The $Z \to c\bar{c} \to \gamma\gamma^*$, $Z \to b\bar{b} \to \gamma\gamma^*$ triangle diagrams and the $Z \to \gamma\psi$, $Z \to \gamma\Upsilon$ decays

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ABSTACT

It is expounded the approach to the $Z \to \gamma\Psi$ and $Z \to \gamma\Upsilon$ decay study, based on the sum rules for the $Z \to c\bar{c} \to \gamma\gamma^*$ and $Z \to b\bar{b} \to \gamma\gamma^*$ amplitudes and their derivatives.

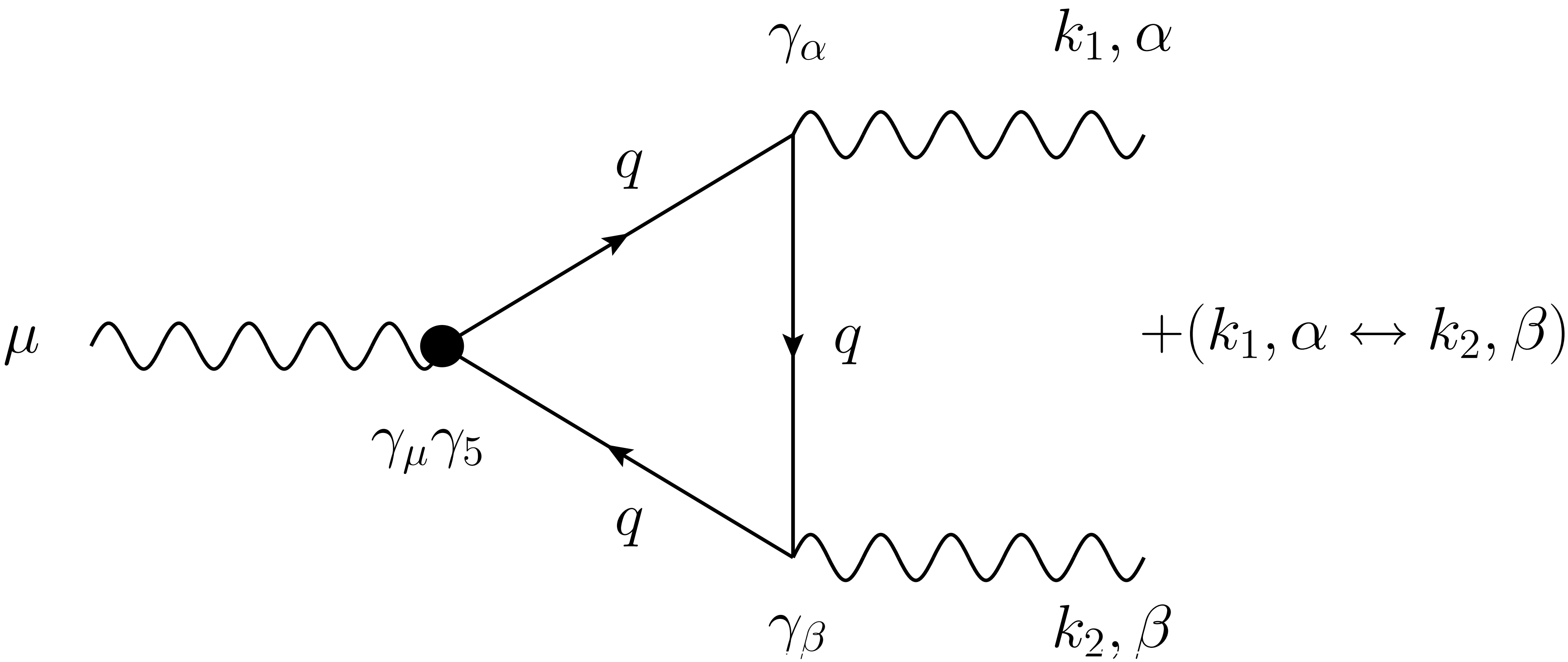
The branching ratios of the $Z \to \gamma\psi$ and $Z \to \gamma\Upsilon$ decays are calculated for different guesses as to saturation of the sum rules.

The angle distributions in the $Z \to \gamma\psi$ and $Z \to \gamma\Upsilon$ decays are calculated also.

OUTLINE

1. The invariant amplitudes of the triangle loop diagrams describing the transition of the axial-vector current $\to q\bar{q} \to \gamma(k_1)\gamma(k_2)$ at $k_1^2 = 0$ and $k_2^2 \neq 0$.
2. The sum rules for the $Z \to c\bar{c}$ (or $b\bar{b}$) $\to \gamma\gamma*$ amplitude.
3. The resonance saturation of the sum rule for the amplitude.
4. The resonance saturation of the sum rule for the amplitude derivative.
5. The simultaneous resonance saturation of the amplitude and its derivative.
6. Summary.
7. The angle distributions in the $Z \to \gamma\psi$ and $Z \to \gamma\Upsilon$ decays.

The Triangle Diagrams



$$T_{\alpha\beta\mu} = A_1 k_1^\sigma \epsilon_{\sigma\alpha\beta\mu} + A_2 k_2^\sigma \epsilon_{\sigma\alpha\beta\mu} + A_3 k_{1\beta} k_1^\delta k_2^\sigma \epsilon_{\delta\sigma\alpha\mu}$$
$$+ A_4 k_{2\beta} k_1^\delta k_2^\sigma \epsilon_{\delta\sigma\alpha\mu} + A_5 k_{1\alpha} k_1^\delta k_2^\sigma \epsilon_{\delta\sigma\beta\mu} + A_6 k_{2\alpha} k_1^\delta k_2^\sigma \epsilon_{\delta\sigma\beta\mu}$$

The Triangle Diagrams

The local gauge invariance

$$k_1^\alpha T_{\alpha\beta\mu} = k_2^\beta T_{\alpha\beta\mu} = 0$$

is ensured by the next constraints:

$$A_1 = k_2^2 A_4 + (k_1 k_2) A_3, \qquad A_2 = k_1^2 A_5 + (k_1 k_2) A_6.$$

Besides that

$$A_3(k_1, k_2) = -A_6(k_2, k_1), \qquad A_4(k_1, k_2) = -A_5(k_2, k_1).$$

A_3, A_4, A_5 and A_6 are the invariant amplitudes free of kinematical singularities. They are well-defined and can be calculated in the analytic form if $k_1^2 = 0$ (or $k_2^2 = 0$).

The Triangle Diagrams

Let us consider the region $k_1^2 = 0$, $Q^2 = -k_2^2 = -E^2 > 0$, $W^2 = -M^2 = -(k_1 + k_2)^2 > 0$ which is suitable for the calculations with the help of the dispersion relations over M^2 (and over E^2). Here is the result of this calculation:

$$A_3 = -A_6 = -\frac{1}{2\pi^2} \cdot \frac{1}{Q^2 - W^2}$$

$$\times \left\{ \frac{Q^2}{Q^2 - W^2} L_1 + \frac{m_q^2}{Q^2 - W^2} L_2 - 1 \right\},$$

$$A_4 = -\frac{1}{2\pi^2} \cdot \frac{1}{Q^2 - W^2} L_1,$$

The Triangle Diagrams

$$A_2 = \frac{1}{4\pi^2}\left\{\frac{Q^2}{Q^2-W^2}L_1 + \frac{m_q^2}{Q^2-W^2}L_2 - 1\right\},$$

$$A_1 = \frac{1}{4\pi^2}\left\{\frac{Q^2}{Q^2-W^2}L_1 - \frac{m_q^2}{Q^2-W^2}L_2 + 1\right\},$$

$$A_5 = -A_4 + \frac{3}{\pi^2}Q^2\frac{d}{dQ^2}\left[\frac{1}{Q^2-W^2}L_1\right] +$$

$$\frac{3}{2\pi^2}Q^4\left(\frac{d}{dQ^2}\right)^2\left[\frac{1}{Q^2-W^2}L_1\right] - -\frac{3}{4\pi^2}Q^2\frac{d}{dQ^2}\left[\frac{1}{Q^2-W^2}L_2\right]$$

$$+\frac{1}{2\pi^2}Q^2m_q^2\left(\frac{d}{dQ^2}\right)^2\left[\frac{1}{Q^2-W^2}L_2\right],$$ where

The Triangle Diagrams

$$L_1 = -\rho \ln \frac{\rho+1}{\rho-1} + \beta \ln \frac{\beta+1}{\beta-1} \ ,$$

$$L_2 = -\ln^2 \frac{\rho+1}{\rho-1} + \ln^2 \frac{\beta+1}{\beta-1} \ ,$$

$$\rho^2 = 1 + \frac{4m_q^2}{W^2}, \qquad \beta^2 = 1 + \frac{4m_q^2}{Q^2} \ .$$

Note that A_5 and A_4 do not contribute into physical values directly (not through the relations (3)) because $k_{1\alpha}$ and $k_{2\beta}$ in Eq. (1) are contracted either with the polarization vectors $(k_{1\alpha} e^{\alpha}(k_1)) = 0$ and $(k_{2\beta} e^{\beta}(k_2)) = 0$ or with the conserved currents $(k_{1\alpha} j^{\alpha}(k_1)) = 0$ and $(k_{2\beta} j^{\beta}(k_2)) = 0$.

The Triangle Diagrams

i) $\mathbf{0 < -W^2 = M^2 < 4m_q^2}$:

$$\rho \to i\sqrt{-\rho^2}, \quad \frac{1}{2}\ln\frac{\rho+1}{\rho-1} \to -i\,\arctan\frac{1}{\sqrt{-\rho^2}}\ ,$$

$2m_q < M$:

$$\sqrt{-\rho^2} \to -i\rho, \quad \arctan\frac{1}{\sqrt{-\rho^2}} \to \frac{\pi}{2} + \frac{i}{2}\ln\frac{1+\rho}{1-\rho}.$$

ii) $\mathbf{0 < -Q^2 = E^2 < 4m_q^2}$:

$$\beta \to i\sqrt{-\beta^2}, \quad \frac{1}{2}\ln\frac{\beta+1}{\beta-1} \to -i\,\arctan\frac{1}{\sqrt{-\beta^2}}\ ,$$

$\mathbf{2m_q < E}$:

$$\sqrt{-\beta^2} \to -i\beta, \quad \arctan\frac{1}{\sqrt{-\beta^2}} \to \frac{\pi}{2} + \frac{i}{2}\ln\frac{1+\beta}{1-\beta}.$$

$Z \to \gamma\,(\psi/\Upsilon)$ in dispersion approach

Let us calculate the amplitude for $Z \to \gamma(k_1)\gamma^*(k_2)$ for $Z \to c\bar{c} \to \gamma\gamma^*$ or $Z \to b\bar{b} \to \gamma\gamma^*$ at $0 \le k_2^2 = E^2 \le 4m_q^2$ $(k_1^2 = 0)$ in the Z boson rest frame, neglecting $\sim (E/M)^2$,

$$T\Big(Z \to q\bar{q} \to \gamma\gamma^*\Big) = M^2 E t_q \Big(\vec{n}\cdot\vec{e}(\gamma^*)\Big)\Big(\vec{n}\cdot\Big[\vec{e}(\gamma)\times\vec{e}(Z)\Big]\Big)\,,$$

where $M \equiv M_Z$; $\vec{n} = \vec{k}_1/\left|\vec{k}_1\right|$; $\vec{e}(Z)$ and $\vec{e}(\gamma^*)$ are the polarization three-vectors of the Z boson and the γ^* quantum in their rest frames ; $\vec{e}(\gamma)$ is the polarization three-vector of the γ quantum. The amplitude t_q takes into account three colors.

$$t_q = -\sigma_q \frac{3}{4}\cdot\frac{e^3 e_q^2}{\sin 2\Theta_W}\,(A_4 + A_6)\,,$$

where $\sigma_c = 1, \sigma_b = -1, e_c = 2/3, e_b = -1/3$.

$Z \to \gamma\,(\psi/\Upsilon)$ in dispersion approach

The t_q amplitude satisfies a dispersion relation without subtractions both in M^2 and in E^2. Consequently, t_q is the amplitude convenient for obtaining sum rules in the E^2 channel. It is most convenient to derive them with the help of the following consideration. The amplitude t_q describes the full amplitude for $Z \to q\bar{q} \to \gamma\gamma^*$ in the region $E^2 \leq 0$ accurate up to higher corrections in QCD and the standard electroweak theory. On the other hand, the full amplitude for $Z \to q\bar{q} \to \gamma\gamma^*$ can be represented with the help of the intermediate hadronic states in the E^2 channel as the sum of resonance contributions and a continuum spectrum contribution:

$Z \to \gamma\,(\psi/\Upsilon)$ in dispersion approach

$$T\Big(Z \to q\bar{q} \to \gamma\gamma^*\Big) = M^2 E t_h^q \Big(\vec{n}\cdot\vec{e}(\gamma^*)\Big)\Big(\vec{n}\cdot\Big[\vec{e}(\gamma)\times\vec{e}(Z)\Big]\Big)$$

where

$$t_h^q = \sum_V \frac{m_V^2}{m_V^2 - E^2}\cdot\frac{e}{f_V} T_V^q + e T_{cont}^q.$$

V is a $(q\bar{q})$ vector quarkonium ; T_{cont}^q is the continuum contribution ($D\bar{D}, D^*\bar{D}, D\bar{D}^*, D^*\bar{D}^*, \cdots$ or $B\bar{B}, B^*\bar{B}, B\bar{B}^*, B^*\bar{B}^*, \cdots$).

There is every reason to believe that where $E^2 \approx 0$

$$t_h^q \approx t_q \approx -\frac{\sigma_q}{M^2}\frac{3e\alpha e_q^2}{2\sin 2\Theta_W}\left(i - \frac{2}{\pi}\ln\frac{M}{m_q} + \frac{1}{\pi} + \frac{\beta}{\pi}\ln\frac{\beta+1}{\beta-1}\right)$$

Let us consider the sum rule for the amplitude and its derivative

$Z \to \gamma\,(\psi/\Upsilon)$ in dispersion approach

$t_q\big|_{E^2=0} = t_h^q\big|_{E^2=0}$ and $\frac{d}{dE^2} t_q\Big|_{E^2=0} = \frac{d}{dE^2} t_h^q\Big|_{E^2=0}$, that is,

$$\sum_V \frac{1}{f_V} T_V^q + T_{cont}^q\Big|_{E^2=0} \equiv T_q(Res) + T_{cont}^q\Big|_{E^2=0} =$$

$$T_q \equiv -\sigma_q \frac{3\alpha e_q^2}{2\sin 2\Theta_W} \cdot \frac{1}{M^2}\left(i - \frac{2}{\pi}\ln\frac{M}{m_q} + \frac{3}{\pi}\right)$$ and

$$\sum_V \frac{1}{f_V m_V^2} T_V^q + \frac{d}{dE^2} T_{cont}^q\Big|_{E^2=0} = D_q(Res) + \frac{d}{dE^2} T_{cont}^q\Big|_{E^2=0}$$

$$= D_q \equiv \frac{\sigma_q}{M^2} \frac{\alpha e_q^2}{4\pi m_q^2 \sin 2\Theta_W}.$$ $$\Gamma(\mathrm{V} \to \mathrm{e}^+\mathrm{e}^-) = \frac{4\pi}{3}\frac{\mathrm{m_V}}{\mathrm{f_V^2}}\alpha^2$$

$Z \to \gamma\,(\psi/\Upsilon)$ in dispersion approach

$$\Gamma(Z \to \gamma V) \approx \frac{1}{24\pi} M^3 m_V^2 \left|T_V^q\right|^2 ,$$

$c1 \equiv J/\psi(1S) \equiv \psi(3097),\ c2 \equiv \psi(3686),\ c3 \equiv \psi(3770),$
$c4 \equiv \psi(4040),\ c5 \equiv \psi(4160),\ c6 \equiv \psi(4415),$

$$f_{c1} : f_{c2} : f_{c3} : f_{c4} : f_{c5} : f_{c6} = 1 : 1.7 : 5 : 2.9 : 3 : 3.7 ,$$

$b1 \equiv \Upsilon(9460),\ b2 \equiv \Upsilon(10023),\ b3 \equiv \Upsilon(10355),$
$b4 \equiv \Upsilon(10579),\ b5 \equiv \Upsilon(10860),\ b6 \equiv \Upsilon(11020),$

$$f_{b1} : f_{b2} : f_{b3} : f_{b4} : f_{b5} : f_{b6} = 1 : 1.5 : 1.8 : 2.4 : 2.2 : 3.5 ,$$

$$f_{c1} = 11.2, \quad f^2_{c1}/4\pi \equiv f^2_{J/\psi(1S)}/4\pi \equiv f^2_{\psi(3097)}/4\pi = 9.9 ,$$
$$\mathrm{f}_{\mathrm{b1}} = 39.7 , \quad \mathrm{f}^2_{\mathrm{b1}}/4\pi \equiv \mathrm{f}^2_{\Upsilon(1\mathrm{S})}/4\pi \equiv \mathrm{f}^2_{\Upsilon(9460)}/4\pi = 125.4 .$$

Sum rule for amplitude

Let us saturate initially the real part of the sum rule for the amplitude with the ground state, that is,

$$T_V^q = f_V Re(T_q)\,,\ \ \text{where}\,,\ V = J/\psi(1S),\ \ \Upsilon(1S).$$

Using $m_c = 1.27$ GeV, $m_b = 4.2$ GeV, $M = 91.19$ GeV, $\Gamma_Z = 2.5$ GeV, $\alpha = 1/137$, and $\sin 2\Theta_W = 0.84$, we find

$$BR(Z \to \gamma J/\psi(1S)) = 7.2 \cdot 10^{-6},$$

$$BR(Z \to \gamma \Upsilon(1S)) = 1.7 \cdot 10^{-5},$$

which are two orders of magnitude higher quark model predictions.

Let us saturate now the real part of the sum rule for the amplitude with the Ψ and Υ families

$$\sum_V \frac{1}{f_V} T_V^q \equiv T_q(Res) = Re(T_q)\,.$$

Sum rule for amplitude

The minimum of $\sum_V \Gamma(Z \to \gamma V)$ is reached when

$$T_V^q = \frac{1}{a_q f_V m_V^2} Re(T_q)\,,\quad \text{where} \quad a_q = \sum_V \frac{1}{f_V^2 m_V^2}\,,$$

$$\Gamma(Z \to \gamma V) = \frac{1}{24\pi} M^3 \left(Re(T_q)\right)^2 \left(f_V m_V a_q\right)^{-2}\,,$$ and

$$\min \sum_V \Gamma(Z \to \gamma V) = \frac{1}{24\pi} M^3 \left(Re(T_q)\right)^2 a_q^{-1}\,.$$

For the Ψ family ($a_c = 1.2 \cdot 10^{-3}$ GeV^{-2})

$$\min \sum_\Psi BR(Z \to \gamma\psi) = 5.05 \cdot 10^{-6},$$

$$BR(Z \to \gamma J/\psi(1S)) = 3.53 \cdot 10^{-6}.$$

For the Υ family ($a_b = 1.43 \cdot 10^{-5}$ GeV^{-2})

$$\min \sum_\Upsilon BR(Z \to \gamma\Upsilon) = 8.58 \cdot 10^{-6},$$

$$BR(Z \to \gamma\Upsilon(1S)) = 4.25 \cdot 10^{-6}.$$

Sum rule for amplitude

When the amplitude is saturated with the ground state

$$D_q(V) = \frac{1}{f_V m_V^2} T_V^q = \frac{1}{m_V^2} Re(T_q)$$

or the resonance family

$$D_q(Res) = \sum_V \frac{1}{f_V m_V^2} T_V^q = \frac{d_q}{a_q} Re(T_q)\,, \quad d_q = \sum_V \frac{1}{f_V^2 m_V^4}$$

$$D_c(J/\psi(1S)) = 0.10 Re(T_c)\ \text{GeV}^{-2} = 5.60 D_c$$

$$D_b(\Upsilon(1S)) = 0.01 Re(T_b)\ \text{GeV}^{-2} = 3.73 D_b$$

$$D_c(Res) = 0.1 Re(T_c)\ \text{GeV}^{-2} = 5 D_c\,,\ d_c = 1.1 \cdot 10^{-4}\ \text{GeV}^{-4}$$

$$D_b(Res) = 0.01 Re(T_b)\ \text{GeV}^{-2} = 3.4 D_b\,,\ d_b = 1.5 \cdot 10^{-7}\ \text{GeV}^{-4}$$

So, the saturation of the amplitudes with the ground states or the resonance families leads to the unacceptably large contributions of the resonances into the amplitude derivatives.

Sum rule for the amplitude derivative

The dispersion integral for T_q is due to the $2m_q \leq E \sim M_Z$ region, that is not a low energy one. Consequently, it is reasonable to study the sum rule for the amplitude derivative because the contribution of low-lying states in the dispersion integral for the amplitude derivative is enhanced as compared to their contribution to the amplitude itself. Note that 90% of the dispersion integral for D_q is determined by the region of low energies $2m_q \leq E \leq 6m_q$.

When the amplitude derivative is saturated with the ground state V, $V = J/\psi(1S),\ \Upsilon(1S)$,

$$T_V^q = f_V m_V^2 D_q\,,\ \text{then}$$

$$BR(Z \to \gamma J/\psi(1S)) = 2.31 \cdot 10^{-7},$$

$$BR(Z \to \gamma \Upsilon(1S)) = 1.24 \cdot 10^{-6}.$$

Sum rule for the amplitude derivative

When the amplitude derivative is saturated with the resonance family,

$$\sum_V \frac{1}{f_V m_V^2} T_V^q \equiv D_q(Res) = D_q\,,$$ **then**

$$T_V^q = \frac{1}{g_q f_V m_V^4} D_q\,,$$ **where** $$g_q = \sum_V \frac{1}{f_V^2 m_V^6}\,,$$ **and**

$$\min \sum_V \Gamma(Z \to \gamma V) = \frac{1}{24\pi} M^3 D_q^2 g_q^{-1}\,.$$

Sum rule for the amplitude derivative

For the ψ family ($g_c = 1.08 \cdot 10^{-5}$ GeV^{-6})

$$\min \sum_{\psi} BR(Z \to \gamma\psi) = 1.95 \cdot 10^{-7}$$

and

$$\mathrm{BR}(\mathrm{Z} \to \gamma \mathrm{J}/\psi(1\mathrm{S})) = 1.64 \cdot 10^{-7}\,,$$
$$BR(Z \to \gamma\psi(3686)) = 2.056 \cdot 10^{-8}\,,$$
$$\mathrm{BR}(\mathrm{Z} \to \gamma\psi(3770)) = 2 \cdot 10^{-9}\,,$$
$$BR(Z \to \gamma\psi(4040)) = 4 \cdot 10^{-9}\,,$$
$$\mathrm{BR}(\mathrm{Z} \to \gamma\psi(4160)) = 3 \cdot 10^{-9}\,,$$
$$BR(Z \to \gamma\psi(4415)) = 1.44 \cdot 10^{-9}\,.$$

Sum rule for the amplitude derivative

For the Υ family ($g_b = 1.52 \cdot 10^{-9}$ GeV^{-6})

$$\min \sum_{\Upsilon} BR(Z \to \gamma \Upsilon) = 7.23 \cdot 10^{-7}$$

and

$$\mathrm{BR}(\mathrm{Z} \to \gamma \Upsilon(1\mathrm{S})) = 4.27 \cdot 10^{-7}\,,$$
$$BR(Z \to \gamma \Upsilon(10023)) = 1.31 \cdot 10^{-7}\,,$$
$$\mathrm{BR}(\mathrm{Z} \to \gamma \Upsilon(10355)) = 7.5 \cdot 10^{-8}\,,$$
$$BR(Z \to \gamma \Upsilon(10579)) = 3.9 \cdot 10^{-8}\,,$$
$$\mathrm{BR}(\mathrm{Z} \to \gamma \Upsilon(10860)) = 3.7 \cdot 10^{-8}\,,$$
$$BR(Z \to \gamma \Upsilon(11020)) = 1.4 \cdot 10^{-8}.$$

Sum rule for the amplitude derivative

The branching ratios for the production of the ground states $\big(BR(Z \to \gamma J/\psi(1S)) = 1.64 \cdot 10^{-7}$ and $BR(Z \to \gamma \Upsilon(1S)) = 4.27 \cdot 10^{-7}\big)$ more or less agree with the quark model predictions $\big(\sim 3.4 \cdot 10^{-8}$ and $\sim 3.4 \cdot 10^{-7}\big)$.

It is appropriate to note here that the branching ratios of all $Z \to \gamma\psi$ decays depend sensibly on the c quark mass, $\sim (1/m_c)^4$, so that the replace of $m_c = 1.27$ GeV by $m_c = 1.5$ GeV halves each of them. In addition, the decrease of the resonance contribution to the amplitude derivative ($D_q(Res) \to yD_q$, $0 < y < 1$) results in the decrease all the branching ratios (results in the y^2 factor before each right-hand side).

Sum rule for the amplitude derivative

When the amplitude is saturated with the ground state

$$T_q(V) \equiv \frac{1}{f_V} T_V^q = m_V^2 D_q$$

or the resonance family

$$T_q(Res) \equiv \sum_V \frac{1}{f_V} T_V^q = \frac{d_q}{g_q} D_q(Res) = \frac{d_q}{g_q} D_q$$

$$T_c(J/\psi(1S)) = 9.59 D_c \text{ GeV}^2 = 0.18 Re(T_c)$$

$$T_b(\Upsilon(1S)) = 89.49 D_b \text{ GeV}^2 = 0.27 Re(T_b)$$

$$T_c(Res) = 10.2 D_c(Res) \text{ GeV}^2 = 10.2 D_c \text{ GeV}^2 = 0.19 Re(T_c)$$

$$T_b(Res) = 96.7 D_b(Res) \text{ GeV}^2 = 96.7 D_b \text{ GeV}^2 = 0.29 Re(T_b)$$

This results specify explicitly that the main body of T_c and T_b is saturated with the continuous spectrum. In addition, they corroborate our idea that the resonances do not contribute to $Im(T_q)$.

Sum rule for amplitude and its derivative

The simultaneous saturation of the amplitude and its derivative with the ground state is provided if only

$$Re(T_q)/D_q = m_V^2\ ,$$ **but in our case**

$$Re(T_c)/D_c = 53.688\ \text{GeV}^2 \neq m^2_{J/\psi(1S)} = 9.59\ \text{GeV}^2$$ **and**

$$Re(T_b)/D_b = 334\ \text{GeV}^2 \neq m^2_{\Upsilon(1S)} = 89.49\ \text{GeV}^2\ .$$

As for the simultaneous saturation of the amplitude and its derivative with the resonance family, it's quite another matter. Considering the resonance contributions in the sum rules for the amplitude, $T_q(Res)$, and its derivative, $D_q(Res)$, as the two constraints we find

$$\min \sum_V \Gamma(Z \to \gamma V) =$$

$$\frac{1}{24\pi} M^3 \cdot \frac{g_q T_q(Res^2)^2 + a_q D_q(Res)^2 - 2 d_q T_q(Res) D_q(Res)}{a_q g_q - d_q^2}$$

Sum rule for amplitude and its derivative

$$T_V^q = \frac{(g_q - d_q/m_V^2)T_q(Res) - (d_q - a_q/m_V^2)D_q(Res)}{f_V m_V^2 (a_q g_q - d_q^2)}$$

This is self-consistent for any $T_q(Res)$, $D_q(Res)$, and m_V^2:

$$\sum_V \frac{1}{f_V} T_V^q = T_q(Res)\,, \quad \sum_V \frac{1}{f_V m_V^2} T_V^q = D_q(Res)\,.$$

The minimum of $\min \sum_V \Gamma(Z \to \gamma V)$, i.e., the lower bound of $\sum_V \Gamma(Z \to \gamma V)$ is reached when

$$T_q(Res) = \frac{d_q}{g_q} D_q(Res).$$

Setting $D_q(Res) = D_q$, we revert to the saturation of the amplitude derivative with the resonance family.

Sum rule for amplitude and its derivative

Let us consider the deviation from the lower bound

$$T_q(Res) = \frac{d_q}{g_q} D_q(Res)\cdot(1+x) = \frac{d_q}{g_q} D_q\cdot(1+x)\,,$$ then

$$\min\sum_V \Gamma(Z\to\gamma V) = \frac{1}{24\pi} M^3 D_q^2 g_q^{-1}\left(1 + x^2\cdot\frac{d_q^2}{\Delta_q}\right)\,,$$

where $\Delta_q = a_q g_q - d_q^2$,

$$T_V^q = \frac{1}{g_q f_V m_V^4} D_q\left[1 + x\cdot\frac{d_q\left(g_q m_V^2 - d_q\right)}{\Delta_q}\right]\,,$$

$$\Gamma(Z\to\gamma V) = \frac{1}{24\pi} M^3 D_q^2\left(f_V m_V^3 g_q\right)^{-2}$$

$$\times\left[1 + 2x\cdot\frac{d_q\left(g_q m_V^2 - d_q\right)}{\Delta_q} + x^2\cdot\frac{d_q^2\left(g_q m_V^2 - d_q\right)^2}{\Delta_q^2}\right].$$

Sum rule for amplitude and its derivative

For the ψ family ($\Delta_c = 4.45 \cdot 10^{-10}$ GeV^{-8})

$$\min \sum_{\psi} BR(Z \to \gamma\psi) = 1.95 \cdot 10^{-7} \cdot (1 + x^2 \cdot 28.17)$$

and

$$\mathrm{BR}(\mathrm{Z} \to \gamma \mathrm{J}/\psi(1\mathrm{S})) = 1.64 \cdot 10^{-7} \cdot (1 - \mathrm{x} \cdot 4.29 + \mathrm{x}^2 \cdot 4.60)$$

$$BR(Z \to \gamma\psi(3686)) = 2.056 \cdot 10^{-8} \cdot (1 + x \cdot 17.4 + x^2 \cdot 76)$$

$$\mathrm{BR}(\mathrm{Z} \to \gamma\psi(3770)) = 2 \cdot 10^{-9} \cdot (1 + \mathrm{x} \cdot 20.79 + \mathrm{x}^2 \cdot 108.07)$$

$$BR(Z \to \gamma\psi(4040)) = 4 \cdot 10^{-9} \cdot (1 + x \cdot 32.24 + x^2 \cdot 259.8)$$

$$\mathrm{BR}(\mathrm{Z} \to \gamma\psi(4160)) = 3 \cdot 10^{-9} \cdot (1 + \mathrm{x} \cdot 37.58 + \mathrm{x}^2 \cdot 353)$$

$$BR(Z \to \gamma\psi(4415)) = 1.44 \cdot 10^{-9} \cdot (1 + x \cdot 49.4 + x^2 \cdot 611)$$

Sum rule for amplitude and its derivative

For the Υ family ($\Delta_b = 2.14 \cdot 10^{-16}$ GeV^{-8})

$$\min \sum_{\Upsilon} BR(Z \to \gamma\Upsilon) = 7.23 \cdot 10^{-7} \cdot (1 + x^2 \cdot 100.46)$$

and

$$\mathrm{BR(Z \to \gamma\Upsilon(1S)) = 4.27 \cdot 10^{-7} \cdot (1 - x \cdot 15.04 + x^2 \cdot 56.6)}$$

$$BR(Z \to \gamma\Upsilon(10023)) = 1.31 \cdot 10^{-7} \cdot (1 + x \cdot 7.74 + x^2 \cdot 15)$$

$$\mathrm{BR(Z \to \gamma\Upsilon(10355)) = 7.5 \cdot 10^{-8} \cdot (1 + x \cdot 21.8 + x^2 \cdot 119)}$$

$$BR(Z \to \gamma\Upsilon(10579)) = 3.9 \cdot 10^{-8} \cdot (1 + x \cdot 31.5 + x^2 \cdot 249)$$

$$\mathrm{BR(Z \to \gamma\Upsilon(10860)) = 3.7 \cdot 10^{-8} \cdot (1 + x \cdot 44.0 + x^2 \cdot 485)}$$

$$BR(Z \to \gamma\Upsilon(11020)) = 1.4 \cdot 10^{-8} \cdot (1 + x \cdot 51.3 + x^2 \cdot 658)$$

Sum rule for amplitude and its derivative

T_q is saturated if $x = 4.26$ for ψ family and $x = 2.45$ for Υ one.

$$\sum_{\psi} \mathrm{BR}(Z \to \gamma\psi) = 10^{-4},$$

$$\mathrm{BR}(Z \to \gamma J/\psi(1S)) = 1.1 \cdot 10^{-5}, \; \mathrm{BR}(Z \to \gamma\psi(3686)) = 3 \cdot 10^{-5},$$

$$BR(Z \to \gamma\psi(3770)) = 4 \cdot 10^{-6}, \; BR(Z \to \gamma\psi(4040)) = 1.9 \cdot 10^{-5},$$

$$\mathrm{BR}(Z \to \gamma\psi(4160)) = 2 \cdot 10^{-5}, \; \mathrm{BR}(Z \to \gamma\psi(4415)) = 1.6 \cdot 10^{-5},$$

and

$$\sum_{\Upsilon} \mathrm{BR}(Z \to \gamma\Upsilon) = 4.36 \cdot 10^{-4},$$

$$\mathrm{BR}(Z \to \gamma\Upsilon(1S)) = 1.28 \cdot 10^{-4}, \; \mathrm{BR}(Z \to \gamma\Upsilon(10023)) = 1.5 \cdot 10^{-5},$$

$$BR(Z \to \gamma\Upsilon(10355)) = 5.9 \cdot 10^{-5}, \; BR(Z \to \gamma\Upsilon(10579)) = 6.3 \cdot 10^{-5},$$

$$\mathrm{BR}(Z \to \gamma\Upsilon(10860)) = 1.12 \cdot 10^{-4}, \; \mathrm{BR}(Z \to \gamma\Upsilon(11020)) = 5.9 \cdot 10^{-5}.$$

But, saturation of T_q with the resonances only is bad idea and

$BR(Z \to \gamma\Upsilon(1S)) = 1.28 \cdot 10^{-4}$ contradicts to experiment.

Sum rule for amplitude and its derivative

When $x = -1$, the resonances do not contribute to T_q at all, then

$$\sum_{\psi} \mathrm{BR}(\mathrm{Z} \to \gamma\psi) = 5.69 \cdot 10^{-6}\,,$$

$$\mathrm{BR}(\mathrm{Z} \to \gamma \mathrm{J}/\psi(1\mathrm{S})) = 1.62 \cdot 10^{-6},\;\; \mathrm{BR}(\mathrm{Z} \to \gamma\psi(3686)) = 1.22 \cdot 10^{-6},$$

$$BR(Z \to \gamma\psi(3770)) = 1.7 \cdot 10^{-7},\;\; BR(Z \to \gamma\psi(4040)) = 9 \cdot 10^{-7},$$

$$\mathrm{BR}(\mathrm{Z} \to \gamma\psi(4160)) = 9.8 \cdot 10^{-7},\;\; \mathrm{BR}(\mathrm{Z} \to \gamma\psi(4415)) = 8 \cdot 10^{-7},$$

and

$$\sum_{\Upsilon} \mathrm{BR}(\mathrm{Z} \to \gamma\Upsilon) = 7.34 \cdot 10^{-5},$$

$$\mathrm{BR}(\mathrm{Z} \to \gamma\Upsilon(1\mathrm{S})) = 3.08 \cdot 10^{-5},\;\; \mathrm{BR}(\mathrm{Z} \to \gamma\Upsilon(10023)) = 1.3 \cdot 10^{-6},$$

$$BR(Z \to \gamma\Upsilon(10355)) = 7.4 \cdot 10^{-6},\;\; BR(Z \to \gamma\Upsilon(10579)) = 8.7 \cdot 10^{-6},$$

$$\mathrm{BR}(\mathrm{Z} \to \gamma\Upsilon(10860)) = 1.65 \cdot 10^{-5},\;\; \mathrm{BR}(\mathrm{Z} \to \gamma\Upsilon(11020)) = 8.7 \cdot 10^{-6}.$$

Sum rule for amplitude and its derivative

Zeros in $BR(Z \to \gamma J/\psi(1S)$ at $x = 0.466$ and in $BR(Z \to \gamma \Upsilon(1S)$ at $x = 0.133$ are striking. In this case

$$\sum_{\psi \neq J/\psi} BR(Z \to \gamma \psi) = 1.39 \cdot 10^{-6}, \quad \sum_{\Upsilon \neq \Upsilon(1S)} BR(Z \to \gamma \Upsilon) = 2.01 \cdot 10^{-6},$$

and

$$T_c(Res) = 0.28 T_c\,, \;\; T_b(Res) = 0.33 T_b\,.$$

The continues spectra dominate the saturation of the T_c and T_b amplitudes, but zeros

$$T^c_{J/\psi(1S)}|_{x=0.466} = 0 \;\; \text{and} \;\; T^b_{\Upsilon(1S)}|_{x=0.133} = 0$$

require a rather bizarre dynamics, as I believe.

Summary

As is evident from the foregoing, the lower bounds of $\sum_\psi BR(Z \to \gamma\psi) = 1.95 \cdot 10^{-7}$ and $\sum_\Upsilon BR(Z \to \gamma\Upsilon) = 7.23 \cdot 10^{-7}$ are reached in the case of the amplitude derivative saturation with the resonances. As this takes place, the branching ratios for the production of the ground states, $BR(Z \to \gamma J/\psi(1S)) = 1.64 \cdot 10^{-7}$ and $BR(Z \to \gamma\Upsilon(1S)) = 4.27 \cdot 10^{-7}$, more or less agree with the quark model predictions, $\sim 3.4 \cdot (10^{-8} - 10^{-7})$.

The angular distributions expected in the center-of-mass system of the $q\bar{q} \to Z \to \gamma V$ and $e^+e^- \to Z \to \gamma V$ reactions

$$W(\theta) = \frac{3}{8} \cdot \frac{1 + \cos^2\theta + (2m_q^2/M^2)\sin^2\theta}{1 + m_V^2/M^2} \approx \frac{3}{8}(1 + \cos^2\theta),$$

where θ is the angle between the γ quantum momentum and the beam axis. Details are below.

Angle distributions

If not to be interested in the photon and V meson polarizations

$$\mathbf{W}\left(\tilde{\mathrm{e}}(\mathrm{Z})\,,\tilde{\mathrm{n}}\right) = (3/4)\left(\left(\tilde{\mathrm{e}}(\mathrm{Z})^* \cdot \tilde{\mathrm{e}}(\mathrm{Z})\right) - \left(\tilde{\mathrm{n}} \cdot \tilde{\mathrm{e}}(\mathrm{Z})^*\right)\left(\tilde{\mathrm{n}} \cdot \tilde{\mathrm{e}}(\mathrm{Z})\right)\right),$$

$$W(S_z = 1\,,\ \theta) = W(S_Z = -1\,,\ \theta) = (3/8)(1 + cos^2\theta)\,,$$

$$W(S_z = 0\,,\ \theta) = (3/4)\sin^2\theta\,,$$

where S_z is the z component of the Z boson spin in its rest frame, θ is the angle between the γ quantum momentum and the z axis.

If to be interested in polarization of the photon only

$$W\left(\vec{e}(Z)\,,\vec{n}\,,\vec{e}(\gamma)\right) = (3/4)\left(\vec{n}\cdot\left[\vec{e}(\gamma)\times\vec{e}(Z)\right]\right)\left(\vec{n}\cdot\left[\vec{e}(\gamma)\times\vec{e}(Z)\right]\right)^*$$

$$W(S_z = 1\,,\ S_\gamma = +1\,,\ \theta) = W(S_z = -1\,,\ S_\gamma = -1\,,\ \theta)$$

$$= (3/16)(1 + \cos\theta)^2\,,$$

$$W(S_z = 1\,,\ S_\gamma = -1\,,\ \theta) = W(S_z = -1\,,\ S_\gamma = +1\,,\ \theta)$$

$$= (3/16)(1 - \cos\theta)^2\,,$$

$$W(S_z = 0\,,\ S_\gamma = +1\,,\ \theta) = W(S_z = 0\,,\ S_\gamma = -1\,,\ \theta) = (3/8)\sin^2\theta\,,$$

where S_γ is the photon helicity.

Angle distributions

Note that Z boson with $S_z = 0$ is not produced if the z axis is the axis of the e^+e^- or $q\bar{q}$ beams in their center-of-mass system.
In that event, the angular distributions in the $e^+e^- \to Z \to \gamma V$ and $q\bar{q} \to Z \to \gamma V$ reactions are

$$\mathbf{W^{e^+e^-}_{S_\gamma=\pm1}}(\theta) = \frac{3}{16\mathbf{N_e}}\left[(1/2-\xi)^2(1\mp\cos\theta)^2+\xi^2(1\pm\cos\theta)^2\right],$$

where $N_e = (1/2-\xi)^2+\xi^2$, $\xi = \sin^2\Theta_W = 0.23$, the z axis is put in the electron momentum direction,

$$W^{u\bar{u}}_{S_\gamma=\pm1}(\theta) = \frac{3}{16N_u}\left[(1/2-e_u\,\xi)^2(1\mp\cos\theta)^2+e_u^2\,\xi^2(1\pm\cos\theta)^2\right],$$

where $N_u = (1/2-e_u\,\xi)^2+e_u^2\,\xi^2$, $e_u = 2/3$; the z axis is put in the u quark momentum direction, and

$$W^{d\bar{d}}_{S_\gamma=\pm1}(\theta) = \frac{3}{16N_d}\left[(1/2-e_d\,\xi)^2(1\mp\cos\theta)^2+e_d^2\,\xi^2(1\pm\cos\theta)^2\right],$$

where $N_d = (1/2-e_d\,\xi)^2+e_d^2\,\xi^2$, $e_d = -1/3$; the z axis is put in the d quark momentum direction.