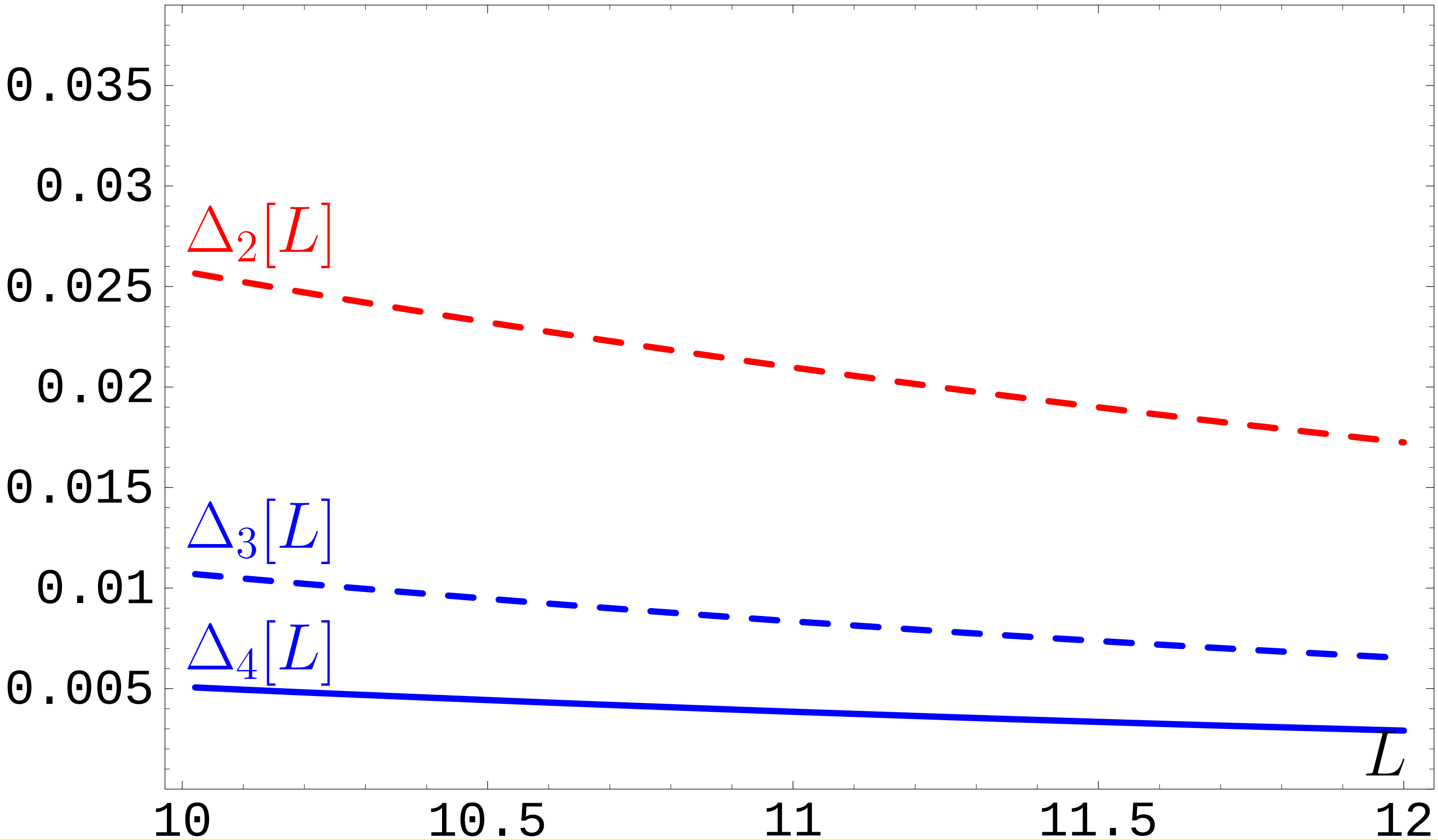

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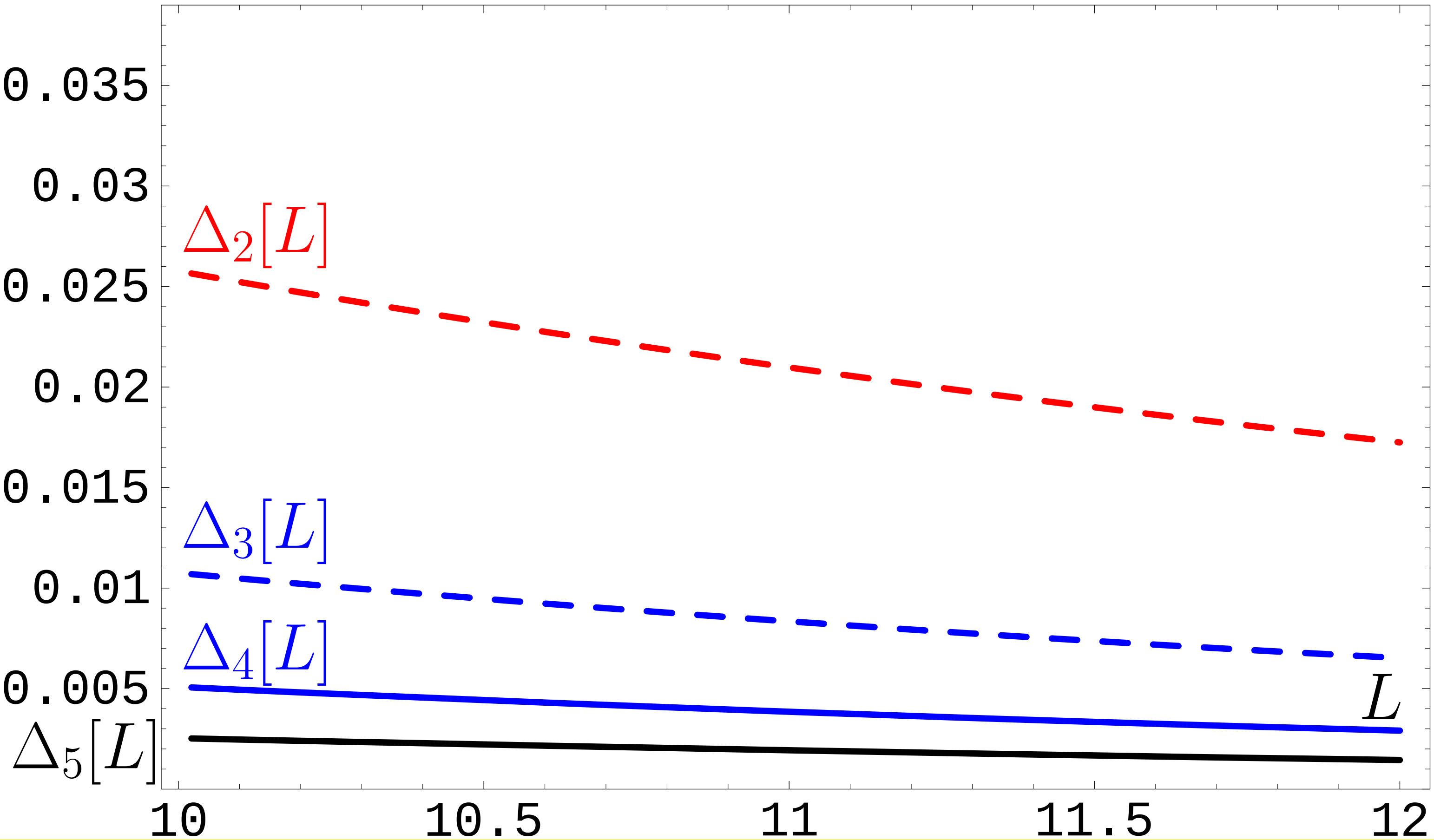

FAPT(M) for $\Gamma_{H\to\bar{b}b}(m_{\boldsymbol{H}})$: Truncation errors

Conclusion: If we need accuracy better than 0.5% — only then we need to calculate the 5-th correction.

FAPT(M) for $\Gamma_{H\to\bar{b}b}(m_H)$: Truncation errors

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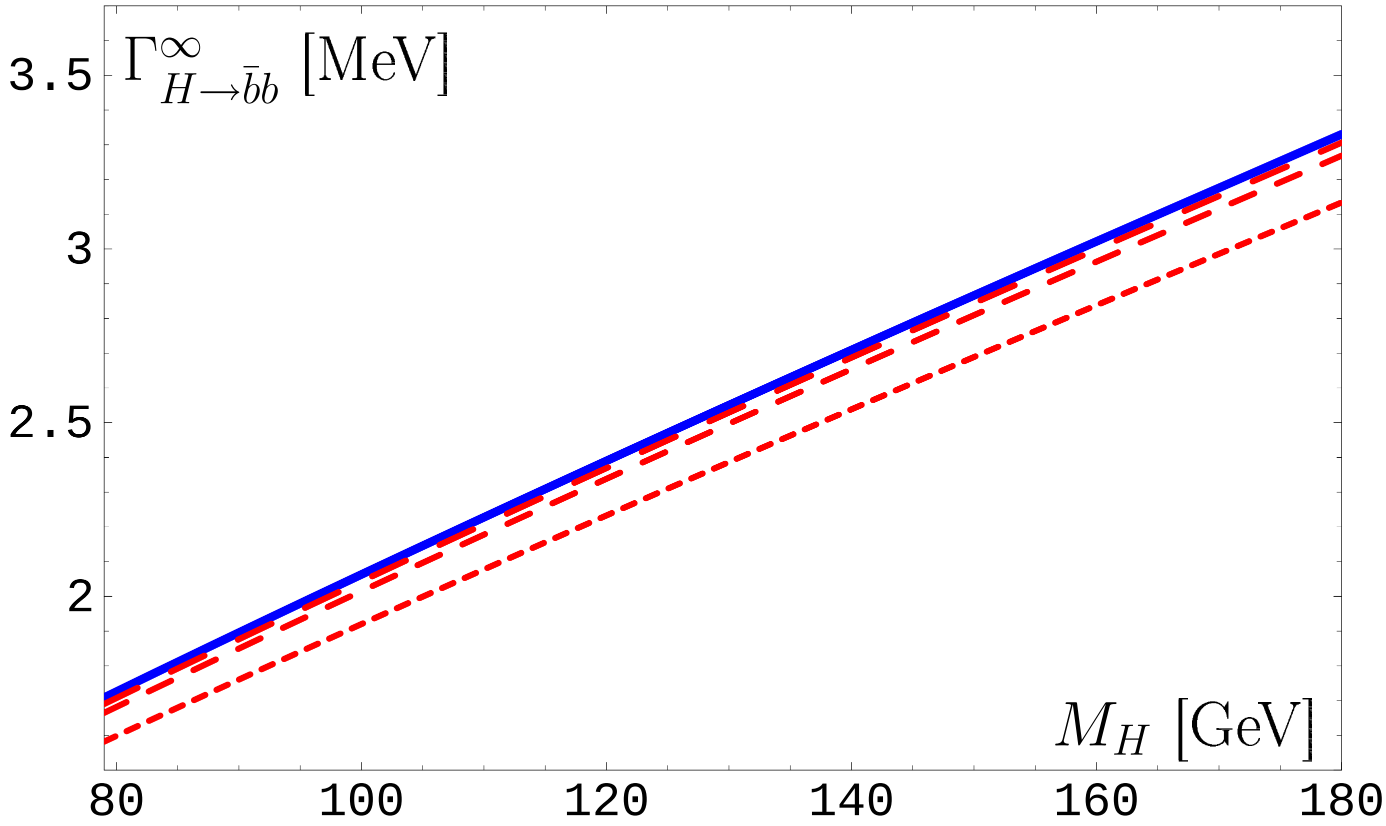
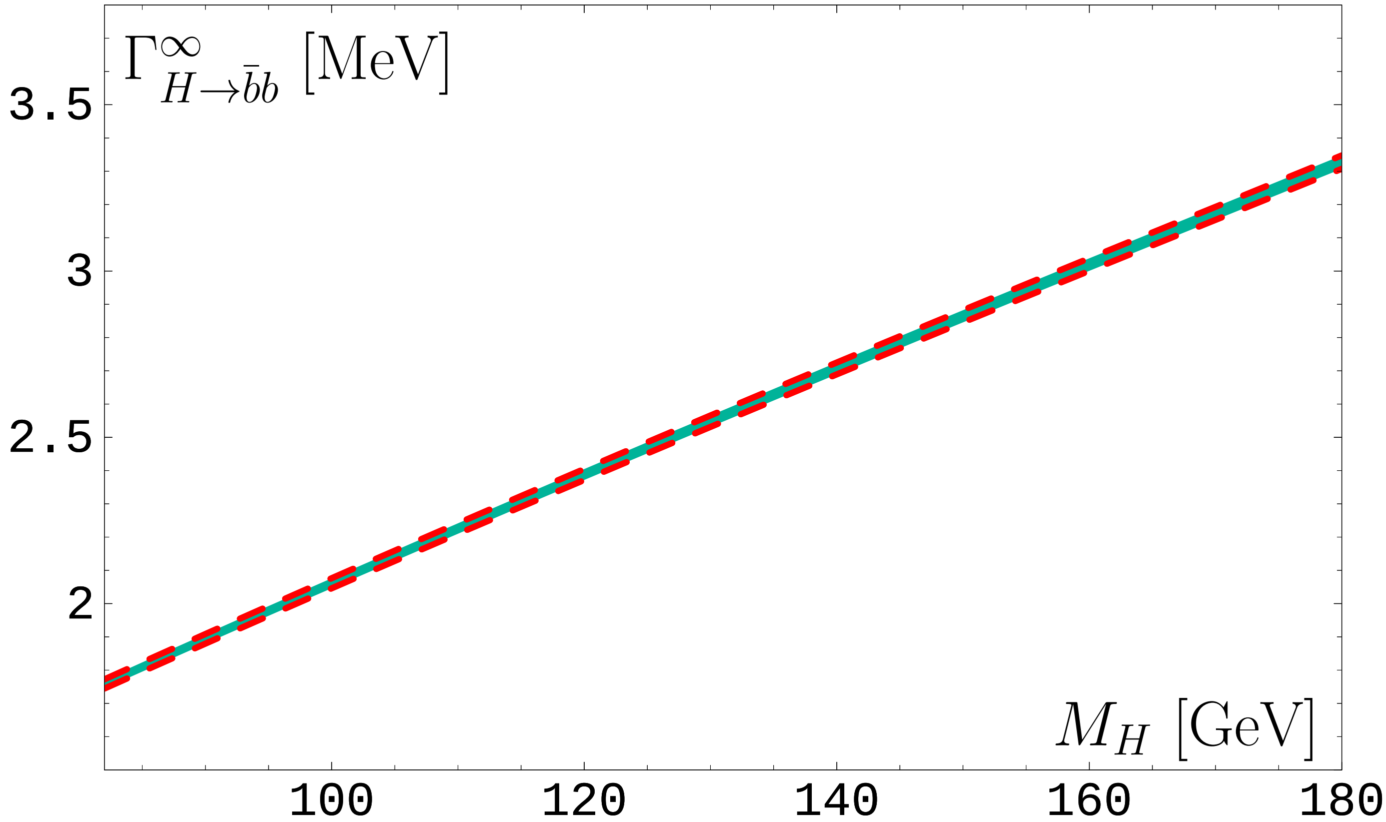
But profit will be tiny — instead of 0.5% one'll obtain 0.3%!



FAPT(M) for $\Gamma_{H\to\bar{b}b}(m_H)$: Truncation errors

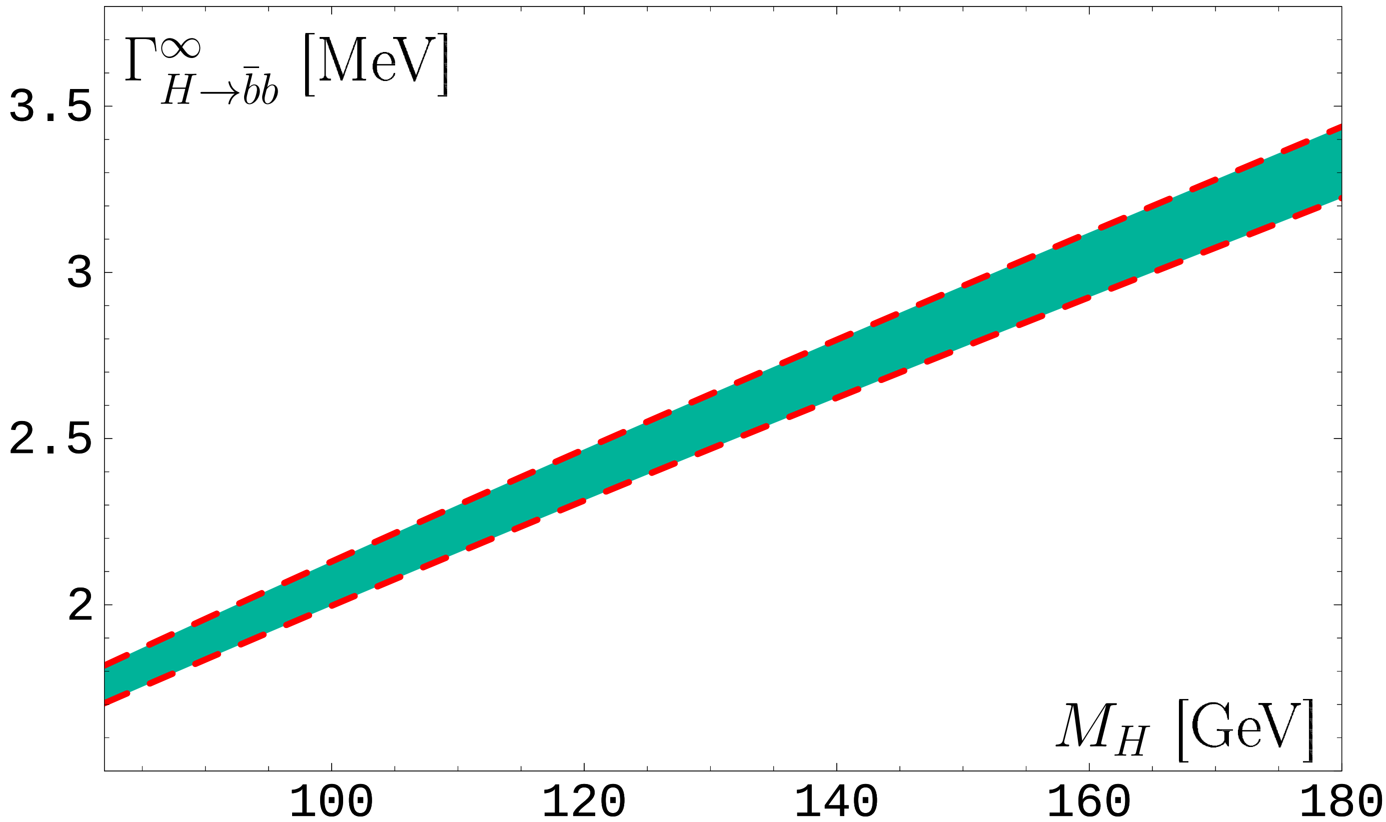
Conclusion: If we need accuracy of the order 0.5% — then we need to take into account up to the 4-th correction.

Note: uncertainty due to $P(t)$-modelling is small $\lesssim 0.6\%$.



FAPT(M) for $\Gamma_{H\to\bar{b}b}(m_H)$: Truncation errors

Conclusion: If we need accuracy of the order 1% — then we need to take into account up to the 3-rd correction — in agreement with Kataev&Kim **[0902.1442]**.
Note: RG-invariant mass uncertainty $\sim 2\%$.



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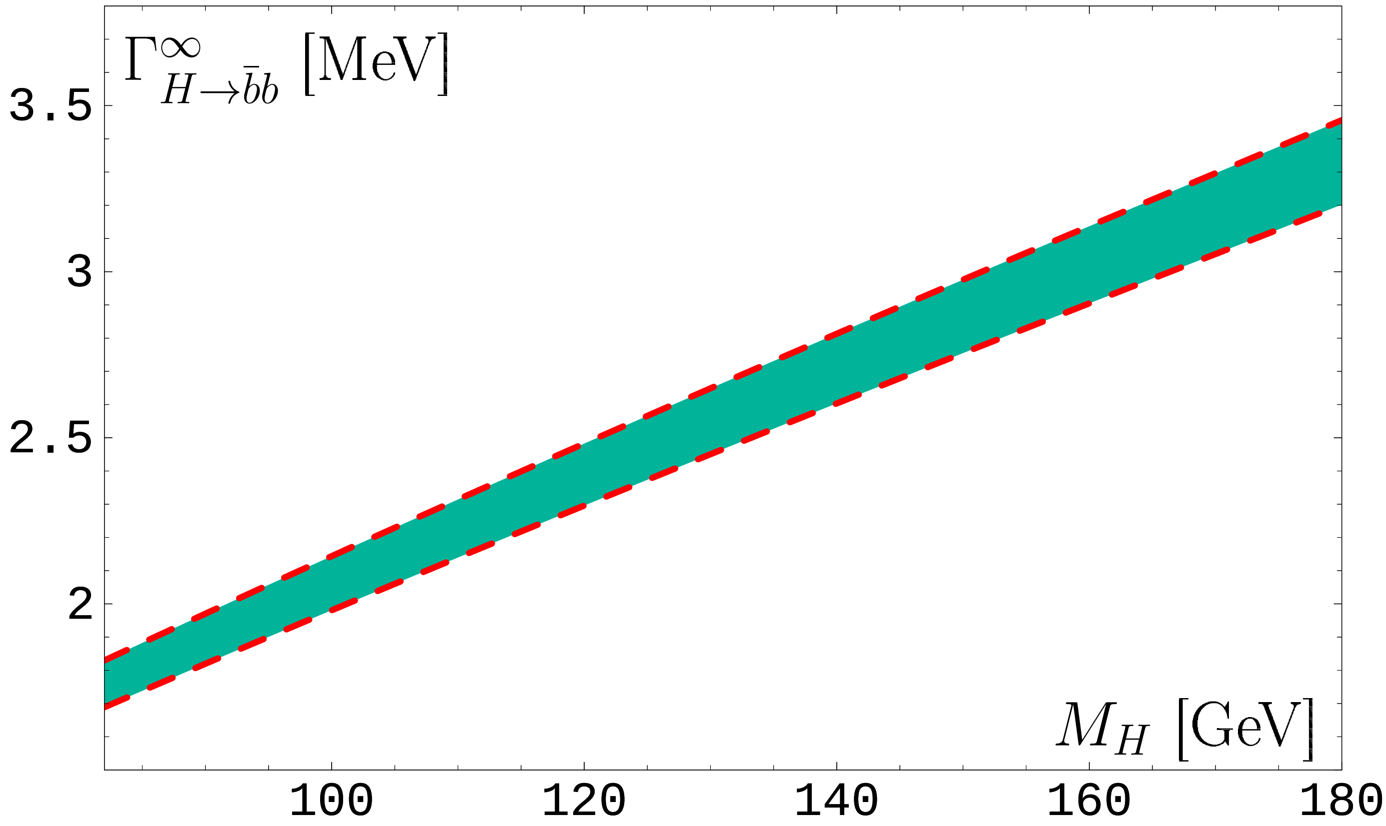
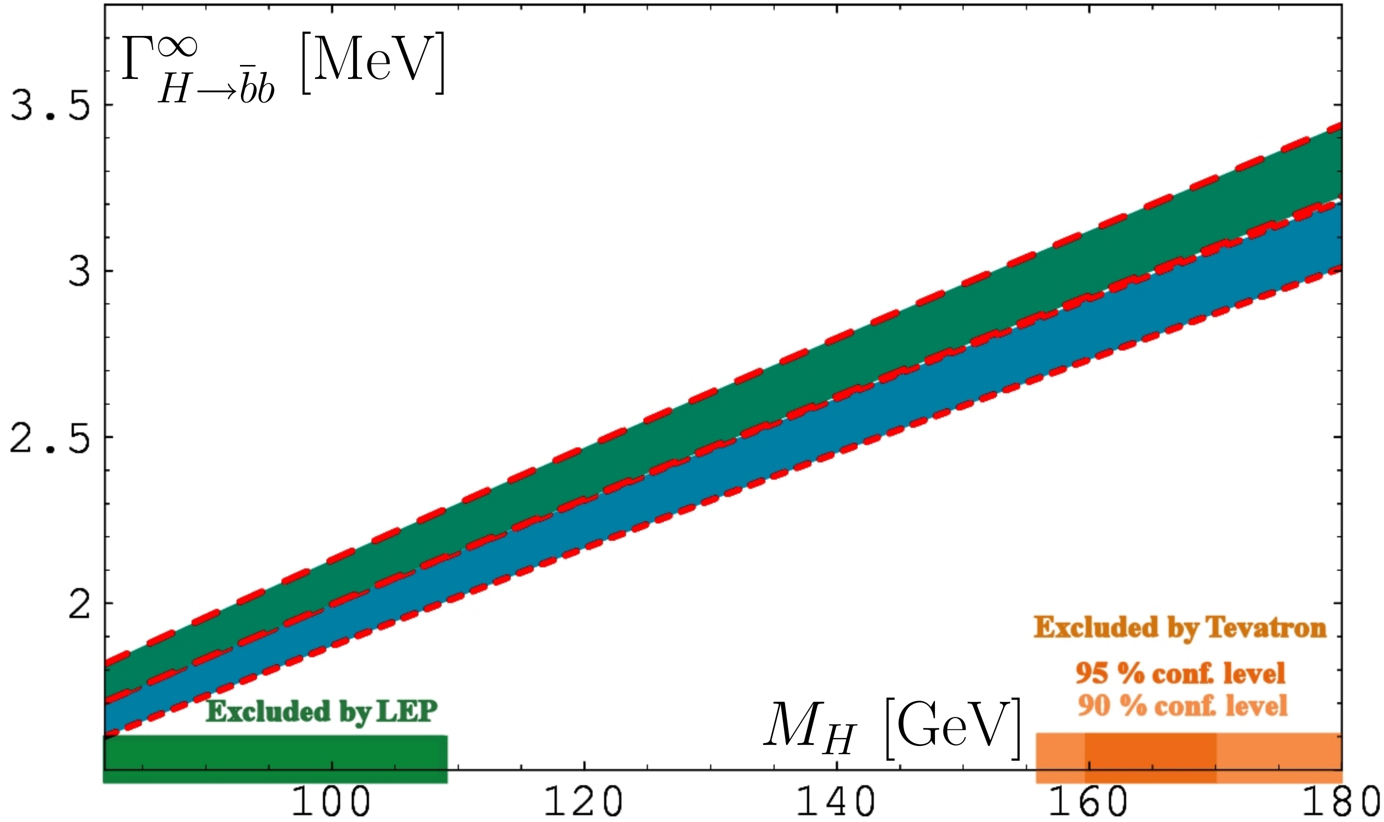
Note: overall uncertainty $\sim 3\%$.



Resummation for $\Gamma_{H\to\bar{b}b}(m_H)$: Loop orders

Comparison of 1- (**upper strip**) and 2- (**lower strip**) loop results.
We observe a 5% reduction of the two-loop estimate.



Resummation
for
Adler function $D(Q^2)$

Adler function $D(Q^2)$ in vector channel

Adler function $D(Q^2)$ can be expressed in QCD by means of the correlator of quark vector currents

$$\Pi_{\mathsf{V}}(Q^2) = \frac{(4\pi)^2}{3q^2}\, i \int dx\, e^{iqx} \langle 0|\, T[\, J_\mu(x) J^\mu(0)\,]\, |0\rangle$$

in terms of discontinuity of its imaginary part

$$R_{\mathsf{V}}(s) = \frac{1}{\pi}\,\mathsf{Im}\,\Pi_{\mathsf{V}}(-s - i\epsilon)\,,$$

so that

$$D(Q^2) = Q^2 \int_0^\infty \frac{R_{\mathsf{V}}(\sigma)}{(\sigma + Q^2)^2}\, d\sigma\,.$$

APT analysis of $D(Q^2)$ and $R_V(s)$

QCD PT gives us

$$D(Q^2) = 1 + \sum_{m>0} \frac{d_m}{\pi^m} \left(\frac{\alpha_s(Q^2)}{\pi}\right)^m .$$

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Model for perturbative coefficients

Coefficients d_m of the PT series:

Model	d_1	d_2	d_3	d_4	d_5
pQCD with $N_f = 4$	$\mathbf{1}$	$\mathbf{1.52}$	$\mathbf{2.59}$		—

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$c = 3.467,\ \beta = 1.325$	1	1.50	2.62		

We use model $d_n^{\text{mod}} = \dfrac{c^{n-1}(\beta^{n+1} - n)}{\beta^2 - 1}\,\Gamma(n)$

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"INNA" model	1	1.44	$[3, 9]$	$[20, 48]$	$[674, 2786]$

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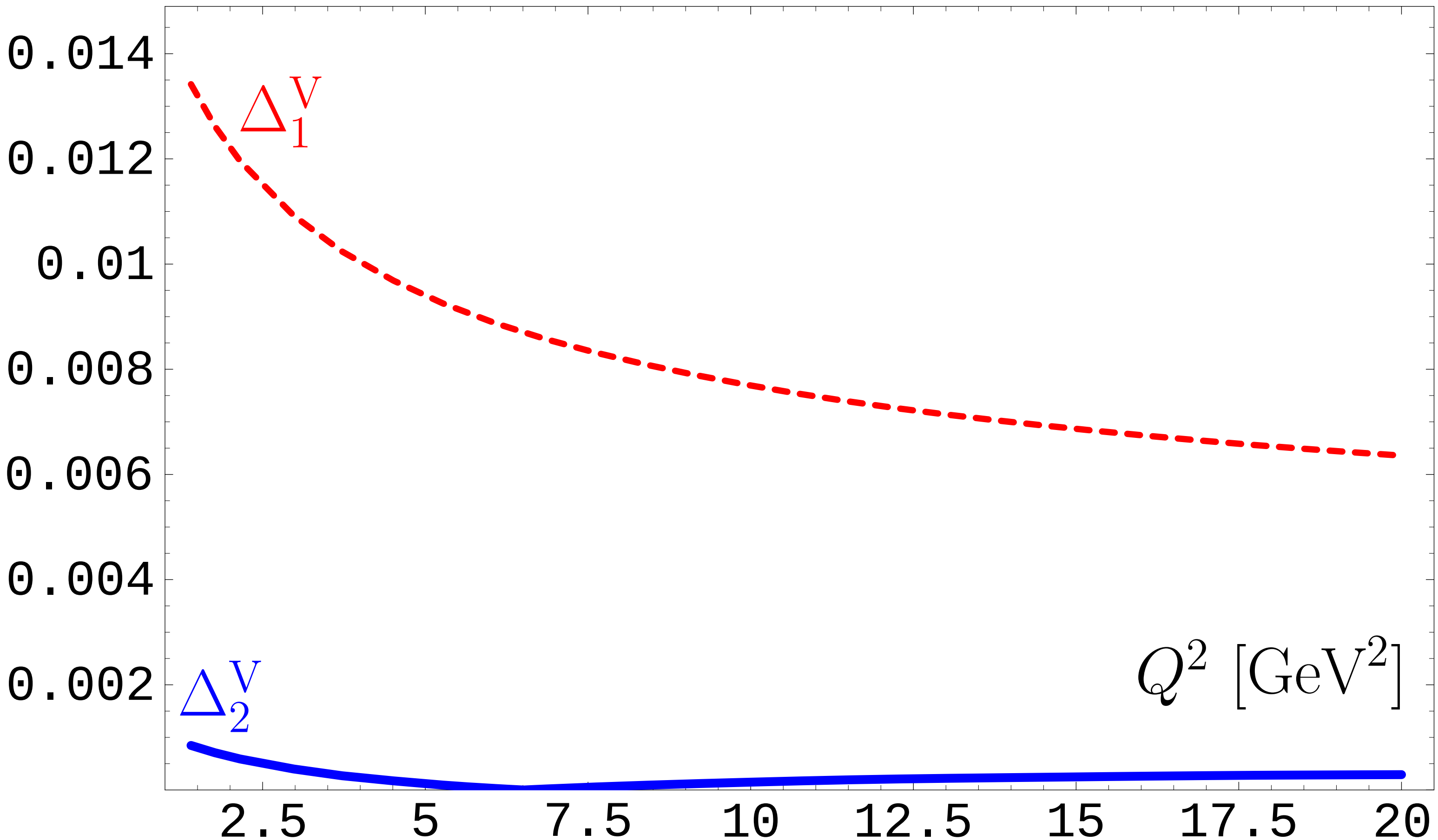

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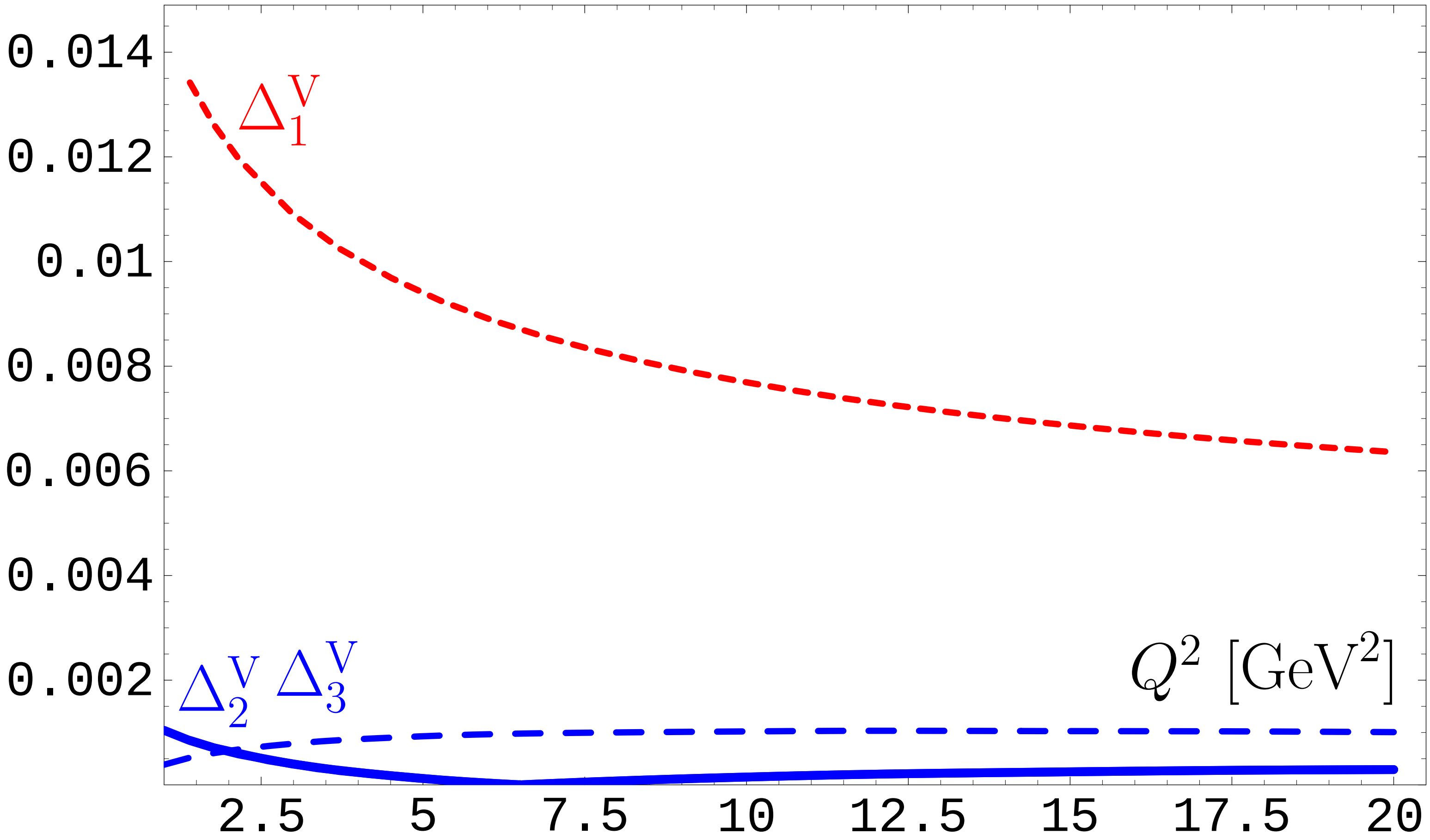
Conclusion: The best accuracy (better than 0.1%) is achieved for **N^{2}LO** approximation.



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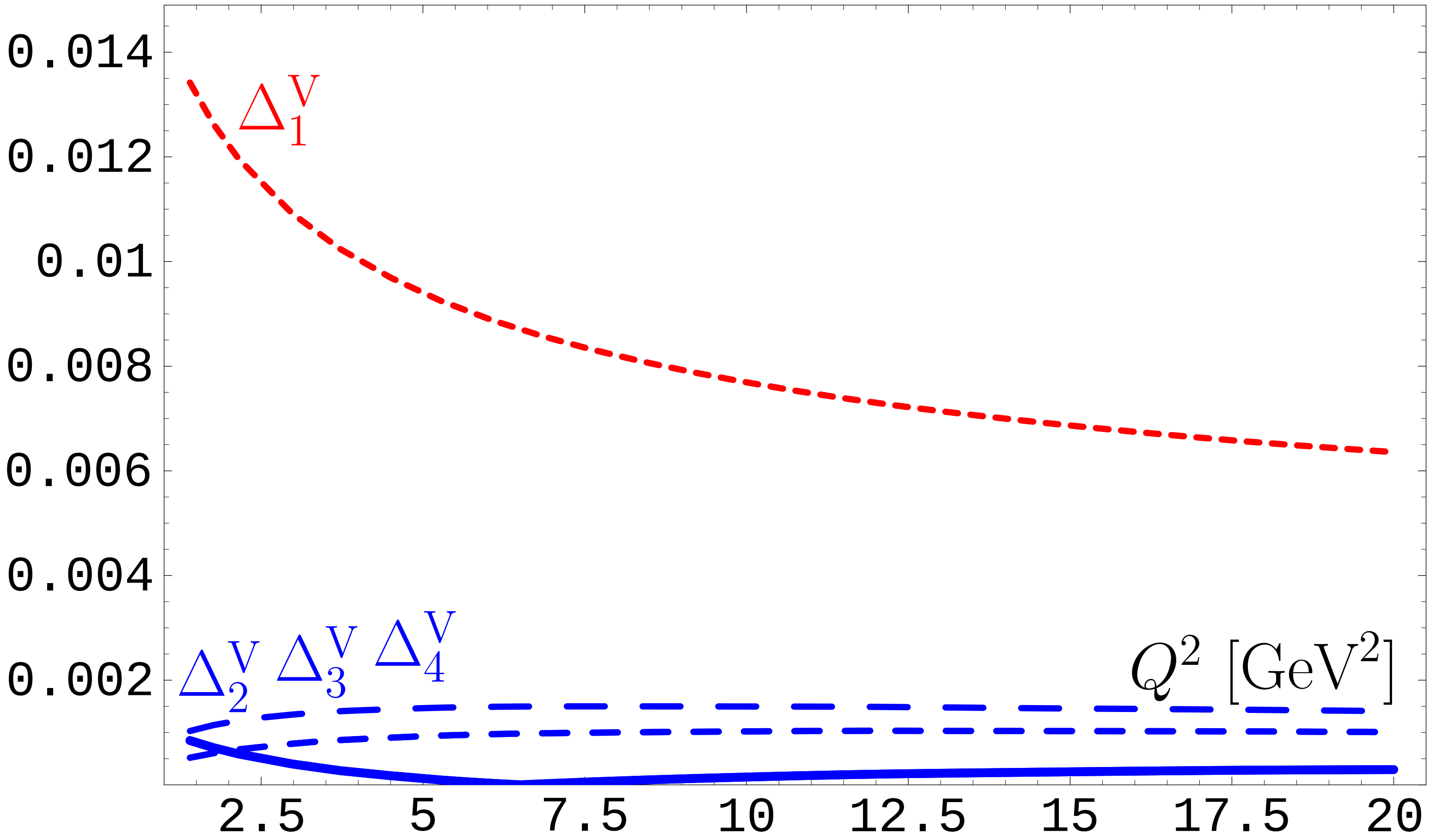
Conclusion: If we add more terms **N^{3}LO** — truncation error increases.



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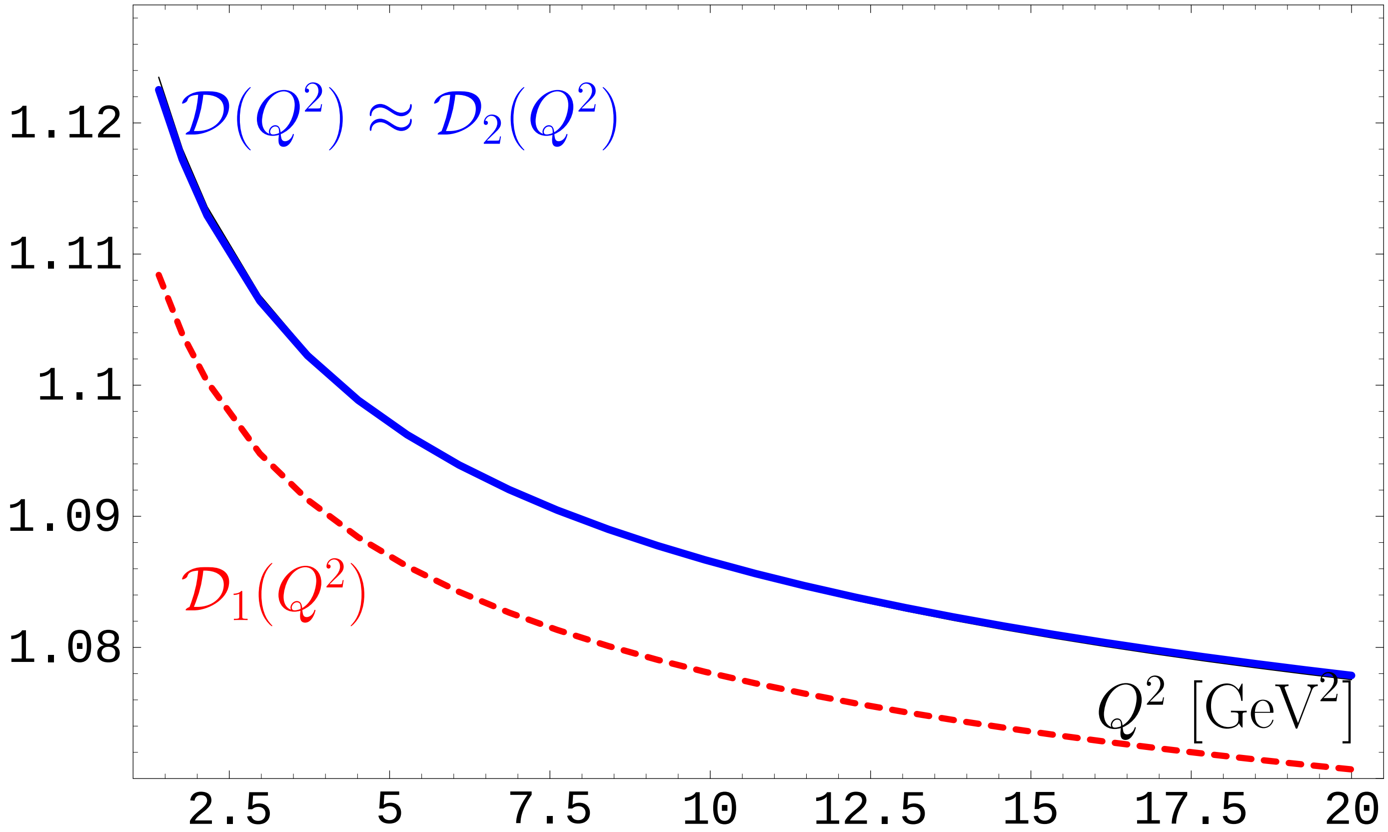

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We use model $d_n^{\text{mod}} = \frac{c^{n-1}(\beta^{n+1} - n)}{\beta^2 - 1}\Gamma(n)$ with parameters $\beta = 1.325$ and $c = 3.456$ estimated by known $\tilde{d}_n$ and with use of **Lipatov** asymptotics.

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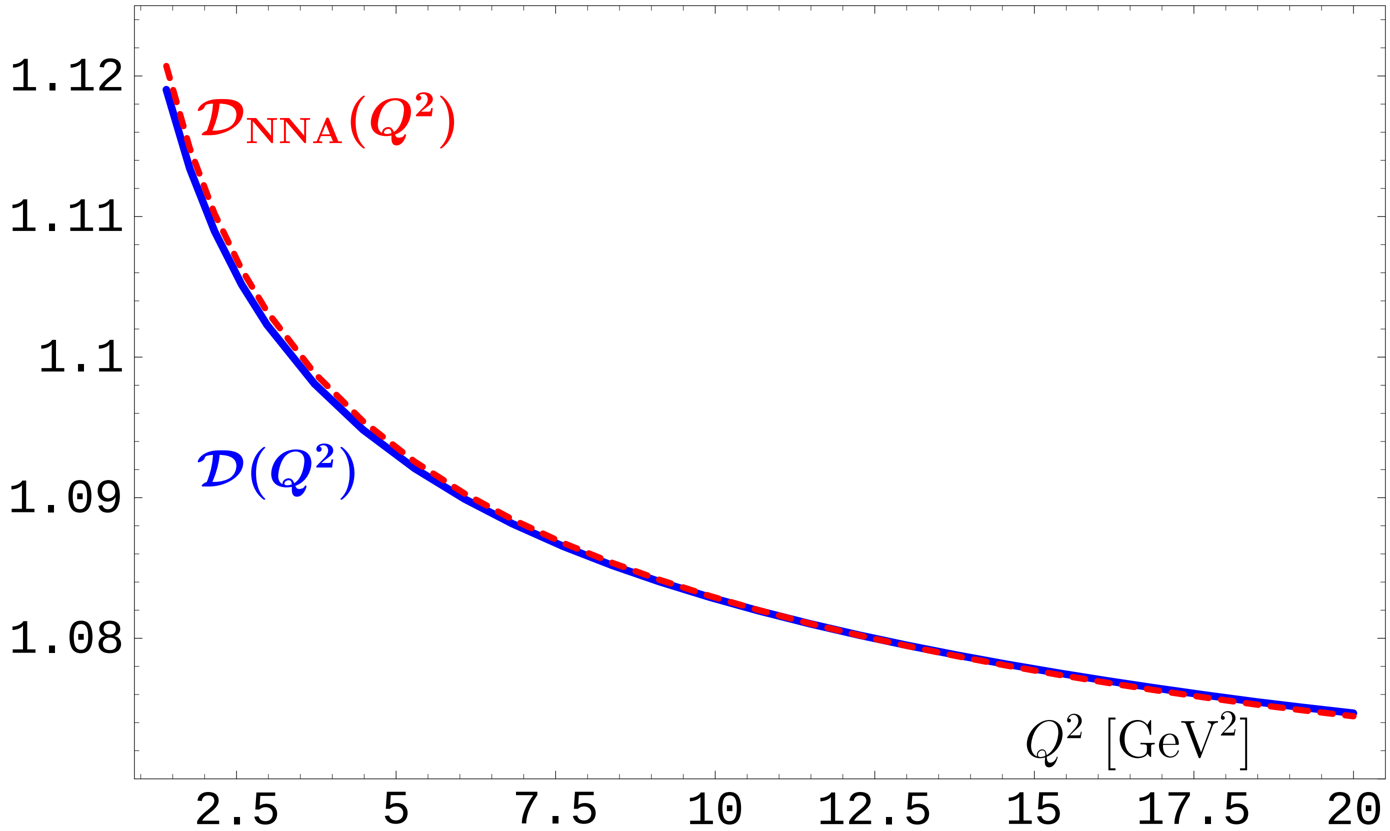
Conclusion: The result of resummation is stable to the variations of higher-order coefficients: deviation is of the order of 0.1%.



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- ...and for Adler function $\mathcal{D}(Q^2)$ — we have accuracy of the order 0.1% already at **N^{2}LO**.