Multiloop Gluon Amplitudes and AdS/CFT

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Abstract

In this talk I review recent progress in the computation of multiloop gluon amplitudes in $\mathcal{N} = 4$ Yang-Mills at weak and strong coupling.

1 Introduction

Gluon scattering amplitudes in QCD and supersymmetric gauge theories are very difficult to compute, so this is fertile ground for new insights and methods. The development of new tools for computing scattering amplitudes is an important topic because the experimental program at the LHC will require many new calculations of QCD-associated processes.

Over the past several years we have learned a lot about remarkably rich mathematical structures in Yang-Mills theories. On the one hand, in the context of AdS/CFT a lot of work has been done exploring Yang-Mills integrable structures and computing anomalous dimensions. An exact S-matrix for planar $\mathcal{N} = 4$ Yang-Mills theory has been proposed in [1]. On the other hand, following the discovery of twistor string theory [2], we have seen a lot of progress in Yang-Mills scattering amplitudes. Of particular interest is the Bern, Dixon and Smirnov iterative relation for planar MHV amplitudes in $\mathcal{N} = 4$ [3]. A quantity known as the cusp anomalous dimension is a player in both games. Alday and Maldacena proposed a prescription for computing scattering amplitudes using AdS string theory at strong coupling [4].

2 Cusp Anomalous Dimension and Integrability

Integrability is a very powerful tool for computing anomalous dimensions of operators. Of particular importance is the cusp anomalous dimension $f(\lambda)$, which governs the behavior of twist-two operators in the limit of very large spin

$$\Delta\left(\mathrm{Tr}[ZD^SZ]\right) = S + f(\lambda)\log S + \mathcal{O}(S^0), \qquad S \gg 1.$$

This quantity has long played an important role in quantitative checks of AdS/CFT. Via the Bethe ansatz equations, the $\mathcal{N} = 4$ Yang-Mills S-matrix gives the entire Yang-Mills spectrum in the large J limit and in particular, it implies a (complicated) integral equation $f(\lambda)$ valid for all λ [1].

At strong coupling, the solution exhibits remarkable agreement with the AdS string [5]

$$f(\lambda) = \frac{\sqrt{\lambda}}{\pi}(1 - \frac{3\log 2}{\sqrt{\lambda}} - \frac{K}{\lambda} + \cdots),$$

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At weak coupling, the solution agrees with perturbative Yang-Mills up to four loops [6, 7]

$$f(\lambda) = \frac{\lambda}{2\pi^2}\left(1 - \frac{\lambda}{48} + \frac{11\lambda^2}{11520} - \left(\frac{73}{1290240} + \frac{\zeta_3^2}{512\pi^6}\right)\lambda^3 + \cdots\right)$$

The rest of the talk will be devoted to the calcualtion of this and other quantities from perturbative scattering amplitudes in $\mathcal{N} = 4$ Yang-Mills.

3 Gluon Scattering Amplitudes

We have learned that Feynman diagrams are not the most efficient way to calculate scattering amplitudes: the final expressions are too messy, have too many terms which moreover hide the structure of amplitudes. There has been a lot of progress on tree amplitude calculations stimulated by twistor string theory [2]. All tree level perturbative amplitides are under control. In the $\mathcal{N} = 4$ theory, all one-loop integrals which appear in any Feynman diagram calculation can be reduced to a set of scalar box integrals [8]. Unitarity methods can be used to determine the coefficients for a desired amplitude [9] at one-loop.

Unitarity based methods for computing the coefficients can be generalized to higher loop amplitudes [10, 3, 11]. The only problem is that the complete basis of integrals is not known even for all two-loop amplitudes. For example, the two-loop four-particle amplitude is given by the sum of only two scalar integrals [12], but in general it is very difficult to determine which integrals contribute to any particular amplitude.

At two and three loops the integrals contributing to four-point amplitude are given by the rung rule [12]. However at four loops the amplitude contains two additional integrals [6]. It seems that non-rung rule contributions can be determined by dual conformality–conformal invariance in momentum space [13].

4 Iterative Ansatz for Multiloop Amplitudes

In planar $\mathcal{N} = 4$ Yang-Mills, MHV amplitudes $M_n^{(L)}(\epsilon) = \frac{A_n^{(L)}}{A_n^{(0)}}$ computed to date satisfy an iteration relation. At two loops, the iteration conjecture expresses n-point amplitudes entirely in terms of one-loop amplitudes and a set of numerical constants. For two-loop MHV amplitudes, the Anastasiou, Bern, Dixon and Kosower conjecture reads

$$M_n^{(2)}(\epsilon) = \frac{1}{2}(M_n^{(1)}(\epsilon))^2 + f^{(2)}(\epsilon)M_n^{(1)}(2\epsilon) + C^{(L)} + \mathcal{O}(\epsilon),$$

where $f^{(2)}(\epsilon) = -(\zeta_2 + \zeta_3\epsilon + \zeta_4\epsilon^2 + \cdots)$, $C^{(2)} = -\zeta_2^2/2$, in dimensional regulation to $D = 4 - 2\epsilon$. Explicit two-loop computations have shown that this relation holds for four [14] and five particles [15].

This iterative structure, together with the exponential nature of IR divergences, suggests an all-orders resummation should be possible. Guided by an explicit calculation of the three-loop four-particle amplitude, Bern, Dixon, and Smirnov were led to the all-loop order BDS proposal

$$\ln M_n = \sum_{l=1}^{\infty} a^l (f^{(l)}(\epsilon)M_n^{(1)}(l\epsilon) + C^{(l)} + \mathcal{O}(\epsilon)) \tag{1}$$

where

$$M_n = \sum_{L=0}^{\infty} a^L M_n^{(L)}(\epsilon), \qquad f^{(l)}(\epsilon) = f_0^{(l)} + \epsilon f_1^{(l)} + \epsilon^2 f_2^{(l)}, \qquad a = \frac{\lambda}{8\pi^2}(4\pi e^{-\gamma})^{\epsilon}. \tag{2}$$

The iterative relations imply that vast majority of the rational coefficients which specify the ϵ expansion of the L-loop amplitude are completely determined in terms of those of lower loop amplitudes. The numbers at orders $1/\epsilon^2$ and $1/\epsilon$ are the L-loop cusp $f(\lambda)$ and collinear $g(\lambda)$ anomalous dimensions respectively, which have been computed perturbatively up to four-loops [6, 7, 16].

5 MHV Amplitudes and Wilson Loops

Recently Alday and Maldacena have given a prescription for using AdS/CFT to calculate gluon scattering amplitudes at strong coupling [4]. The prescription is computationally equivalent to evaluating a certain Wilson loop composed of null line segments. Their calculation confirmed the strong coupling prediction from the BDS iteration ansatz for the four-point amplitude.

The authors [17] showed that the lowest-order contributions to a light-like rectangular Wilson loop agree with the BDS ansatz for gauge theory four-particle amplitudes. This relation between MHV amplitudes and Wilson loops is very surprising. It works for four and five-point amplitudes at one and two loops. But four and five-point results are fixed by dual conformal symmetry [18]. What happens with a larger number of legs? It seems very difficult to find explicit string solutions beyond four points corresponding to strong coupling. Instead, Alday and Maldacena have shown that in the limit of a large number of legs, the Wilson loop calculation does not agree with the BDS ansatz [18].

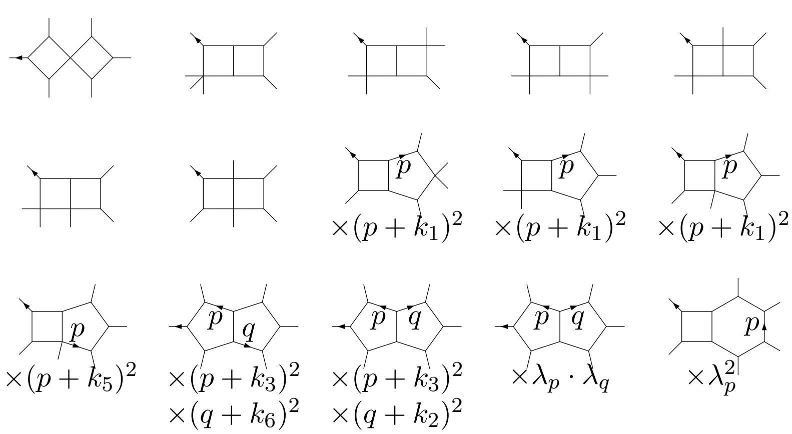
In the limit of very large T and L and for $T >> L$, the expectation value of the rectangular Wilson loop

$$\log\langle W_{\text{rect}}^{AM}\rangle = \frac{4\sqrt{\lambda}\pi^2}{\Gamma(1/4)^4}\frac{T}{L} \neq \log\langle W_{\text{rect}}^{BDS}\rangle = \frac{\sqrt{\lambda}}{4}\frac{T}{L}.$$

Evidently either the connection between Wilson loops and the amplitudes breaks down, or the BDS ansatz for amplitudes breaks down beyond, possibly beginning with the six-point amplitude. To answer this question one needs a calculation of the two-loop six-point Wilson loop and amplitude.

Recently we found the complete expression for the parity-even part of the two-loop six-particle amplitude [19]. We performed the calculation using the unitarity-based method, employing a variety of cuts to express the amplitude in terms of selected set of six-point two-loop Feynman integrals depicted below. We evaluated the integrals using the AMBRE and MB packages and computed the amplitude numerically against BDS ansatz, and against values for the corresponding Wilson loop.

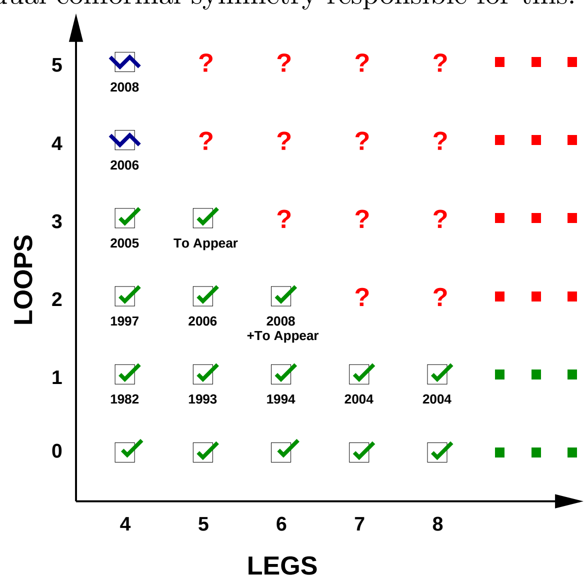
We found the discrepancy with BDS ansatz, but precise agreement with Wilson loop calculations [20]. This reslt provides non-trivial confirmation of a proposed n-point equivalence between Wilosn loops and planar MHV amplitudes, and suggests that an additional mechanism besides dual conformal symmetry fixes their form at six points and beyond.



kinematics	(u_1, u_2, u_3)	Δ_A	Δ_W
$K^{(1)}$	$(1/4, 1/4, 1/4)$	-0.0181 ± 0.017	$< 10^{-5}$
$K^{(2)}$	$(0.547253,\ 0.203822,\ 0.88127)$	-2.753 ± 0.012	-2.7553
$K^{(3)}$	$(28/17, 16/5, 112/85)$	-4.74445 ± 0.00653	-4.7446
$K^{(4)}$	$(1/9, 1/9, 1/9)$	4.1161 ± 0.10	4.0914
$K^{(5)}$	$(4/8, 4/81, 4/81)$	9.9963 ± 0.50	9.7255

6 Conclusions

How far does one need to calculate before unlocking all the structure? How much is gained by adding one more loop, or one more leg? Amazingly, almost every new calculation so far has led to a new surprise! In the case of loops, there were strong reasons to suspect that special things would start happening at four loops (and they did!) so there was great interest in the calculation of the four-loop cusp anomalous dimension. At five loops it would perhaps be interesting to see cancellation of $\zeta(6,2)$ which generically appear in the amplitudes but does not appear in the solution of the BES equation. In the case of legs, starting at six-points the BDS ansatz breaks down while Wilson loop/amplitude duality holds, suggesting that there should be an additional mechanism besides dual conformal symmetry responsible for this.



We developed some techniques to aid in direct tests of the conjectured planar $\mathcal{N}=4$ Yang-Mills S-matrix and multiloop iterative relations. The motivation behind this research is the desire to explore and uncover the rich mathematical structure underlying $\mathcal{N}=4$ Yang-Mills theory. Discovering such structures also has the pleasant side benefit of making previously difficult calculations much simpler. Prospects are great for continued progress, both in supersymmetric gauge theories as well as QCD. There is definitely a lot more to learn and discover.

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