The Leading Singularity Method at Two Loops

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Abstract

I provide a brief introduction to the leading singularity method for determining multi-loop scattering amplitudes, focusing in particular on the two-loop six-particle MHV amplitude in maximally supersymmetric Yang-Mills theory.

1 Singularities of the S-Matrix

It has long been known that much of the structure of scattering amplitudes can be unlocked by understanding just their singularities. One of the most surprising features of both Yang-Mills theory and gravity, that has only fully emerged in the last few years, is that their tree-level amplitudes are completely determined by their behavior near only a small subset of their singularities [4, 5, 9].

In $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) theory, the problem of computing any one-loop amplitude can be reduced to that of computing tree amplitudes [3]. The key to such simplification is that although one-loop amplitudes have many poles and branch cuts with a complicated structure of intersections, they are completely determined by their highest codimension singularities, called the leading singularities.

Higher loop amplitudes have even more complicated singularities and it is not yet completely understood how to fully exploit the structure of these singularities. In this talk I describe, and then apply, a method [11] based on leading singularities which has three attractive features:

- it provides a natural basis of integrals to work with,
- the coefficients of the integrals are determined by solving linear equations,
- and those linear equations are almost trivial to write down.

This method is quite general, but the specific target of our current interest is a calculation of the two-loop six-particle MHV amplitude in $\mathcal{N} = 4$ super-Yang-Mills theory. The parity-even part of this amplitude was recently computed by other means in [10]. The leading singularity method successfully reproduces this result, and also computes the parity-odd part of the amplitude with no additional work.

2 The Leading Singularity Method

2.1 Background

Any L-loop scattering amplitude can, in principle, be obtained by summing over all contributing Feynman diagrams:

$$\mathcal{A}^{(L)}(p) = \int d\ell_1 \cdots d\ell_L \sum_j F_j(p, \ell), \qquad (1)$$

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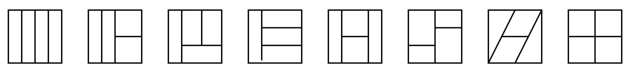
where p and ℓ denote the external and loop momenta respectively. However, in practice this is a hopeless exercise due to the enormously large number of Feynman diagrams, as well as their complexity, in Yang-Mills theory.

Rather, calculations typically proceed by first finding a representation of the amplitude in terms of a relatively simple basis is integrals $\{I_i\}$:

$$\mathcal{A}^{(L)}(p) = \sum_i c_i(p) \int d\ell_1 \cdots d\ell_L \, I_i(p, \ell), \tag{2}$$

where the coefficients $c_i(p)$ (which, importantly, depend only on the external momenta) are computed by other means, such as the unitarity-based method or maximal cuts [8].

As a state of the art example, unitarity-based methods were used to express the four-loop four-particle amplitude in $\mathcal{N} = 4$ super-Yang-Mills theory as the sum of just eight integrals [7]:



It is important to emphasize that these figures do not represent individual Feynman diagrams—indeed there would be far too many to draw individually, including many with topologies different from just the eight shown here. Rather, the calculation of [7] indicates that when one adds up the enormously large number of Feynman diagrams of any other topologies, they magically add up to zero, leaving simple contributions from only the integrals shown above.

One important difficulty in implementing this procedure in general is that there is no known basis of integrals for a general n-particle L-loop amplitudes, even in a theory as simple as $\mathcal{N} = 4$ SYM. Two special cases include

- one loop, where scalar box integrals provide a complete basis for any n, and
- a very plausible conjecture exists for four particles at any number of loops, which has emerged from the study of dual conformal symmetry.

Another difficulty is that even when a suitable basis of integrals is known (or can be guessed) for a particular amplitude, determining the coefficients can require a difficult calculation.

2.2 The Leading Singularity Method

The idea behind the leading singularity method is to equate (1) and (2) at the level of the integrand,

$$\sum_i c_i(p) I_i(p, \ell) = \sum_j F_j(p, \ell), \tag{3}$$

and instead of integrating over the real ℓ-axis in $\mathbb{C}^{4L}$ as usual, we instead choose to integrate (3) over closed contours $\Gamma \subset \mathbb{C}^{4L}$ to obtain linear equations for the desired coefficients:

$$\sum_i c_i(p) \int_\Gamma I_i(p, \ell) = \int_\Gamma \sum_j F_j(p, \ell). \tag{4}$$

We can require this to be true for any contour Γ, so by choosing many different contours we can get many different linear equations.

If we choose a random contour in $\mathbb{C}^{4L}$ we would get the useless equation

$$0 = 0. \tag{5}$$

In order to get useful equations, we should identify the isolated poles of the integrand on the right-hand side of (3), and for each one choose a contour $\Gamma \subset \mathbb{C}^{4L}$ such that integrating over Γ computes the residue of the integrand at the corresponding pole.

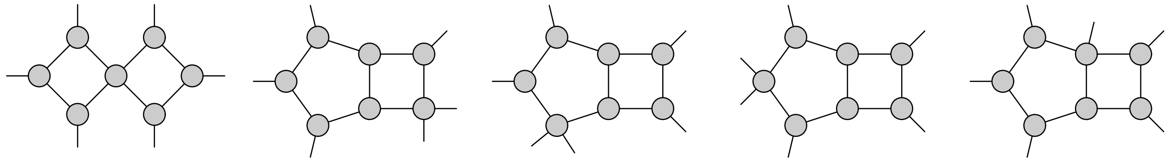
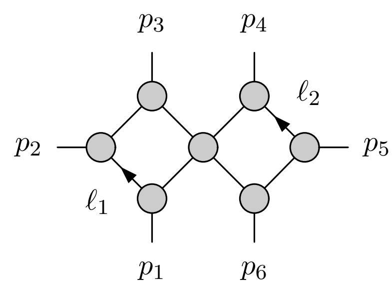


Figure 1: The eight-topology propagators.

Fortunately it is easy to identify the isolated poles of the right-hand side of (3): the poles in Feynman diagrams occur when internal propagators go on-shell. In order to completely localize the loop integral at L-loops we are interested in poles of order $4L$. These poles are the leading singularities.

For the two-loop six-particle amplitude there are five obvious topologies (shown in Fig. 1) where eight different propagators can go simultaneously on-shell and hence are associated with leading singularities. For example, the first diagram



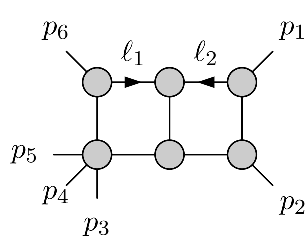
represents the sum over the subset of all Feynman diagrams in which all eight of the indicated propagators are present. This set of Feynman diagrams has isolated poles at the locus

$$S = \{(\ell_1, \ell_2) \in \mathbb{C}^8 : \quad \begin{array}{l} \ell_1^2 = 0, (\ell_1 + p_1)^2 = 0, (\ell_1 - p_2)^2 = 0, (\ell_1 - p_2 - p_3)^2 = 0, \\ \ell_2^2 = 0, (\ell_2 - p_4)^2 = 0, (\ell_2 + p_5)^2 = 0, (\ell_2 + p_5 + p_6)^2 = 0\}. \end{array} \tag{6}$$

For generic external momenta p_i this set consists of four distinct points in $\mathbb{C}^8$. At each of these four points, the integrand has an isolated pole of degree 8.

To calculate the residue at one of these poles (i.e., the result of integrating the right-hand side of (3) over one of the associated contours Γ) is simple: just take the product of seven on-shell tree-level amplitudes, at each of the grey circles, and evaluate this product at the corresponding solution (ℓ_1, ℓ_2) for the loop momenta.

In addition to these obvious ones, there are other more subtle leading singularities. To see how these arise, consider the topology



Although it looks like the maximal singularity is a degree-7 pole (because there are only 7 propagators apparent), in fact there is a hidden pole of order 8 in this topology. In order to expose it, consider first a contour integral which fixes the loop momenta in the right-hand box.

$$\int_\Gamma d^4\ell_2 \frac{1}{\ell_2^2(\ell_1+k_1)^2(\ell_1+k_1+k_2)^2(\ell_1+\ell_2)^2} = \frac{1}{2}\frac{1}{(k_1+k_2)^2(\ell_2-k_1)^2}$$

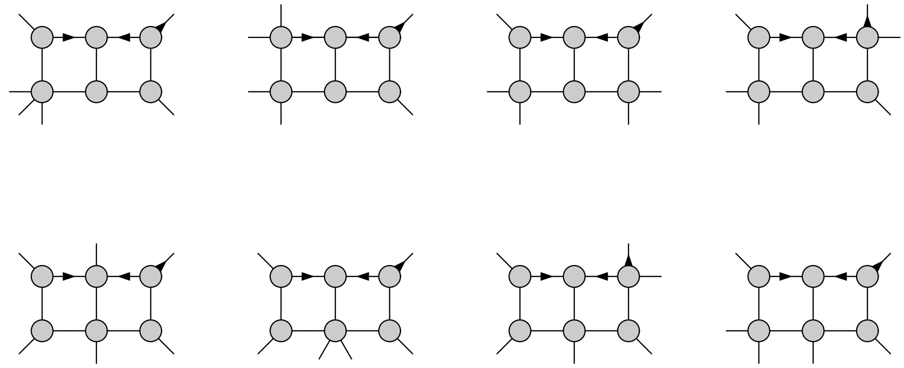


Figure 2: The seven-topology propagators.

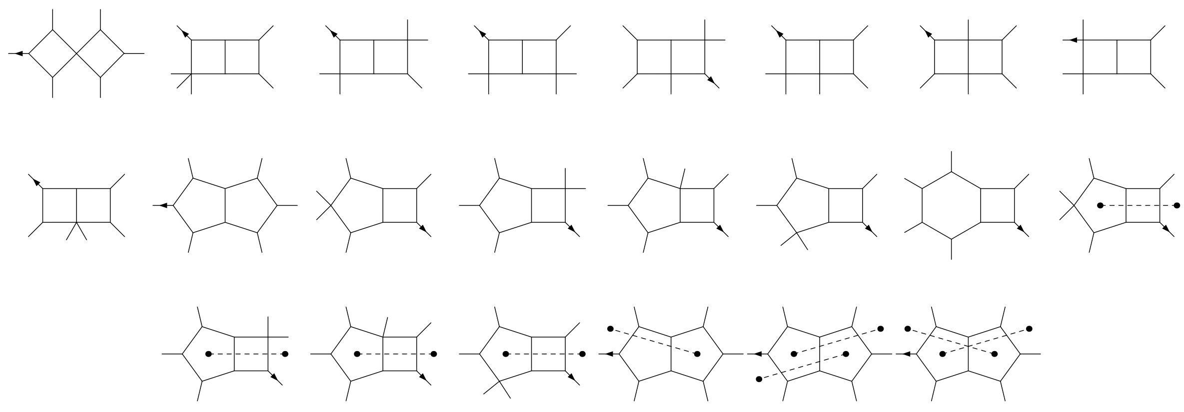


Figure 3: A basis for the two-loop six-particle amplitude in SYM.

where the right-hand side is just the Jacobian evaluated at the location of the singularity. Now this Jacobian has itself an additional singularity $1/(\ell_2 - k_1)^2$. The conclusion is that there do exist isolated poles of order 8 in such double-box topologies. There are a total of 8 different topologies of this type, shown in Fig. 2.

2.3 Constructing a Basis of Integrals

Next we need to construct a set of integrals $\{I_i\}$ in terms of which to express the amplitude. The construction proceeds as follows. We begin with the 13 scalar integrals appropriate to the 13 different topologies shown above. It turns out that with just this set of integrals, the linear equations have no solution, so we must add additional integrals to the set $\{I_i\}$. There is a systematic procedure to implement this, which finishes when the basis is large enough that a solution to all of the linear equations can be found.

It can happen that when this procedure finishes, one ends up with a set $\{I_i\}$ that is over-complete. This happens because loop integrals for 6 or more external particles can frequently be expressed as linear combinations of other integrals [1]. If this happens, then the equations do not have a unique solution: given any solution $\{c_i\}$ one can add any set of coefficients $\{\tilde{c}_i\}$ that is actually zero due to a reduction identity.

Ultimately we find a representation of the 2-loop six-particle MHV amplitude in terms of the integrals shown in Fig. 3. Several of the coefficients can be set to zero by using reduction identities. The parity-even part of the amplitude agrees with the recent result of [10] obtained via other methods. Note that the full coefficients, both the parity even and parity odd parts, emerge from solving the same linear equations in the leading singularity method—in fact it is unnatural to separate the two parts, and we have only done this in order to make the comparison and check our results.

3 The ABDK/BDS Conjecture

One of the reasons for recent interest in multi-loop amplitudes in $\mathcal{N} = 4$ super-Yang-Mills theory is the ABDK/BDS conjecture [2, 6] for multi-loop MHV amplitudes, which at two loops takes the form

$$M_n^{(2)}(\epsilon) = \frac{1}{2}(M^{(1)}(\epsilon))^2 - (\zeta(2) + \zeta(3)\epsilon + \zeta(4)\epsilon^2)M^{(1)}(2\epsilon) - \frac{\pi^4}{72} + \mathcal{O}(\epsilon) \tag{7}$$

in dimensional regularization to $D = 4 - 2\epsilon$, where $M_n^{(L)} = \mathcal{A}_n^{(L)}/\mathcal{A}_{\text{tree}}^{(L)}$.

Although this conjecture is now known to be false in general—this was one of the main results of the recent paper [10], we find numerical evidence that the parity-odd part of the two-loop six-particle amplitude does in fact satisfy 7. In fact it is reasonable to believe that the parity-odd part always satisfies ABDK/BDS, but this remains unproven.

It is important to note that since the leading singularity method outlined above takes all loop momenta ℓ to be precisely four-dimensional, it is insensitive to any terms in the dimensionally-regulated amplitude which vanish in $D = 4$. For $n = 6$ it has been shown that these so-called μ-terms drop out of (7).

4 Conclusion

The motivation for our work was two-fold: to unlock previously hidden mathematical richness lurking deep inside multi-loop amplitudes in $\mathcal{N} = 4$ SYM, and to exploit that structure to help simplify otherwise formidable computations. The leading singularity method provides a relatively simple way to find representations of complicated amplitudes in terms of a simple basis of integrals by just solving linear equations.

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