Rigorous Definition of Quantum Field Operators and Test Functions Space in Noncommutative Quantum Field Theory

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Abstract

The space, on which quantum field operators are given, is constructed in noncommutative quantum field theory. This construction is the central point of the Wightman reconstruction theorem.

1 Introduction

Quantum field theory (QFT) as a mathematically consistent theory was formulated in the framework of the axiomatic approach in the works of Wightman, Jost, Bogoliubov, Haag and others ([1] - [4]).

Within the framework of this theory on the basis of most general principles such as Poincaré invariance, local commutativity and spectrality, a number of fundamental physical results, for example, the CPT-theorem and the spin-statistics theorem were proven [1] - [3].

Noncommutative quantum field theory (NC QFT) being one of the generalizations of standard QFT has been intensively developed during the past years (for reviews, see [5, 6]). The idea of such a generalization of QFT ascends to Heisenberg and was initially developed in Snyder's work [7]. The present development in this direction is connected with the construction of noncommutative geometry [8] and new physical arguments in favor of such a generalization of QFT [9]. Essential interest in NC QFT is also due to the fact that in some cases it is a low-energy limit of string theory [10].

The simplest and at the same time the most studied version of noncommutative theory is based on the following Heisenberg-like commutation relations between coordinates:

$$[\hat{x}_\mu, \hat{x}_\nu] = i\,\theta_{\mu\nu}, \tag{1}$$

where $\theta_{\mu\nu}$ is a constant antisymmetric matrix.

The relation (1) breaks the Lorentz invariance of the theory, while the symmetry under the $SO\,(1,1) \otimes SO\,(2)$ subgroup of the Lorentz group survives [11].

NC QFT can be formulated also in commutative space by replacing the usual product of operators by the star (Moyal-type) product:

$$\varphi(x) \star \varphi(x) = \exp\left(\frac{i}{2}\,\theta^{\mu\nu}\,\frac{\partial}{\partial x^\mu}\,\frac{\partial}{\partial y^\nu}\right)\varphi(x)\varphi(y)|_{x=y}. \tag{2}$$

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This product of operators can be extended to the corresponding product of operators in different points:

$$\varphi(x_1) \star \cdots \star \varphi(x_n) = \prod_{a<b\leq n} \exp\left(\frac{i}{2}\theta^{\mu\nu}\frac{\partial}{\partial x_a^\mu}\frac{\partial}{\partial x_b^\nu}\right)\varphi(x_1)\ldots\varphi(x_n). \tag{3}$$

Wightman approach in NC QFT was formulated in [12] and [13]. For scalar fields the CPT theorem and the spin-statistics theorem were proven in the case $\theta^{0\nu} = 0$. In [14] - [16] it was shown that the main classical axiomatic results are valid or have analog in NC QFT at least if time commutes with spatial variables, i.e. $\theta^{0\nu} = 0$.

In [12] it was proposed that Wightman functions in the noncommutative case can be written down in the standard form

$$W\left(x_1, \ldots, x_n\right) = \langle\, \Psi_0, \varphi(x_1)\ldots\varphi(x_n)\, \Psi_0 \,\rangle, \tag{4}$$

where Ψ_0 is the vacuum state. However, unlike the commutative case, these Wightman functions are only $SO\left(1,1\right) \otimes SO\left(2\right)$ invariant. In fact in [12] the CPT theorem has been proven in the commutative theory, where Lorents invariance is broken up to $SO\left(1,1\right) \otimes SO\left(2\right)$ symmetry, as in the noncommutative theory it is necessary to use the $\star$-product at least in coinciding points.

In [13] it was proposed that in the noncommutative case the usual product of operators in the Wightman functions has to be replaced by the Moyal-type product (3) both in coinciding and different points. Such a product of operators reflects the natural physical assumption, that noncommutativity should change the product of operators not only in coinciding points, but also in different ones.

Actually it seems very natural to use in different points not the $\star$-product itself but the following generalization of the $\star$-product in coinciding points:

$$\varphi\left(x\right) \star \varphi\left(x\right) \rightarrow \xi\left(x-y\right)\varphi\left(x\right) \star \varphi\left(y\right), \tag{5}$$

where $\xi\left(x-y\right)$ is some function falling rapidly if $\left|x-y\right|^2 \gg \theta$. For example, $\xi\left(x-y\right)$ can be arbitrary continuous function, satisfying the inequality:

$$\left|\xi\left(x-y\right)\right| < C\,\exp\left(-\frac{\left|x-y\right|^2}{\theta}\right), \tag{6}$$

where C is a some positive number,

$$\theta \equiv \max_{\mu,\nu}\left|\theta^{\mu\nu}\right|, \quad \left|x-y\right| \equiv \max_{i=0,1,2,3}\left|x_i - y_i\right|.$$

In [14] it was shown that in the derivation of axiomatic results, the concrete type of product of operators in various points is insignificant.

The Wightman functions can be generally written down as follows [14]:

$$W\left(x_1, \ldots, x_n\right) = \langle\Psi_0, \varphi\left(x_1\right)\tilde{\star}\cdots\tilde{\star}\,\varphi\left(x_n\right)\Psi_0\rangle. \tag{7}$$

The meaning of $\tilde{\star}$ depends on the considered case. In particular,

$$\varphi\left(x\right)\tilde{\star}\varphi\left(y\right) = \varphi\left(x\right)\star\varphi\left(y\right), \tag{8}$$

$$\varphi\left(x\right)\tilde{\star}\varphi\left(y\right) = \xi\left(x-y\right)\varphi\left(x\right)\star\varphi\left(y\right), \tag{9}$$

$$\begin{gathered}\varphi\left(x\right)\tilde{\star}\varphi\left(y\right) = \varphi\left(x\right)\varphi\left(y\right),\ x \neq y;\\ \varphi\left(x\right)\tilde{\star}\varphi\left(x\right) = \varphi\left(x\right)\star\varphi\left(x\right)\end{gathered} \tag{10}$$

Let us stress that actually the field operator given at a point cannot be a well-defined operator [3]. Well-defined operator is a smoothed operator:

$$\varphi_f \equiv \int \varphi(x) f(x)\, dx, \tag{11}$$

where $f(x)$ is a test function. In QFT the standard assumption is that all $f(x)$ are test functions of tempered distributions. On the contrary, in the NC QFT the corresponding generalized functions can not be tempered distributions as the $\star$-product contains infinite number of derivatives. As is well known (see, for example, [1]) that there could be only a finite number of derivatives in any tempered distribution (proof of this statement see in [17]).

The formal expression (7) actually means that the scalar product of the vectors $\Phi_k = \varphi_{f_k} \cdots \varphi_{f_1} \Psi_0$ and $\Psi_n = \varphi_{f_{k+1}} \cdots \varphi_{f_n} \Psi_0$ is the following:

$$\langle \Phi_k, \Psi_n \rangle =$$

$$\int W(x_1, \ldots, x_n)\, \overline{f_1(x_1)}\,\tilde{\star} \cdots \tilde{\star}\overline{f_k(x_k)}\tilde{\star} f_{k+1}(x_{k+1})\tilde{\star} \cdots \tilde{\star}\, f_n(x_n)$$

$$dx_1 \ldots dx_n, \quad W(x_1, \ldots, x_n) = \langle \Psi_0, \varphi(x_1) \cdots \varphi(x_n) \Psi_0 \rangle. \tag{12}$$

In paper [18] it was shown that the series

$$f(x) \star f(y) = \exp\left(\frac{i}{2}\,\theta^{\mu\nu}\,\frac{\partial}{\partial x^\mu}\,\frac{\partial}{\partial y^\nu}\right) f(x) f(y) \tag{13}$$

converges if $f(x) \in S^\beta$, $\beta < 1/2$, S^β is a Gel'fand-Shilov space [17]. The similar result was obtained also in [19].

In this report we give a rigorous definition of quantum field operator in NC QFT. For reader's convenience we shortly prove that the test functions space is a Gel'fand-Shilov space S^β with $\beta < 1/2$ (detailed proof see in [18]).

2 Rigorous Definition of Quantum Field Operators in NC QFT

Let us define rigorously quantum field operator φ_f. To this end we construct a closed and nondegenerate space J such that operators φ_f be well defined on dense domain of J.

As in the commutative case every vector of J can be approximated with arbitrary accuracy by the vectors of the type

$$\varphi_{f_1} \cdots \varphi_{f_n} \Psi_0,$$

where Ψ_0 is a vacuum vector. In other words the vacuum vector Ψ_0 is cyclic.

The difference of noncommutative case from the commutative one is that the action of the operator φ_f is defined by the $\star$-product, at least, at coincident arguments of the appropriate functions.

Construction of space J we begin with the introduction of set M of breaking sequences of the following kind

$$g = \{g_0, g_1, \ldots g_k\}, \tag{14}$$

where $g_0 \in \mathbb{C}$, $g_1 = g_1^1(x_1), x_1 \in \mathbb{R}^4, g_i = g_i^1(x_1)\,\tilde{\star} \cdots \tilde{\star} g_i^i(x_i)$, $x_j \in \mathbb{R}^4, 1 \leq j \leq i$, particularly $\tilde{\star}$ is determined by the formulae (8) - (10), k depends on g. Addition and multiplication by complex numbers of the above mentioned sequences are defined component by component, i.e.

$$\{g_0, g_1, \ldots g_k\} + \{h_0, h_1, \ldots h_m\} = \{h_0 + g_0, h_1 + g_1, + \ldots\},$$

$$C\, g = \{C\, g_0, C\, g_1, \ldots C\, g_k\}.$$

It is obvious, that $C\, g \in M$, if $g \in M$.

The every possible finite sums of the sequences belonging M form space J_0' on which action of the operator φ_f, $f = f\,(x)$, $x \in \mathbb{R}^4$ will be determined.

Certainly, to determine the $\tilde{\star}$-product, functions g_k should have sufficient smoothness. As stated above the $\star$-product is well-defined, if g_k belongs to one of Gel'fand-Shilov spaces S^β, $\beta < 1/2$ [17]. Moreover, $f \star g_k \in S^\beta$ with the same β [18].

The operator φ_f is defined as follows

$$\varphi_f\, g = \{f g_0, f\tilde{\star}g_1, \dots f\tilde{\star}g_k\}, \tag{15}$$

where $f\tilde{\star}g_i = f\,(x)\tilde{\star}g_i^1\,(x_1)\,\tilde{\star}\cdots\tilde{\star}g_i^i\,(x_i.)$

We assume that

$$f\,\tilde{\star}\,(g_i + h_i) = f\,\tilde{\star}\,g_i + f\,\tilde{\star}\,h_i. \tag{16}$$

If $\tilde{\star}$ is defined by one of the equations (8) - (10), then it is evident that eq. (16) is fulfilled. In accordance with eq. (16) any vector of space J_0' is a sum of the vectors belonging to set M, the operator φ_f is determined on any vector of space J_0' and $\varphi_f \Phi \in J_0', \forall\, \Phi \in J_0'.$

The scalar product of vectors in J_0' we define with the help of Wightman functions $W\,(x_1, \dots, x_n) \equiv \langle \Psi_0, \varphi\,(x_1)\,\dots\,\varphi\,(x_n)\,\Psi_0\rangle$. We consider firstly a chain of vectors: vacuum vector $\Psi_0 = \{1, 0, \dots 0\}, \Phi_1 = \varphi_{f_1}\Psi_0, \dots \Phi_k = \varphi_{f_k}\dots\varphi_{f_1}\Psi_0,\ f_i = f_i\,(x_i),\ x_i\ \in \mathbb{R}^4$.
According to (15), $\Phi_k = \{0, \dots f_k\tilde{\star}\dots\tilde{\star}f_1, 0\dots 0\}$.
Similarly, $\Psi_n = \varphi_{f_{k+1}}\dots\varphi_{f_n}\Psi_0 = \{0, \dots f_{k+1}\tilde{\star}\dots\tilde{\star}f_n, 0\dots 0\}$. It is obvious, that J_0' is a span of the vectors of such a type.

In what follows we consider the case $\tilde{\star} = \star$. Let us point out that the definition of $\tilde{\star}$ by formulas (9) and (10) leads to the final results same with this case.

The scalar product of vectors Φ_k and Ψ_n is

$$\langle\, \Phi_k, \Psi_n\,\rangle = \langle\, \Psi_0, \varphi_{\bar{f}_1}\,\dots\,\varphi_{\bar{f}_k}\,\varphi_{f_{k+1}}\,\dots\,\varphi_{f_n}\,\Psi_0\,\rangle =$$

$$\int W\,(x_1, \dots, x_n)\,\overline{f_1\,(x_1)}\,\star\cdots\star\,\overline{f_k\,(x_k)}\,\star\,f_{k+1}\,(x_{k+1})\,\star\cdots\star\,f_n\,(x_n)\,d\,x_1\dots d\,x_n. \tag{17}$$

The adjoined operator φ_f^* is defined by the standard formula. If operator φ_f is Hermitian then $\varphi_f^* = \varphi_{\bar{f}}$. In this paper we consider only Hermitian (real) operators, but the construction can be easily extended to complex fields.

Let us show now that a condition

$$\langle\, \Phi_k, \Psi_n\,\rangle = \overline{\langle\, \Psi_n, \Phi_k\,\rangle} \tag{18}$$

is fulfilled, if (as well as in commutative case [1]),

$$W\,(x_1, \dots, x_n) = \overline{W\,(x_n, \dots, x_1)}. \tag{19}$$

Really, in accordance with (17)

$$\langle\, \Psi_n,\, \Phi_k\,\rangle = \langle\, \Psi_0, \varphi_{\bar{f}_n}\,\dots\,\varphi_{\bar{f}_{k+1}}\,\varphi_{f_k}\,\dots\,\varphi_{f_1}\,\Psi_0\,\rangle =$$

$$\int W\,(x_n, \dots, x_1)\,\overline{f_n\,(x_n)}\,\star\cdots\star\,\overline{f_{k+1}\,(x_{k+1})}\,\star\,f_k\,(x_k)\,\star\cdots\star\,f_1\,(x_1)\,d\,x_1\dots d\,x_n. \tag{20}$$

The required condition is satisfied, since owing to antisymmetry of $\theta^{\,\mu\nu}$

$$\overline{f_n\,(x_n)}\star\overline{f_{n-1}\,(x_{n-1})}\star\dots\star\overline{f_1\,(x_1)} = \overline{f_1\,(x_1)\star\dots\star f_n\,(x_n)}.$$

According to the formula (17), the scalar product of any vectors $g \in M$ and $h \in M$ is

$$\langle\, g, h\,\rangle = \sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\int\ d\,x_1\dots d\,x_k\,d\,y_1\dots d\,y_m$$

$$W\left(x_1,\dots x_k,y_1,\dots y_m\right)\bar{g_k}^1\left(x_1\right)\star\dots\star\bar{g_k}^k\left(x_k\right)\star h_m{}^1\left(y_1\right)\star h_m{}^m\left(y_m\right). \tag{21}$$

As any vector of space J_0' is a finite sum of the vectors belonging to the set M, formula (21) defines the scalar product of any two vectors in J_0'.

As well as in commutative case, we need to pass from J_0' to nondegenerate and closed space J.

The space J_0' can contain isotropic, i.e. orthogonal to J_0' vectors which, as is known, form subspace [20]. Designating isotropic space as $\tilde{J}_0$ and passing to factor-space $J_0 = J_0'/\tilde{J}_0$, we obtain nondegenerate space, i.e. a space which does not contain isotropic vectors. Let us note, that if $g\in\tilde{J}_0$ then $\varphi_f g\in\tilde{J}_0$. For closure of space J_0 we assume, as well as in commutative case, that J_0 is a normalized space. If the metrics of J_0 is positive, norm $\Phi\equiv\|\Phi\|$ can be defined by the formula $\|\Phi\| = \langle\Phi,\Phi\rangle^{1/2}$. $\bar{J}_0$ is a closure of J_0, it is carried out with the help of standard procedure - closure to the introduced norm. In gauge theories in case of covariant gauge it is necessary to introduce indefinite metric (see, e.g., [21] and [22]). Let us recall that axiomatic formulation was extended on gauge theories, first of all, in papers of Morchio and Strocchi [23]. The standard space, which is considered in axiomatic theory of gauge fields, is a Krein space [20], [24].

Let us recall that nondegenerate space admits fundamental decomposition if

$$J_0 = J_0^+\oplus J_0^-, \tag{22}$$

where $J_0^\pm$ is a space with the definite positive (negative) metric, $J_0^+\perp J_0^-$. In other words, if $x\in J_0$, then $x = x^+ + x^-$, $x^\pm\in J_0^\pm$, $x^+\perp x^-$, that is $\langle x^+,x^-\rangle = 0$.
It is obvious that

$$\langle\, x,y\,\rangle = \langle\, x^+,y^+\,\rangle + \langle\, x^-,y^-\,\rangle. \tag{23}$$

It is easy to see that we can introduce in J the positive-definite scalar product $(\cdot,\cdot)$, namely

$$(x,y) = \langle\, x^+,y^+\,\rangle - \langle\, x^-,y^-\,\rangle. \tag{24}$$

We can introduce norm using this product $\|x\| = (x,x)^{1/2}$.
Evidently, $\|x^+\| = \langle\, x^+,x^+\,\rangle^{1/2}$, $\|x^-\| = (-\langle\, x^-,x^-\,\rangle)^{1/2}$.
The closure of $J_0 : \bar{J}_0 = \bar{J}_0^+\oplus\bar{J}_0^-$ is a Krein space.

The space $\bar{J}_0$, in turn, can contain isotropic subspace $\tilde{J}$. Factor-space $J = \bar{J}_0/\tilde{J}$ evidently is a nondegenerate space.

Thus, we have constructed closed and nondegenerate space J such that operators φ_f are determined on dense domain J_0. Hence, the axiom of cyclicity of vacuum is fulfilled.

Let us construct in J the scalar product of any two vectors Φ and Ψ. It is obvious, that there exist sequences of vectors $\Phi^n\in J_0$ and $\Psi^m\in J_0$ such that

$$<\Phi,\Psi> = \lim_{n,m\to\infty} <\Phi^n,\Psi^m>. \tag{25}$$

We shall note, that condition $\theta^{0\nu} = 0$ was not used and thus given above construction is valid in the general case as well.

Let us stress that if the $\star$-product acts only in coinciding points and is substituted by usual one in different points then given construction can also be fulfilled, only in the different points we have to put $\theta^{\mu\nu} = 0$. But in this case the function $f\left(x,y\right) = f\left(x\right)f\left(y\right), x\neq y, f\left(x,x\right) = f\left(x\right)\star f\left(x\right)$ is not continuous when $x = y$. In order to overcome this difficulties let us proceed from the function defined by eq. (10) to its regularization:

$$f_\alpha\left(x,y\right) = \xi\left(x-y\right)\varphi\left(x\right)\star\varphi\left(y\right),$$

$$\xi\left(x-y\right) = 1,\quad \text{if}\quad \left|x-y\right|^2 < \alpha-\varepsilon,\qquad \xi\left(x-y\right) = 0,\quad \text{if}\quad \left|x-y\right|^2\geq\alpha, \tag{26}$$

α is arbitrary, ε can be taken arbitrary small without loss of continuity of $\xi\,(x-y)$ (see, e.g. [25]).

It is evident that if $\alpha \gg \theta$, then expressions (9) and (26) practically coincide.

Now let us pass to the limit $\theta^{\mu\nu}=0$, that is proceed to the commutative case. In this case $f\,(x)\tilde{\star}f\,(y) \to f\,(x)\,f\,(y)$ and we come to the construction of space J in the commutative case. Let us point out that the first step in the standard construction of this space [1], [2] is the introduction of sequences g determined by the formula (14), in which, however, $g_i \equiv g_i\,(x_1,\dots x_i)$ are smooth functions of variables $x_j \in \mathbb{R}^4$. We shall note that in the commutative case, starting with J_0', we shall come to the same space J. Really, as space of functions of a type $g_i^1\,(x_1)\,g_i^2\,(x_2)\dots g_i^i\,(x_i)$ is dense in space of functions $g_i\,(x_1,\dots x_i)$ [1], [2], we can complete J_0' up to the space of the above mentioned sequences and then carry out the standard construction of space J.

Remark *In fact we have obtained a very general construction, which is valid not only in the case of NC QFT, but also for any case, when usual product of the functions is substituted by the new one, if the following conditions are satisfied:*

i *Corresponding functions belong to the some space, such that Wightman functions are defined as generalized functions (functionals) over this space;*

ii *The scalar product in this space is defined by eq. (17), but condition (19) may be substituted by the new one;*

iii *There exists the passage to the standard multiplication.*

3 Test Functions Space

Now according to [18] let us determine the spaces in which the $\star$-multiplication is well-defined. Evidently the space of tempered distributions cannot be the space compatible with the $\star$-multiplication, as each function of this space contains only a finite number of derivatives. Gel'fand and Shilov proved that if $f\,(x) \in S^\beta$ (see ineq. (27)) then the series of derivatives of infinite order can be well-defined in such a space. Thus we assume that $f\,(x) \in S^\beta$ and prove that the $\star$-product is well-defined only if each f_i belongs to the Gel'fand-Shilov space S^β, $\beta < 1/2$. The $\star$-product can be also well-defined if $\beta = 1/2$, but only for functions which satisfy inequality (27) with sufficiently small B.

Let us recall the definition and basic properties of Gel'fand-Shilov spaces S^β. In the case of one variable $f\,(x)$, $x \in \mathbb{R}^1$ belongs to the space S^β, if the following condition is satisfied:

$$\left| x^k \,\frac{\partial^{\,q} f\,(x)}{\partial x^q}\right| \leq C_k\, B^{\,q}\, q^{\,q\beta}, \quad -\infty < x < \infty, \quad k, q \in \mathbb{N}, \tag{27}$$

where the constants C_k and B depend on the function $f\,(x)$. Below we use the inequality (27) only at $k=0$:

$$\left| \frac{\partial^{\,q} f\,(x)}{\partial x^q}\right| \leq C\, B^{\,q}\, q^{\,q\beta}, \quad -\infty < x < \infty, \quad q \in \mathbb{N}. \tag{28}$$

In the case of a function of several variables, the latter inequality (28) holds for any partial derivative:

$$\left| \frac{\partial^{\,q} f\,(x^1,\dots x^k)}{(\partial x^i)^q}\right| \leq C\, B^{\,q}\, q^{\,q\beta}, \quad -\infty < x_i < \infty, \quad q \in \mathbb{N}. \tag{29}$$

As our results do not depend on constant C, in what follows we put $C=1$.

We point out that if the $\star$-product is well-defined for

$$f_i\,(x_i)\,\star\, f_{i+1}\,(x_{i+1}),$$

it is also well-defined for product of arbitrary number of functions.

Let us study

$$f\,(x) \star f\,(y) = \exp\left(\frac{i}{2}\,\theta^{\mu\nu}\,\frac{\partial}{\partial x^\mu}\,\frac{\partial}{\partial y^\nu}\right) f\,(x)\,f\,(y). \tag{30}$$

We have to find the conditions under which the series

$$\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{i}{2}\,\theta^{\mu\nu}\,\frac{\partial}{\partial x^\mu}\,\frac{\partial}{\partial y^\nu}\right)^n f\,(x)\,f\,(y) \equiv \sum_{n=0}^{\infty}\frac{D_n}{n!} \tag{31}$$

converges. After simple calculations [18] we come to the inequality:

$$|D_n| < \left(4\,\theta\,B^2\right)^n\,n^{2\,n\,\beta}. \tag{32}$$

Using this inequality and the fact that, according to the Stirling formula, $\frac{1}{n!} < \left(\frac{e}{n}\right)^n$, we come to the estimate

$$\left|\frac{D_n}{n!}\right| < \tilde{B}^n\,n^{-2\,n\,\gamma}, \tag{33}$$

where $\tilde{B} = 4\,e\,\theta\,B^2,\ \gamma = 1 - 2\,\beta$.

For any $\tilde{B}$ the series

$$\sum_{n=0}^{\infty}\tilde{B}^n\,n^{-2\,n\,\gamma} \tag{34}$$

converges if $\gamma > 0$, i.e. $\beta < 1/2$, and diverges if $\beta > 1/2$. If $\beta = 1/2$ the series converges if $\tilde{B} < 1$.

Thus we come to the conclusion that the series (31) for arbitrary B and C is a convergent one if $\beta < 1/2$ and divergent if $\beta > 1/2$. If $\beta = 1/2$ the series converges at sufficiently small B.

Similarly we can prove that the function $f_\star\,(x, y) \equiv f\,(x) \star f\,(y)$ belongs to the same Gel'fand-Shilov space S^β, $\beta < 1/2$ as $f\,(x)$ [18].

4 Conclusions

We have rigorously constructed field operators in NC QFT. This construction is important for any rigorous treatment of the axiomatic approach to NC QFT via NC Wightman functions and the derivation of rigorous results such as CPT and spin-statistics theorems.

The carried out construction of the closed and nondegenerate space, such that operators φ_f are determined on its dense domain, corresponds to the theorem named as "main" in [2] and opens a way to derivation of the reconstruction theorem in noncommutative field theory, that we are going to make.

Acknowledgements

Yu.S. Vernov is partly supported by the grant of the President of the Russian Federation NS-7293.2006.2 (Government Contract 02.445.11.7370).

References

[1] R. F. Streater and A. S. Wightman, *PCT, Spin and Statistics and All That*, Benjamin, NewYork (1964).

[2] 2. R. Jost, *The General Theory of Quantum Fields* ,Amer. Math.Soc., Providence, R.I. (1965).

[3] N. N. Bogoliubov, A. A. Logunov and I. T. Todorov, *Introduction to Axiomatic Quantum Field Theory*, W. A. Benjamin, Inc., New York, 1975;
N. N. Bogoliubov, A. A. Logunov, A. I. Oksak and I. T. Todorov, *General Principles of Quantum Field Theory*, Kluwer, Dordrecht (1990).

[4] R. Haag, *Local Quantum Physics*, Springer, Berlin (1996).

[5] M. R. Douglas and N. A. Nekrasov, *Rev. Mod. Phys.* **73** 977 (2001), hep-th/0106048.

[6] R. J. Szabo,*Phys. Rept.* **378** 207 (2003), hep-th/0109162.

[7] H. S. Snyder, *Phys. Rev.* **71** 38 (1947).

[8] A. Connes, *Noncommutative Geometry*, Academic Press, New York (1994).

[9] S. Doplicher, K. Fredenhagen and J. E. Roberts, *Phys. Lett. B* **331** 39 (1994); *Comm. Math. Phys.*, **172**, 187 (1995).

[10] N. Seiberg and E. Witten, *JHEP* **9909** 32 (1999), hep-th/9908142.

[11] L. Álvarez-Gaumé and J. L. F. Barbon, *Int. J. Mod. Phys. A*, **16** 1123 (2001), hep-th/0006209.

[12] L. Álvarez-Gaumé and M. A. Vázquez-Mozo, *Nucl. Phys. B*, **668** 293 (2003), hep-th/0305093.

[13] M. Chaichian, M. N. Mnatsakanova, K. Nishijima, A. Tureanu and Yu. S. Vernov, hep-th/0402212.

[14] Yu.S. Vernov, M.N. Mnatsakanova, *Theor. Math. Phys.* **142** 337 (2005).

[15] M. Chaichian, P. Prešnajder and A. Tureanu, *Phys. Rev. Lett.*, **94** 151602 (2005), hep-th/0409096.

[16] M. Chaichian, M. N. Mnatsakanova, A. Tureanu and Yu. S. Vernov, *Classical Theorems in Noncommutative Quantum Field Theory*, hep-th/0612112.

[17] I. M. Gelfand and G.E. Shilov, *Generalized Functions*, Vol. 2, Academic Press Inc., New York (1968).

[18] M. Chaichian, M. N. Mnatsakanova, A. Tureanu and Yu. S. Vernov, *JHEP* **09** 125 (2008), hep-th/0706.1712v1 (2007).

[19] M. A. Soloviev, *Theor. Math. Phys.* **153**, 1351 (2007), math-ph/0708.0811.

[20] J.Bognár, *Indefinite Inner Product Spaces*, Springer-Verlag, Berlin, (1974).

[21] T. Kugo and I. Ojima, *Prog. Theor. Phys.Suppl.* **66**, 1 (1979).

[22] F. Strocchi, *World Sci.Lect.Notes Phys.* **51**, 1 (1993).

[23] G. Morchio and F. Strocchi, *Ann. Inst. H.Poincaré A*, **33**, 251 (1980).

[24] T.Ya.Azizov, I.S.Iokhvidov, *Linear Operators in Spaces with an Indefinite Metric*, J.Wiley and Sons. 1989 (in Russian: M.:Nauka, 1986).

[25] L. Hëormander, *The Analysis of Linear Partial Differential Operators*, vol. 1, Springer, Berlin, 1983.