

On Cancellations of Ultraviolet Divergences in Supergravity Amplitudes

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Abstract

Concrete calculations have pointed out that amplitudes in perturbative gravity exhibit unanticipated cancellations taming their ultraviolet behaviour independently of supersymmetry. Similar ultraviolet behaviour of $\mathcal{N} = 4$ super-Yang-Mills and $\mathcal{N} = 8$ maximal supergravity has explicitly been observed until three loops. These cancellations can be connected to two manifest features of gravitational theories: firstly gauge invariance from diffeomorphism symmetries and secondly that amplitudes are colourless and exhibits crossing symmetry. We will give a simple physical explanation of the cancellations exhibited in gravity amplitudes. We will discuss these two properties in turn as well as the rôle of supersymmetry and string theory dualities in the structure of multiloop amplitudes in supergravity.

1 Introduction

The theoretical construction of unification models for particle physics has led to remarkable progress in the understanding of fundamental interactions in Nature. However a complete theory for gravity is still illusive and it is expected that subtle quantum gravity effects will play an important role in understanding the outstanding fundamental problems of modern cosmology and models for particle physics. Since the discovery of quantum mechanics in the last century, physicists have been pursuing the construction of a consistent theory for quantum gravity in order to gain a complete understanding of quantum gravitational effects at all scales. Field theories with point-like interactions for gravity in four dimensions are non-renormalisable because of the dimensionality of the gravitational coupling constant. No known symmetry has so far been shown capable of regulating the ultraviolet divergences for such a theory *although* such constructions have not been proven either to be impossible. Interestingly unique quantum corrections to gravity can be extracted from treating general relativity as a point-like effective field theory [1].

String theory provides a consistent framework for quantum gravity and its supersymmetric extensions. Within this formalism various gravity amplitudes can be computed [2, 3]. Expressions for field theory amplitudes preserving supersymmetry can be derived in the infinite

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tension limit ($\alpha' \rightarrow 0$) of the string. String theory rules for graviton amplitudes that hold at tree level have been formulated very elegantly by Kawai-Lewellen-Tye [4]. Interestingly such rules also hold in a number of different scenarios [5, 6] with various matter contents [7]. At one-loop level string based rules have been formulated for amplitude calculations in both gauge theory and gravity [8, 9]. The effect of massive string modes on the low energy effective action of various compactifications of string theory leads to important quantum corrections [10] which are relevant for particle physics unification, moduli stabilisation [11] and cosmology [12].

String theory combines the effect of a hard ultraviolet momentum cutoff (determined by the extension of the string while keeping gauge invariance) and the decoupling of unphysical states thanks to the modular invariance of its world-sheet theory. Although the theory is perturbatively finite, its complete degrees of freedom are provided by the non-perturbative U-duality symmetries [13, 14, 15].

Power counting arguments based on known symmetries indicate that supergravity theories have ultraviolet divergences in four dimensions and candidates for explicit counter-terms at three-loop order have been constructed [16, 17, 18, 19, 20]. However contrary to the statements from power counting arguments it has recently been shown by explicit computation that one-loop amplitudes in $\mathcal{N} = 8$ supergravity [21, 22, 23, 24, 25] can be constructed from the same basis of scalar integrals as $\mathcal{N} = 4$ super Yang-Mills. Furthermore divergences in four dimensions in maximal $\mathcal{N} = 8$ supergravity have been shown to be explicitly absent until three-loop order by direct computation [26]. It has been suggested that the absence of divergences might persist to higher loop order [27, 28, 29] with the consequence that four dimensional $\mathcal{N} = 8$ supergravity could be a perturbatively finite theory [22, 28, 26, 23, 27, 29].

The discrepancy between power counting and explicit computation emphasises the lack of knowledge of the consequences of physical effects such as (diffeomorphism) gauge invariance in amplitude calculations [23, 30, 31, 32, 33, 34, 35]. This together with suggestions of possible finiteness of $\mathcal{N} = 8$ supergravity is a motivation for reconsidering the ultraviolet behaviour of (super)gravity theories and their relation to string theory.

This analysis aims to answer the following questions:

How can $\mathcal{N} = 8$ supergravity amplitudes be finite?

What rôle does string theory symmetries and dualities play in the possible finiteness?

2 One-loop amplitudes in gravity

A one-loop n -graviton amplitude in $D = 4 - 2\epsilon$ dimensions takes the generic form

$$\begin{aligned} M_{n;1}^{(D)} &= \mu^{2\epsilon} \int d^D \ell \frac{\prod_j^{2n} (q_{\mu_j}^{(2n,j)} \ell^{\mu_j}) + \prod_j^{2n-1} (q_{\mu_j}^{(2n-1,j)} \ell^{\mu_j}) + \dots + K}{\ell_1^2 \dots \ell_n^2} \\ &\equiv \mu^{2\epsilon} \int d^D \ell \frac{\mathcal{P}_{2n}(\ell)}{\ell_1^2 \dots \ell_n^2} \end{aligned} \quad (1)$$

where $\ell_i^2 = (\ell - k_1 - \dots - k_i)^2$ are the propagators along the loop and $q_{\mu_j}^{(i,j)}$ are functions of external momenta and polarisations. Because of the two derivative nature of the cubic gravitational coupling, the numerator $\mathcal{P}_{2n}(\ell)$ is a polynomial with at most $2n$ power of loop momentum $\ell \equiv \ell_n$.

A one-loop amplitude can be expanded via a succession of Passarino-Veltman reductions [36] in a linear basis of n -points scalar integrals

$$I_n^{(D)} = \int \frac{d^D \ell}{\ell_1^2 \dots \ell_n^2} \quad (2)$$

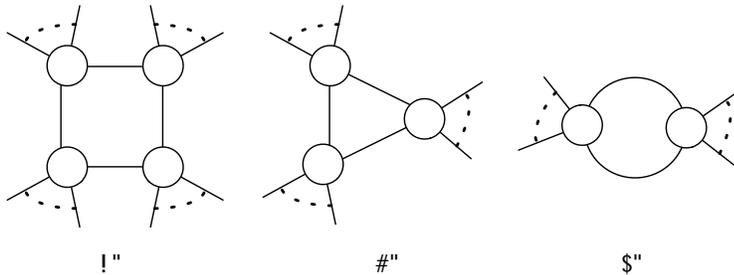


Figure 1: Basis of one-loop scalar integrals given by a) a scalar box, b) scalar triangle and c) a scalar bubble integral. In $D = 4 - 2\epsilon$ dimensions these diagrams carry all the ultraviolet and infrared divergences of the amplitudes.

where $\ell_i^2 = (\ell - K_1 - \dots - K_i)^2$ and where $K_p = k_{i_1} + \dots + k_{i_p}$ is the sum of momenta flowing into the corner p . A loop amplitude in four dimensions with $2n$ powers of loop momenta from each vertex can be shown to generically contain scalar box, triangle and bubble integrals and as well as polynomial (non-logarithmic) rational terms [37].

Explicit evaluation of one-loop gravity amplitudes in $D = 4 - 2\epsilon$ dimensions in [21, 38, 39, 22] show that only scalar box integrals enter in the decomposition of gravity one-loop amplitudes. This ‘only boxes’ property (or the ‘no triangle hypothesis’) indicates that the *highest* total power of the loop momentum polynomial in the numerator in the generic one-loop amplitudes has to be the same as in the $\mathcal{N} = 4$ super Yang-Mills theory (*i.e.* of order $n - 4$ and not as naïve power counting suggests $2n - 8$). For theories with less supersymmetries it was argued in [23] that the highest power of loop momentum is given by

$$\mathcal{P}_{2n}(\ell) \sim \ell^{2n - \mathcal{N} - (n-4)}, \quad \text{for } \ell \gg 1 \quad (3)$$

This behaviour displays two types of cancellations of loop momenta,

- i) There is a cancellation of \mathcal{N} powers of loop momenta due to the effect of the \mathcal{N} linearised on-shell supersymmetries (counting the number of supersymmetries in units of four dimensional Majorana supercharges). This cancellation is independent of the number of external states and the dimension as long as the number of supersymmetries is preserved.
- ii) There are $n - 4$ extra ‘unexpected’ [23] cancellations which depend on the number of external legs.

An $\mathcal{N} = 4$ super-Yang-Mills n -point one-loop amplitude contains two kind of contributions. One comes with at most $n - 4$ powers of loop momentum where n powers of loop momentum come from the derivatives in the cubic vertices and four cancellations are due to supersymmetry

$$\int d^D \ell \frac{\mathcal{P}_{n-4}(\ell)}{\ell_1^2 \dots \ell_n^2}, \quad \text{with } n \geq 4 \quad (4)$$

Another contribution comes with up to $2n - 8$ powers of loop momenta and has many trivial cancellations due to explicit powers of ℓ_i^2 in the numerator. Such contributions lead to trivial cancellations such as

$$\int d^D \ell \frac{\mathcal{P}_{n-4}(\ell) \ell_{n+1}^2 \dots \ell_{n+p}^2}{\ell_1^2 \dots \ell_{n+p}^2} = \int d^D \ell \frac{\mathcal{P}_{n-4}(\ell)}{\ell_1^2 \dots \ell_n^2}, \quad \text{with } n \geq 4 \quad (5)$$

These types of contributions arise from φ^4 -type of vertices.

Since the one-loop $\mathcal{N} = 8$ supergravity amplitude have the same maximum number of loop momenta as $\mathcal{N} = 4$ super-Yang-Mills, they can be expanded in the *same* basis of elementary

scalar master integrals (in the dimensional regularisation scheme).¹ The tensorial structure multiplying these integrals in gravity are related to the ones of the corresponding super-Yang-Mills amplitudes by the Kawai-Lewellen-Tye relations [4, 40, 38]. At the level of the effective action the connection is not immediate because of the different nature of gauge interactions. There are no particular reasons for the higher-derivative corrections to the supergravity effective action to be related directly via KLT to the corresponding contributions of the super-Yang-Mills effective action. The relation between the two theories is however a useful guide for the explicit construction of higher-derivative gravitational corrections [10, 41].

3 Origin of the cancellations

For a theory with \mathcal{N} on-shell linearly realised supersymmetries the integration over the fermionic zero modes leads to the cancellation of \mathcal{N} powers of loop momenta.

At one-loop level the extra cancellations of powers of loop momenta was shown in [32, 34] to be accounted for firstly (a) by the summation over all the permutations of the external legs due to the absence of the concept of colour in gravity and secondly (b) by the decoupling of longitudinal modes from the diffeomorphism gauge invariance.

- a) The absence of colour which forces a summation over all the orderings of the external legs leads to various cancellations for on-shell amplitudes. At higher-loop order this implies that one should sum over all planar and non-planar contributions. From this, four dimensional gravity amplitudes have a better infrared behaviour than the corresponding (coloured) ordered QCD amplitudes [42]. This gives a set of reduction formulas needed for the reduction of loop integrals in a basis of elementary scalar box integrals [32, 34].
- b) The diffeomorphism symmetries $\varepsilon_{\mu\nu} \rightarrow \varepsilon_{\mu\nu} + \partial_\mu v_\nu + \partial_\nu v_\mu$ and the decoupling of the longitudinal modes in the amplitudes allows the cancellation of the highest powers of loop momenta in the loop integral by using unordered integral reduction formulas [32, 34].

The string based rules [9, 32, 34] give very compact and well organised expressions for amplitudes. The position of the external legs are in this formalism labelled by ν_i which take values over the range $[0, 1]$. The unordered scalar n -point amplitude is given by

$$\mu^{2\epsilon} \int \frac{d^D \ell}{\pi^{\frac{D}{2}}} \prod_{i=1}^n \frac{1}{(\ell - k_1 - \dots - k_i)^2} = \Gamma\left(\frac{D-1}{2}\right) \Gamma\left(n - \frac{D}{2}\right) \int_0^1 d\nu_1 \dots d\nu_{n-1} \delta(\nu_n = 1) Q_n^{\frac{D}{2}-n} \quad (6)$$

where μ is an infrared regulator and

$$Q_n = \sum_{1 \leq i < j \leq n} (k_i \cdot k_j) [(\nu_i - \nu_j)^2 - |\nu_i - \nu_j|] \quad (7)$$

For a given orderings of the external legs the absolute value in eq. (7) can be lifted. The absolute value forces the n -point integral to break into various regions of analyticity in the complex energy momentum plane corresponding to the possible physical orderings of the external legs. By keeping the absolute value in Q_n and by integrating the ν_i freely over the range $[0, 1]$ all ordering are included.

¹In field theory we work using the dimensional regularisation scheme. The momentum cutoff scheme is more natural from the string/M-theory point of view, and will be used later on. Such a scheme is however difficult to implement in field theory without breaking gauge invariance. In the cutoff regularisation scheme the basis of integrals could be different in particular for the integrals containing the finite part of the amplitude.

For gravitational amplitudes the numerator $\mathcal{P}(\varepsilon_{ij}, k_i, \ell)$ of the integrand of the loop amplitude in eq. (1) depends on the polarisation vectors ε_{ij} , the external momenta k_i and the loop momentum ℓ , with the counting given by eq. (3)

$$M_{n;1}^{(D)} = \int \frac{d^D \ell}{(2\pi)^D} \frac{\mathcal{P}(\varepsilon_{ij}, k_i, \ell)}{\ell^2 \dots (\ell - k_1 - \dots - k_n)^2} \quad (8)$$

Within the string based rules, $\mathcal{P}(\varepsilon_{ij}, k_i, \ell)$ contains two types of contributions. One contribution $\mathcal{P}^{(1)}(\varepsilon_{ij}, k_i, \ell)$ which does *not* involve the the zero mode contribution of the bosonic correlators and one contribution $\mathcal{P}^{(2)}(\varepsilon_{ij}, k_i, \ell)$ which does (defining $\varepsilon_{ij} \equiv (h_i \bar{h}_j + h_j \bar{h}_i)/2$)

$$\langle h_i \cdot \partial_{\nu_i} X(\nu_i) h_j \cdot \partial_{\nu_j} X(\nu_j) \rangle = (h_i \cdot h_j) [\delta(\nu_i - \nu_j) - 1/T] \quad (9)$$

This second contribution gives rise to dimension shifted integrals [32, 34].

By expanding the polarisations of the external states in a basis of independent momenta [32, 34], the factor $\mathcal{P}^{(1)}(\varepsilon, k_i, \ell)$ can be rewritten as a homogeneous polynomial of order $2n - \mathcal{N}$ in supergravity (or $n - \mathcal{N}$ in super-Yang-Mills) in the first derivative of Q_n and the ‘‘fermionic propagator’’ $G_F(x) = \text{sign}(x)$

$$\mathcal{P}^{(1)}(\varepsilon, k_i, \ell) = \sum_{r+s=2n-\mathcal{N}} c_{r,s} \prod_{i=1}^r \partial_{\nu_i} Q_n \prod_{s \text{ pairs } (pq)} G_F(\nu_p - \nu_q) \quad (10)$$

Only the contributions with $r > n - \mathcal{N}/2$ contain triangle contributions. The coefficients $c_{r,s}$ are functions of the external momenta for which exact expressions are not needed for showing the absence of triangles in the amplitude. We remark that because

$$\partial_{\nu_i} Q_n = -2 \sum_{j=1}^n (k_i \cdot k_j) \nu_j - \sum_{j=1}^n (k_i \cdot k_j) \text{sign}(\nu_i - \nu_j) \quad (11)$$

only the first derivative in Q_n bring dependence on the loop momentum (through its dependence on the ν_i variables). The second derivative in Q_n is given by

$$\partial_{\nu_i} \partial_{\nu_j} Q_n = 2(k_i \cdot k_j) (\delta(\nu_i - \nu_j) - 1), \quad i \neq j \quad (12)$$

The first contribution produces an integral with one less propagator generating a massive external leg as represented in figure 1 (such contributions arise from the reducible contributions when two vertex operators are colliding in string theory [32]).

In the amplitude for each term in the sum eq. (10) one can reduce the number of loop momenta by integration by parts. For the unordered integrals the boundary terms vanish, but the integration by parts generates ultraviolet and infrared finite contributions given by the dimension shifted scalar integrals

$$\begin{aligned} I_n^{(D+2\delta)} &= \mu^{2\epsilon} \int \frac{d^D \ell d^{2\delta} \ell_{\perp}}{\prod_{i=1}^n (\ell - k_1 - \dots - k_i)^2 + \ell_{\perp}^2} \\ &= \Gamma\left(\frac{D+2\delta-1}{2}\right) \int_0^{\infty} \frac{dT}{T} T^{n+\delta-D/2} \int_0^1 d\nu_1 \dots d\nu_{n-1} \delta(\nu_n - 1) e^{-TQ_n} \end{aligned} \quad (13)$$

These contributions have only four dimensional external momenta and combine with the polarisation dependent contributions from $\mathcal{P}^{(2)}(\varepsilon_{ij}, k_i, \nu_i)$ in eq. (9) into gauge invariant expressions. There were shown in [32, 34] to cancel from the physical amplitude.

The origin of the absence of triangles and bubbles can be traced back to the gauge invariance of the amplitude (the ‘cancelled propagator argument’ [2, 3]), where the longitudinal polarisation decouple. Substituting h_j with $i k_j$ in eq. (9) one has

$$\langle h_i \cdot \partial_{\nu_i} X(\nu_i) \partial_{\nu_j} (e^{ik_j \cdot X}) \rangle \Big|_{\text{linear in } k_j} \quad (14)$$

which gives a total derivative that cancels against the one generated by integrating by parts in $\mathcal{P}^{(1)}(\varepsilon_{ij}, k_i, \nu_i)$.

4 Consequences for the ultraviolet properties of gravity amplitudes

4.1 One-loop amplitudes

The behaviour in eq. (3) indicates that the n -graviton one-loop amplitude has ultraviolet divergences in dimensions

$$D^{1\text{-loop}} \geq D_c^{1\text{-loop}} = \mathcal{N} + n - 4 \quad (15)$$

For more than four gravitons the critical dimension in eq. (15) indicates that one-loop gravity amplitudes are more converging than naively expected from supersymmetric cancellations alone. This leads to the critical dimension for ultraviolet divergences $D_c^{\text{susy}} = \mathcal{N}$.

For $\mathcal{N} = 8$ supergravity the one-loop gravity amplitude would be finite in eight dimensions for at least five gravitons and finite in ten dimensions for at least seven gravitons. For $\mathcal{N} = 4$ supergravity one-loop amplitudes are finite in four dimensions for at least five gravitons.

4.2 Higher-loop amplitudes

At L loop order linearised on-shell supersymmetry implies that the critical dimension for ultraviolet divergences in the four-graviton amplitude is given by

$$D \geq 2 + \frac{c_{\mathcal{N}}}{L} \quad (16)$$

This implies that supergravity theories are finite in two dimensions and that they are not finite in four dimensions. The loop order for the appearance of the first logarithmic divergence is determined by the value of $6 \leq c_{\mathcal{N}} \leq 18$ depending on the implementation of the linearised on-shell supersymmetries determining the mass dimension of the first possible counter-term to the supergravity theory [16, 17, 18, 19, 29].

A L loop n -graviton amplitude has mass dimension

$$[M_{n;L}^{(D)}] = \text{mass}^{(D-2)L+2} \quad (17)$$

The low energy limit of the four-graviton amplitude at L loops reads

$$[M_{4;L}^{(D)}] = \text{mass}^{(D-2)L-(6+2\beta_L)} \partial^{2\beta_L} R^4 \quad (18)$$

where we have used that $\mathcal{N} = 8$ supergravity four-graviton amplitudes have a factor of R^4 and allowed $2\beta_L$ powers of derivatives distributed on the four Riemann tensors. The behaviour in eq. (18) indicates that the amplitude should be expanded in a basis of L -loop integrals with the mass dimension²

$$[I_{4;L}^{(D)}] = \text{mass}^{(D-2)L-(6+2\beta_L)} \quad (19)$$

and a critical dimension for ultraviolet divergences given by

$$D \geq 2 + \frac{6 + 2\beta_L}{L} \quad (20)$$

When $\beta_L = L$ at each loop order two extra powers of the external momenta are factorised and the critical dimension for ultraviolet divergences is given by [27, 29]

$$D \geq D_c = 4 + \frac{6}{L} \quad (21)$$

²This basis contains planar and non-planar contributions and some integrals will have numerators with momentum factors [28].

This is the same critical dimension as $\mathcal{N} = 4$ super-Yang-Mills and would imply finiteness in four dimensions if valid at all loop orders.

As soon as β_L is bounded after some loop order, the relevant critical dimension is given by (20) and the theory will have an ultraviolet divergence in four dimensions. The pure spinor formalism gives a counting of supersymmetric zero modes valid in all dimensions between $4 \leq D \leq 11$ where $\mathcal{N} = 8$ supergravity can be defined. This construction implies [43] that $\beta_L = 12$ for $L \geq 6$ and a critical dimension for ultraviolet divergence given by $D \geq 2 + 18/L$ according (20) which indicates that in four dimensions the first divergence would occur at nine-loop [29] with a counter-term given by the expression (25).

The rule $\beta_L = L$ is the optimal one for finiteness in four dimensions. If β_L grows slower than L the theory will not be finite in four dimensions. For instance $\mathcal{N} = 4$ supergravity is expected to satisfy the rule $\beta_L = L/2$ and have a critical dimension for ultraviolet divergences given by $D \geq 3 + 6/L$, and a first divergence at $L = 6$ loops in four dimensions. If β_L grows faster than L , the theory would be too finite. For instance the L loop (planar and non planar) ladder diagrams of the four-graviton amplitudes are all two-particle cut constructible and are given by scalar φ^3 diagrams with a prefactor satisfying the rule $\beta_L = 2(L - 1)$. These diagrams are ultraviolet finite for $D \leq 6$ which means that the leading ultraviolet divergences of $\mathcal{N} = 8$ amplitudes are not contained in these ladder diagrams.

When the rule $\beta_L = L$ is satisfied at each loop order the four-graviton amplitudes get a new ultraviolet primitive divergence of order Λ^{D-4} which is typical of “effective” interactions of the type of $\mathcal{N} = 4$ super-Yang-Mills. Amplitudes satisfying this rule should be expandable in the same basis of integrals as $\mathcal{N} = 4$ super-Yang-Mills, but since gravity has no colour, one must include the planar and non-planar diagrams.

The absence of triangles and bubbles at one-loop order implies via general factorisation theorems that higher-loop amplitudes cannot contain diagrams factorisable in one-loop amplitudes containing triangles or bubbles. This constraint affects the structure of the higher loop amplitude [28] but is not a sufficient condition for perturbative finiteness which requires further subtle cancellations between triangle free contributions [26].

5 Relation to string/M-theory and string theory dualities

In the previous section we only discussed the effects of on-shell supersymmetries, diffeomorphism invariance and the absence of colours. In this section we will discuss the rôle of string theory dualities.

The rule $\beta_g = g$ implies that the $\partial^{2g}R^4$ couplings to the ten-dimensional string theory effective action receive perturbative contributions up to genus g [44, 45, 46, 27, 47]. This rule has been directly shown up to genus six using the pure spinor formalism [48, 43].

The eleven dimensional incarnation of $\mathcal{N} = 8$ supergravity is non-renormalisable with a two-loop logarithmic divergence as indicated by the formula (21). The associated counter-term is the dimension twenty operator $\partial^{12}R^4$ (see as well ref. [49]). After having reduced the gravity integrals to the scalar integral basis, one can regulate the integrals with a local ultraviolet cutoff Λ in eleven dimensions without breaking gauge invariance [44, 45, 46, 47]. This is more suitable for extracting the contributions to the effective action. The cutoff should be determined by the microscopic degree of freedom of M-theory and is related to the tension of the M2-brane $T_{M2} \sim 1/\ell_P^3$ or the M5-brane $T_{M5} \sim 1/\ell_P^6$. The $\mathcal{N} = 8$ supersymmetric cancellations of loop momenta in the one-loop amplitudes imply that the highest power of the one-loop sub-divergences is given by Λ^3 and more generally one gets the following infinite series

of counter-terms to the four-point M-theory effective action [44, 45, 46, 47]

$$S_{M\text{-theory}} = \frac{1}{\ell_P^9} \int d^{11}x \left[\mathcal{R}_{(11)} + \sum_{k \geq 0} c_k \ell_P^{6k+6} \nabla^{6k} R^4 \right] \quad (22)$$

given by powers of covariant derivatives distributed on the Riemann tensors. This is precisely the series of higher-derivative corrections to the M-theory effective action that is selected by the strong coupling limit of string theory [50]. The coefficients are constrained by the microscopic degrees of freedom of M-theory and its duality symmetries, and have been determined up to order $k = 2$ in [44, 45, 46, 47]. Using the renormalisation scheme where the value of the counter-term is fixed by the relation between multiloop amplitudes in M-theory and string theory and its duality symmetries, one finds that the R^4 counter-term is fixed by the value of the type II genus one four-graviton contribution, the $\partial^6 R^4$ by the genus two and the $\partial^{12} R^4$ by the genus three contribution [44, 45, 46, 47].

We consider a Kaluza-Klein expansion of the eleven dimensional multiloop amplitudes on a circle of radius $R_{11} \ell_P$. Using the M-theory conjecture [14] which identifies $R_{11}^3 = g_s^2$ and $\ell_P = \sqrt{R_{11}} \ell_s$ one finds [27] that in the string weak coupling limit, where $R_{11} \rightarrow 0$, the higher-derivative $\partial^{2g} R^4$ couplings to the low-energy expansion of type II superstring satisfy the non-renormalisation condition $\beta_g = g$ to all orders in perturbation.

6 Conclusion and discussion

We have discussed cancellations that could be enough for making the ungauged $\mathcal{N} = 8$ supergravity theory perturbatively finite. It is interesting to note that a non-renormalisable theory with a dimensionful coupling constant still can have a surprisingly good perturbative ultraviolet behaviour. Ungauged $\mathcal{N} = 8$ supergravity is unlikely to lead directly to relevant phenomenology because of the absence of chiral matter. However since most of the cancellations in eq. (3) take place independent of supersymmetry it is expected that interesting results can be obtained in theories with less supersymmetry and with more phenomenological relevance.

We have not discussed the properties of gauged supergravity theories [51, 52, 53, 54] which are phenomenologically more promising and seem to enjoy nice quantum properties [55, 56, 57]. The non-Abelian structure of the gauging naturally contain duality multiplets under the full U-duality group of supergravities [58] which bring along new effects in the loop amplitude and need a separate analysis.

The local $SU(8)$ R-symmetry of the ungauged $\mathcal{N} = 8$ allows one only to consider superfields of at least mass dimension at least 1/2 leading to possible counter-terms starting from eight-loops [16, 17]

$$\delta L = \kappa_{(4)}^{d+12} \int d^4x d^{32}\theta \det(E) \mathcal{L}(R, T) \quad (23)$$

where $\mathcal{L}(R, T)$ is a superspace density of mass dimension d . In the full superspace this density has a least mass dimension 2, since it can only be constructed from superfields of at least mass dimension 1/2. For instance using the mass dimension 1/2 gravitino superfield χ_{ijk}^α and the dimension 1 vector superfield $W_{\alpha\beta}^{ij}$ invariants under the full $E_7 \times SU(8)$ symmetry of the ungauged $\mathcal{N} = 8$ supergravity, one can construct the possible eight- or nine-loop counter-terms [16, 17, 18, 19, 29, 20, 59] given by the following four-point higher-derivative supersymmetric invariants

$$\delta L = \kappa_{(4)}^{14} \int d^4x d^{32}\theta \det(E) (\chi_{ijk}^\alpha \bar{\chi}_\alpha^{ijk})^2 \sim \kappa_{(4)}^{14} \int d^4x \sqrt{-g} (\nabla^{10} R^4 + \text{susy completion}) \quad (24)$$

$$\delta L = \kappa_{(4)}^{16} \int d^4x d^{32}\theta \det(E) (W_{\alpha\beta}^{ij})^4 \sim \kappa_{(4)}^{16} \int d^4x \sqrt{-g} (\nabla^{12} R^4 + \text{susy completion}) \quad (25)$$

given by the supersymmetric completion of powers of covariant derivatives distributed on the Riemann tensors.

The $SO(8) \times SU(8)$ gauged $\mathcal{N} = 8$ supergravity has an on-shell superspace formulation [60] which leads to the possible counter-terms [60]

$$\delta L = \kappa_{(4)}^{d+12} \int d^4 x d^{32} \theta \det(E) \mathcal{L}(R, T) f(g) \quad (26)$$

with $f(g)$ constructed from the gauge fields. This expression reduces to the ungauged counter-terms in the limit $g \rightarrow 0$ with $f(g) \rightarrow 1$. But there exist as well an infinite set of new counter-terms which do not reduce to counter-terms of the ungauged theory. As in the ungauged case only superfields of mass dimension 1/2 are invariant under the local $SU(8)$ R-symmetry (the new spin 1 superfield from the gauging are of mass dimension 1), and the first possible counter-term can only arise at eight loops. It is interesting to understand in more details the structure of the loop amplitudes in these different versions of $\mathcal{N} = 8$ supergravity.

This analysis illuminates the importance of string theory dualities and symmetries and their rôle in the cancellations. These dualities and symmetries appear to be very important in the possible ultraviolet finiteness of $\mathcal{N} = 8$ supergravity together with physical effects such as diffeomorphism invariance of amplitudes, although further research is needed to fully clarify these matters at higher loop order.

Acknowledgements

We would like to thank G. Dvali, and M. B. Green for useful discussions and B. de Wit for some comments on the draft. PV would like to thank the organizers of the 3rd meeting of the RTN “Forces Universes” in Valencia and the organizers of Quarks 2008 for giving him the opportunity of presenting these results. The authors would like to thank the Niels Bohr Institute in Copenhagen for hospitality and financial support. The research of (NEJBB) was supported by grant DE-FG0290ER40542 of the US Department of Energy. The research (PV) was supported in part the RTN contracts MRTN-CT-2004-005104 and by the ANR grant BLAN06-3-137168.

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