Some interesting features of quantum correction in $N=1$ supersymmetric theories

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Abstract

We discuss calculation of the Gell-Mann–Low function in supersymmetric theories with the higher covariant derivative regularization and find that all integrals, defining this function, are possibly integrals of total derivatives. For the contribution of matter superfields this is partially explained by substituting solutions of Slavnov–Taylor identities into the Schwinger–Dyson equations. We find that the exact Gell-Mann–Low function coincides with the Novikov, Shifman, Vainstein and Zakharov β-function only if there is an identity for Green functions, which does not follow from any known symmetry of a supersymmetric theory.

In $N=1$ supersymmetric theories it is possible to suggest [1] the form of exact β-function

$$\beta(\alpha) = -\frac{\alpha^2\Big[3C_2 - C(R)\Big(1-\gamma(\alpha)\Big)\Big]}{2\pi(1-C_2\alpha/2\pi)}, \tag{1}$$

where $\gamma(\alpha)$ is the anomalous dimension of the matter superfield. This function is called the exact Novikov, Shifman, Vainstein and Zakharov (NSVZ) β-function. Let us try to derive the matter contribution to this function using the usual methods of the perturbation theory exactly to all orders. We will consider $N=1$ supersymmetric Yang–Mills theory, which is described by the action

$$S = \frac{1}{2e^2}\mathrm{Re}\,\mathrm{tr}\int d^4x\,d^2\theta\,W_a C^{ab} W_b + \\ +\frac{1}{4}\int d^4x\,d^4\theta\,\Big(\phi^+ e^{2V}\phi + \tilde\phi^+ e^{-2V^t}\tilde\phi\Big) + \Big(\frac{1}{2}m\int d^4x\,d^2\theta\,\tilde\phi^t\,\phi + h.c.\Big), \tag{2}$$

where ϕ and $\tilde\phi$ are chiral scalar matter superfields, V is a real scalar gauge superfield, and the supersymmetric gauge field stress tensor is given by

$$W_a = \frac{1}{8}\bar D^2(e^{-2V}D_a e^{2V}). \tag{3}$$

The action is invariant under the gauge transformations

$$e^{2V}\to e^{i\Lambda^+}e^{2V}e^{-i\Lambda};\qquad \phi\to e^{i\Lambda}\phi;\qquad \tilde\phi\to e^{-i\Lambda^t}\tilde\phi. \tag{4}$$

Quantization of this model can be made by standard methods [2]. In particular, we use the background field method, which allows preserving the background gauge invariance and considerably simplifies calculations of quantum corrections: $e^{2V}\to e^{2V'}\equiv e^{\mathbf{\Omega}^+}e^{2V}e^{\mathbf{\Omega}}$, where $\mathbf{\Omega}$ is

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the background field. Using the background gauge invariance it is possible to set $\Omega = \Omega^+ = \boldsymbol{V}$. The gauge is fixed by adding the following term:

$$S_{gf} = -\frac{1}{32e^2}\,\mathrm{tr}\int d^4x\, d^4\theta\,\Big(V\boldsymbol{D}^2\bar{\boldsymbol{D}}^2 V + V\bar{\boldsymbol{D}}^2\boldsymbol{D}^2 V\Big), \qquad (5)$$

where we defined the background covariant derivatives

$$\boldsymbol{D} \equiv e^{-\boldsymbol{\Omega}^+}\frac{1}{2}(1+\gamma_5)De^{\boldsymbol{\Omega}^+};\quad \bar{\boldsymbol{D}} \equiv e^{\boldsymbol{\Omega}}\frac{1}{2}(1-\gamma_5)De^{-\boldsymbol{\Omega}};\quad \boldsymbol{D}_\mu \equiv -\frac{i}{4}(C\gamma^\mu)^{ab}\Big\{\boldsymbol{D}_a, \bar{\boldsymbol{D}}_b\Big\}. \qquad (6)$$

The corresponding ghost Lagrangian is

$$S_c = i\,\mathrm{tr}\int d^4x\, d^4\theta\,(\bar c + \bar c^+)V\Big[(c+c^+) + \mathrm{cth}\,V(c-c^+)\Big]. \qquad (7)$$

Also it is necessary to add the Nielsen-Kallosh ghosts

$$S_B = \frac{1}{4e^2}\mathrm{tr}\int d^4x\, d^4\theta\, B^+ e^{\boldsymbol{\Omega}^+} e^{\boldsymbol{\Omega}}\, B. \qquad (8)$$

In order to calculate quantum corrections we also introduce additional sources

$$S_{\phi_0} = \frac{1}{4}\int d^4x\, d^4\theta\,\Big(\phi_0^+ e^{2V}\phi + \tilde\phi_0^+ e^{-2V^t}\tilde\phi\Big) + h.c. \qquad (9)$$

where ϕ_0 and $\tilde\phi_0$ are scalar superfields. Differentiation with respect to additional sources allows calculating vacuum expectation values of some composite operators.

We regularize model (2) by the higher covariant derivatives [3]. For this purpose we add to the action the higher derivative term

$$S_\Lambda = \frac{1}{2e^2}\mathrm{tr}\,\mathrm{Re}\int d^4x\, d^4\theta\, V\frac{(\boldsymbol{D}_\mu^2)^{n+1}}{\Lambda^{2n}}V, \qquad (10)$$

which is invariant under the background gauge transformations, but breaks the BRST-invariance. Therefore, calculating quantum corrections it is necessary to use a special subtraction scheme, which restores the Slavnov-Taylor identities in each order [4]. In order to cancel the remaining one-loop divergences we should also insert to generating functional the Pauli–Villars determinants [5]:

$$Z[J,\boldsymbol{\Omega}] = \int D\mu\prod_i\Big(\det PV(V,\mathbf{V},M_i)\Big)^{c_i}\exp\Big\{iS + iS_\Lambda + iS_c + iS_B + iS_{gh} + \mathrm{Sources}\Big\}, \qquad (11)$$

where the coefficients c_i satisfy the conditions $\sum\limits_i c_i = 1$; $\sum\limits_i c_i M_i^2 = 0$.

We will calculate the Gell-Mann–Low function. If $\mathbf{V}$ denotes the background gauge field and

$$\Gamma^{(2)} = -\frac{1}{16\pi}\mathrm{tr}\int\frac{d^4p}{(2\pi)^4}\, d^4\theta\,\mathbf{V}(-p)\,\partial^2\Pi_{1/2}\mathbf{V}(p)\, d^{-1}(\alpha,\mu/p) +$$
$$+\frac{1}{4}\int\frac{d^4p}{(2\pi)^4}\, d^4\theta\,\Big(\phi^+(-p,\theta)\,\phi(p,\theta) + \tilde\phi^+(-p,\theta)\,\tilde\phi(p,\theta)\Big)\, ZG(\alpha,\mu/p), \qquad (12)$$

where α is a renormalized coupling constant and Z is a renormalization constant for the matter superfield, then the Gell-Mann–Low function $\beta(\alpha)$ and the anomalous dimension $\gamma(\alpha)$ are defined by

$$\beta\Big(d(\alpha,\mu/p)\Big) = \frac{\partial}{\partial\ln p}d(\alpha,\mu/p);\quad \gamma\Big(d(\alpha,\mu/p)\Big) = -\frac{\partial}{\partial\ln p}\ln ZG(\alpha,\mu/p). \tag{13}$$

Calculation of the matter contribution to the Gell-Mann–Low function can be made substituting solutions of Slavnov–Taylor identities to the Schwinger–Dyson equation for the two-point Green function of the gauge superfield [6]. The Schwinger–Dyson equations (without subtraction diagrams) can be written as a sum of two effective diagrams, presented in Fig. 1. Here (in the massless case for simplicity) the vertexes are given by

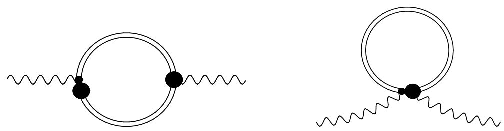

Figure 1: Effective diagrams, obtained from Schwinger–Dyson equations for the two-point Green function of the gauge superfield

$$\frac{\delta^2\Gamma}{\delta\phi_{0x}^+\delta\phi_y} = -\frac{1}{8}G(\partial^2)\bar D_x^2\delta^8_{xy};\qquad \frac{\delta^3\Gamma}{\delta\mathbf{V}_x^a\delta\phi_z^+\delta\phi_w};\qquad \frac{\delta^3\Gamma}{\delta\mathbf{V}_x^a\delta\phi_{0z}^+\delta\phi_w}, \tag{14}$$

and the propagator is

$$\frac{\delta^2\Gamma}{\delta\phi_x^+\delta\phi_y} = -\frac{D_x^2\bar D_x^2}{4\partial^2 G(\partial^2)}\delta^8_{xy}. \tag{15}$$

Expressions for vertexes in the limit $p\to 0$, where p is a momentum of the external gauge superfield, can be found by solving Slavnov-Taylor identities:

$$\frac{\delta^3\Gamma}{\delta\mathbf{V}_y^a\delta\phi_{0z}^+\delta\phi_x}\Bigg|_{p=0} = eT^a\Big[-2F\partial^2\Pi_{1/2y}\Big(\bar D_y^2\delta^8_{xy}\delta^8_{yz}\Big) + \frac{1}{8}fD^bC_{bc}\bar D_y^2\Big(\bar D_y^2\delta^8_{xy}D_y^c\delta^8_{yz}\Big) +$$
$$+\frac{i}{16}\partial_x^\mu G'\bar D\gamma_\mu\gamma_5 D_y\Big(\bar D_y^2\delta^8_{xy}\delta^8_{yz}\Big) - \frac{1}{4}G\bar D_y^2\delta^8_{xy}\delta^8_{yz}\Big]; \tag{16}$$
$$\frac{\delta^3\Gamma}{\delta\mathbf{V}_y^a\delta\phi_z^+\delta\phi_x}\Bigg|_{p=0} = eT^a\Big[F\partial^2\Pi_{1/2y}\Big(\bar D_y^2\delta^8_{xy}D_y^2\delta^8_{yz}\Big) - \frac{i}{32}\partial_x^\mu G'\bar D\gamma_\mu\gamma_5 D_y\Big(\bar D_y^2\delta^8_{xy}D_y^2\delta^8_{yz}\Big) +$$
$$+\frac{1}{8}G\bar D_y^2\delta^8_{xy}D_y^2\delta^8_{yz}\Big]. \tag{17}$$

where all functions depend on ∂_x^2. The functions F and f can not be found from the Slavnov–Taylor identities because they are multiplied by the transversal part of the gauge superfield. Actually vertexes (16) and (17) are defined by the same diagrams, but in the first case one of the external matter lines is not chiral.

In order to find the Gell-Mann–Low function we consider the massless theory and calculate the expression

$$\frac{d}{d\ln\Lambda}d_0^{-1}(\alpha_0,\Lambda/p)\Bigg|_{p=0}, \tag{18}$$

where d_0 is defined exactly as the function d, but supposing that $Z=1$. (The dependance on Z can be easily restored.) Then, substituting propagators and vertexes into the effective diagrams, presented in Fig. 1, we obtain

$$-\frac{1}{16\pi}\frac{d}{d\ln\Lambda}d_0^{-1}\Big|_{p=0} = \ldots - C(R)\int\frac{d^4q}{(2\pi)^4}\frac{d}{d\ln\Lambda}\Big\{\frac{1}{2q^2}\frac{d}{dq^2}\ln\Big(q^2G^2\Big) - \frac{8f}{q^2G}\Big\} - (PV), \qquad (19)$$

where dots denote contributions of the gauge field and ghosts and (PV) denotes a contribution of the Pauli– Villars fields. (Explicit form of (PV) can be found, for example, in [6, 7].) The first term in Eq. (19) is an integral of the total derivative and can be easily calculated using the identity

$$\int\frac{d^4k}{(2\pi)^4}\frac{1}{k^2}\frac{d}{dk^2}f(k^2) = \frac{1}{16\pi^2}\Big(f(k^2=\infty) - f(k^2=0)\Big). \qquad (20)$$

The corresponding contribution to the β-function is in agreement with NSVZ expression (1). However, if the second term in Eq. (19) is not 0, then the Gell-Mann-Low function differs from NSVZ beta-function in the substitution

$$\gamma(\alpha) \to \gamma(\alpha) + \lim_{p\to 0}\frac{16f(p^2)}{p^2G(p^2)}. \qquad (21)$$

Nevertheless, the explicit calculations in three- [8] and, partially, four-loop approximation [9] always show that in Abelian case the second term in Eq. (19) is also an integral of the total derivative and is always equal to 0. This allows suggesting existence of the new identity

$$\int\frac{d^4q}{(2\pi)^4}\frac{d}{d\ln\Lambda}\frac{f}{q^2G} = 0, \qquad (22)$$

which is not a consequence of supersymmetric or gauge Slavnov–Taylor identities. (The derivative with respect to $\ln\Lambda$ is needed in order to make all integrals well defined.) In the massive case similar identity [6] can be written in the graphical form, presented in Fig. 2. (The symbol $D_a\bar D^2$ denotes the combination of derivatives in the $\phi^+ - \phi$ propagator. The propagator $\phi - \tilde\phi$ should be also modified, see Ref. [6] for more details.)

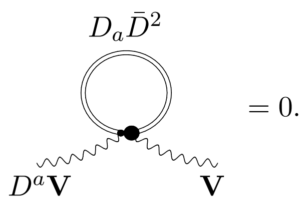

Figure 2: Graphical form of identity (22), which is also valid in the massive case

In order to verify if the new identity takes place in the non-Abelian case, we consider [10] the three-loop graph, presented in Fig. 3, and construct a group of diagrams, which are obtained from this graph by attaching two external gauge lines to the matter line by all possible ways. The corresponding contribution to the function f should be found calculating the vertex diagrams, which are obtained from the graph, presented in Fig. 4, by attaching one external gauge line to the matter line by all possible ways. (This graph is found by cutting the graph, presented in Fig. 3.) After calculating this diagrams and substituting the result to the left hand side of identity (22) we obtain

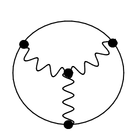

Figure 3: We consider diagrams, which are obtained from this graph by attaching to external gauge lines to the matter line by all possible ways

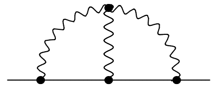

Figure 4: Graph, defining the function f for the considered group of diagrams. In order to obtain usual diagrams it is necessary to attach one external gauge line to the matter line by all possible ways

$$\int \frac{d^4q}{(2\pi)^4}\,\frac{d}{d\ln\Lambda}\,\frac{f(q^2)}{q^2\,G(q^2)} = \alpha^2\pi^2\,C_2\Big(C_2(R)-\frac{1}{2}\,C_2\Big)\int\frac{d^4q\,d^4k\,d^4l}{(2\pi)^{12}}\,\frac{\partial}{\partial q^\mu}\Bigg\{\Lambda\frac{d}{d\Lambda}\Big[\frac{1}{(k+q)^2}\times$$
$$\times\frac{(k+q+l)^\mu}{(k+q+l)^2\Big(1+k^{2n}/\Lambda^{2n}\Big)\Big(1+l^{2n}/\Lambda^{2n}\Big)q^2k^2l^2(k+l)^2\Big(1+(k+l)^{2n}/\Lambda^{2n}\Big)}\Big]\Bigg\} = 0. \quad (23)$$

Therefore, the new identity and the factorization of integrands to total derivatives seem also to take place in the non-Abelian case.

In order to check if the factorization of integrands to total derivatives is a general feature of supersymmetric theories, we calculate the two-loop β-function for the $N=1$ supersymmetric Yang-Mills theory without matter [11]. It well known that in this case

$$d_0^{-1}(\alpha,\Lambda/p) = d_2\ln\frac{\Lambda}{p} + \mathrm{const} + O(\alpha^2). \quad (24)$$

Two-loop diagrams, defining the Gell-Mann–Low function, are presented in Fig. 5. The function d_0 can be obtained by calculating the diagrams without counterterms insertions. Then in the limit $p\to 0$ we find (in the Euclidean space after the Weak rotation)

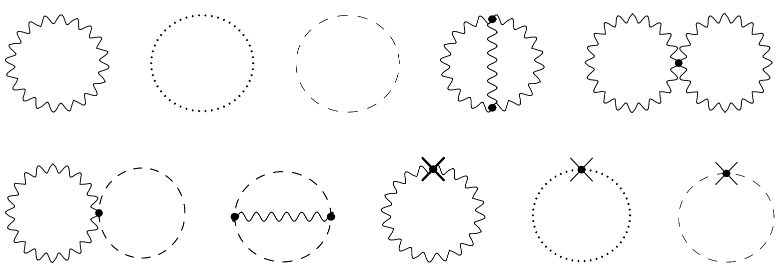

Figure 5: Diagrams, defining two-loop Gell-Mann–Low function for $N=1$ supersymmetric Yang–Mills theory

$$d_2 = 8\pi\cdot 6\pi\,\alpha_0\,\frac{d}{d\ln\Lambda}\int\frac{d^4k}{(2\pi)^4}\,\frac{1}{k^2}\,\frac{d}{dk^2}\int\frac{d^4q}{(2\pi)^4}\Big(q^2(1+q^{2n}/\Lambda^{2n})(q+k)^2\Big)^{-1}\times$$
$$\times(1+(q+k)^{2n}/\Lambda^{2n})^{-1}\Big[2(n+1)\Big(1+k^{2n}/\Lambda^{2n}\Big)^{-1} - 2n\Big(1+k^{2n}/\Lambda^{2n}\Big)^{-2}\Big]. \quad (25)$$

This integral can be calculated by Eq. (20). The corresponding two-loop Gell-Mann-Low function coincides with the NSVZ expression (1) in the considered approximation:

$$\beta(\alpha) = -\frac{3C_2\alpha^2}{2\pi} - \frac{3\alpha^3C_2^2}{(2\pi)^2} + O(\alpha^4). \quad (26)$$

Therefore, factorization of integrands to total derivatives seems to be a general feature of all supersymmetric theories. However, the reason is so far unclear. Actually new identity for Green functions (22) is a consequence of this fact. This identity is not a consequence of the supersymmetric or gauge Slavnov-Taylor identities. The corresponding terms in the effective action are invariant under rescaling. What symmetry leads to this identity? How it can be proven? So far I can not answer these questions.

Acknowledgments

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