Quasi-classical Hawking Temperatures and Black Hole Thermodynamics

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Abstract

The semi-classical derivation of Hawking radiation for axially symmetric, stationary spacetimes with a Killing horizon is examined following the recent quasi-classical tunneling analysis [1] and a simple formula is found for the inverse Hawking temperature $\beta = 1/T_H$. The formula is invariant under canonical transformations and is shown to be equivalent to the integral of a closed differential 1-form around the horizon enclosing a pole. The Hawking temperature is then given in terms of the winding number in the first homotopy group of the torus formed from the compactified imaginary parts of the analytically continued radial and time variables.

1 Introduction

The semi-classical WKB formula [2, 3] for deriving the Hawking temperature in spacetime backgrounds with a Killing horizon has attracted much attention in the recent literature. The original analysis was motivated by the idea that the particles constituting the Hawking radiation should arise via tunneling out of the black hole. Thus, one should be able to explicitly calculate the temperature in this way. This semi-classical 'tunneling' formula has evolved significantly since its inception due to several problems arising when generalizing it to various black hole backgrounds and alternate coordinate systems. We will begin by describing these problems and their resolutions.

The first problem with the semi-classical tunneling method is that it is not actually tunneling in the usual sense. Instead it is closer to an over-the-barrier reflection problem [4] since the particle momentum is never imaginary. However, there does appear an imaginary contribution to the tunneling amplitude due to the fact that the classical path of the particle has a pole at the horizon and so it is analytically continued around the pole.

The next problem is that, although the original semi-classical method [3] works in the case of the Schwarzschild black hole in Painlevé coordinates, it gives incorrect temperatures in many other backgrounds and coordinates [5].

Finally, the original formula is not canonically invariant as it must be in order that it describe proper quantum mechanical observables [6]. We showed [5] that this last problem can be corrected by using a canonically invariant formula, but with the result that all temperatures, including that found in Painlevé coordinates, then differed by a factor of two from the standard Hawking temperature. This was the so-called factor two problem. Recently this factor two problem has been solved. The solution came via the discovery of a previously overlooked contribution from the transformation of the *time variable* [1] coming from the fact that the time coordinate changes sign across the horizon. This new temporal contribution must be incorporated in order to get the correct Hawking temperatures. Although this new contribution fixes

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the factor of two problem, it also destroys the close analogy to semi-classical tunneling which motivated the tunneling analysis in the first place. In other words, *it is not the usual quantum mechanical tunneling that produces Hawking radiation*, the difference comes from the fact that general relativity treats time in a different way than conventional quantum mechanics. We refer to the semi-classical method, including the new temporal contribution, as quasi-classical tunneling.

In the following, we begin with a summary of the semi-classical derivation of the Hawking temperature along with the new contribution coming from the time variable. Next, these contributions are shown to combine into the integral of a differential 1-form around the horizon which is therefore a canonically invariant result. However, the path is not closed in the time variable. We remedy this by changing coordinates to compactify the imaginary part of the time coordinate at the horizon allowing us to derive the hawking temperature in terms of the integral of the new canonical momentum 1-form around the horizon. We then use the 1-form to show that the temperature can be found in terms of the periodicity in the Euclidean time as you circle the horizon. Finally a connection is made with black hole thermodynamics using a method similar to that given in [7] wherein the fluctuations in the black hole mass lead to a similar formula.

2 Quasi-classical tunneling

In this section we review the details of the canonically invariant semi-classical method. The purpose is two-fold. First, many researchers continue to use the original, incorrect, semi-classical formula and so it is deemed necessary to show explicitly why the canonically invariant one must be used. Second, the resulting formulas lead, in the following section, to the formulation of quasi-classical tunneling which is the subject of this work.

Consider an axially symmetric, stationary background with Killing horizon at $r = r_H$ where we choose the 'mostly plus' metric signature and natural units so that all of the constants are 1. A scalar field moving in this background is given, in the semi-classical approximation, by $\phi = e^{-\frac{iS}{\hbar}}$ where S is the action. The Klein-Gordon equation

$$-\frac{\hbar^2}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi\right) + m^2\phi = 0 \tag{1}$$

then reduces, at zeroth order in $\hbar$, to the Hamilton-Jacobi equation

$$g^{\mu\nu}\partial_\mu S\partial_\nu S + m^2 = 0, \tag{2}$$

where $g_{\mu\nu}$ is an arbitrary 2-dimensional metric for the (t, r) section of the spacetime at the equator $\theta = \pi/2$.

We now split the action into a temporal and spatial part, $S = Et + S_0(\vec{x})$, and substitute into the Hamilton-Jacobi equation. The solution is

$$\partial_r S_0 = \frac{1}{g_{00}}\left[g_{01}E \pm \sqrt{-g}\sqrt{E^2 + g_{00}m^2}\right] \tag{3}$$

which will be used below to find the black hole decay rate.

Let us now examine the lagrangian coming from the metric:

$$\mathcal{L} = \frac{1}{2}g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}, \tag{4}$$

where τ is some affine parameter along the geodesic which may be the proper time in the case of massive particles and the usual affine parameter in the case of null geodesics. For massless particles $\mathcal{L} = 0$ and for particles of mass m, $2\mathcal{L} = -m^2$.

This lagrangian defines the canonical momenta in the usual way as

$$p_r \equiv \frac{\partial \mathcal{L}}{\partial \dot{r}} = g_{01}\dot{t} + g_{11}\dot{r}$$
$$p_t \equiv -\frac{\partial \mathcal{L}}{\partial \dot{t}} = -g_{00}\dot{t} - g_{01}\dot{r} \equiv E, \tag{5}$$

where the dot denotes the derivative with respect to the affine parameter τ. We can use p_t to eliminate $\dot{t}$ from (4) leaving an expression for the coordinate velocity in the radial direction

$$\dot{r} = \pm\sqrt{\frac{g_{00}m^2 + E^2}{-g}} \tag{6}$$

where $+$ is for out-going particles and $-$ for in-going and we have denoted the determinant of the metric by g. Thus, the momentum component p_r can be written as

$$p_r = -\frac{1}{g_{00}}\left[g_{01}E \pm \sqrt{-g}\sqrt{E^2 + g_{00}m^2}\right]. \tag{7}$$

Comparing this expression with (3) we see that $\partial_r S_0 = -p_r$, and we can solve for the spatial part of the action

$$\Delta S_0 = -\int p_r dr \tag{8}$$

The decay rate is given by

$$\Gamma = |\langle x_f|\, x_i\rangle|^2 = \langle x_i|\, x_f\rangle \langle x_f|\, x_i\rangle$$
$$\sim e^{-\text{Im}\ \Delta S} = e^{\text{Im}\ \left(E\Delta t - \oint p_r dr\right)} \tag{9}$$

where the canonically invariant integral is over the closed path as shown in [5].

In [1] it was shown using Kruskal-Szekeres coordinates that the time variable changes as $t \to t - 2\pi i M$ as one crosses the horizon and so, for the closed path in (9), we will have $\Delta t = -4\pi i M$. Thus the decay rate (9) is

$$\Gamma \sim e^{-4\pi ME} e^{-\ \text{Im}\ \oint p_r dr} \equiv e^{-\beta E}. \tag{10}$$

The inverse temperature $\beta = 1/T_H$ is then given by

$$\beta E = 4\pi ME +\ \text{Im}\ \oint p_r dr. \tag{11}$$

Now insert the expression for the canonical momentum (7) to find

$$\beta = 4\pi M - \frac{1}{E}\ \text{Im}\ \oint \frac{g_{01}E \pm \sqrt{-g}\sqrt{E^2 + g_{00}m^2}}{g_{00}} dr. \tag{12}$$

One can see that there is a pole at the horizon $r = r_H$ since $g_{00}(r_H) = 0$ and so we shift the pole into the upper half plane $r_H \to r_H + i\epsilon$ and write $g_{00} = g'_{00}(r_H)(r - r_H - i\epsilon)$ for the out-going part of the path and into the lower half plane for the in-going part of the path. The result is a positively oriented (counter-clockwise) path around the horizon. The result is

$$\beta = 4\pi M - \lim_{\epsilon\to 0} \text{Im}\ \oint \frac{g_{01} \pm \sqrt{-g}}{g'_{00}(r_H)(r - r_H \pm i\epsilon)} dr \tag{13}$$

where we have set $\epsilon = 0$ in the mass term and the upper/lower signs correspond to out-going/in-going particles respectively.

Let us examine this expression for β in detail in order to show how it corrects the differences that were encountered between Painlevé and other coordinates systems in the original (incorrect) semi-classical method. We split the integral into the out-going and in-going parts

$$\begin{aligned}\beta = 4\pi M - &\frac{1}{g_{00}'(r_H)}\lim_{\epsilon\to 0} \\ &\text{Im}\left\{\int_{r_i}^{r_f}\frac{g_{01}+\sqrt{-g}}{r-r_H-i\epsilon}+\int_{r_f}^{r_i}\frac{g_{01}-\sqrt{-g}}{r-r_H+i\epsilon}\right\}dr\end{aligned} \tag{14}$$

where $r_i < r_H$ and $r_f > r_H$. We see explicitly by this expression that, for metrics with an off-diagonal component g_{01}, there is a difference in sign between the in-going term and the out-going term. This causes the g_{00} terms to cancel out between the out-going and in-going terms and is precisely the reason why the amplitude is different for the in-going and out-going paths in coordinates, such as Painlevé, where $g_{01} \neq 0$. Thus if, as is often done, one simply squares the amplitude coming from the out-going path, then the g_{01} component will not be canceled as it should be and the result is a factor of 2 multiplying β. This can be seen by noticing that on the horizon we have $g_{01} = 1$ and $\sqrt{-g} = 1$ in Painlevé coordinates. Thus, although the original method gives the correct Hawking temperature, this is a coincidence and only works in special coordinate systems. Continuing one step further, we can use

$$\lim_{\epsilon\to 0}\text{Im}\ \frac{1}{r-r_H\pm i\epsilon} = \pi\delta(r-r_H) \tag{15}$$

to get a simple expression for the inverse temperature

$$\beta = 4\pi M - \left.\frac{2\pi\sqrt{-g}}{g_{00}'}\right|_{r=r_H} \tag{16}$$

Later on, we will suggest a connection between quasi-classical tunneling and black hole thermodynamics. To make this connection more apparent let us here take note of an alternate way of writing the inverse temperature given in (14). That is

$$\begin{aligned}\beta &= 4\pi M - 2\ \text{Im}\int_{r_i}^{r_f}\frac{\sqrt{-g}}{g_{00}}dr \\ &= 4\pi M + i\oint\frac{\sqrt{-g}}{g_{00}}dr.\end{aligned} \tag{17}$$

The inverse temperature written in this way will appear when we examine the first law of back hole thermodynamics using the method of [7] in the penultimate section below.

Let us now return to (11) and parameterize the path in terms of a parameter θ as follows

$$\begin{aligned}\beta E &= -\text{Im}\ E\Delta t + \text{Im}\oint p_r\frac{dr}{d\theta}d\theta \\ &= -\text{Im}\oint E\frac{dt}{d\theta}d\theta + \text{Im}\oint p_r\frac{dr}{d\theta}d\theta \\ &= \text{Im}\oint\left(-p_t\frac{dt}{d\theta}+p_r\frac{dr}{d\theta}\right)d\theta \\ &= \text{Im}\oint P_\mu\frac{dx^\mu}{d\theta}d\theta \\ &= \text{Im}\oint P_\mu dx^\mu\end{aligned} \tag{18}$$

where we have defined $P_0 = -p_t$ and $P_1 = p_r$. We have abused the notation in the above formula since the path in the t variable is not closed. Both r and t are continued to complex values and both depend on the path via a parameter θ. Explicitly, we parameterize r as $r = 2M + \epsilon e^{i\theta}$ as in [5] then

$$t = \mathrm{Re}(t) - 2Mi\theta \tag{19}$$

so that

$$\frac{dt}{d\theta}d\theta = -2Mid\theta. \tag{20}$$

When θ traverses the closed path, $\theta \to \theta + 2\pi$, the r variable returns to its original value, but the time coordinate changes by $t \to t - 4\pi iM$ and we see that the path is not closed in t. Although this is not a problem *per se* in that our formula gives the correct temperatures, we can close the path in the imaginary part of the time variable and thereby give a more precise topological reason for our result.

The coordinate time diverges at the horizon, which is the reason for the sign change in t as you cross the horizon and is the source of the temporal contribution to the tunneling amplitude. We have seen this contribution by transforming to Kruskal coordinates in [1]. Transforming the (complexified) t variable[1] as

$$T = e^{-t} = e^{-\mathrm{Re}(t)}e^{2Mi\theta} \equiv T_0 e^{2Mi\theta} \tag{21}$$

has the effect of compactifying the imaginary part of the path in t. We then have have a closed path in the imaginary parts of both the t and the r variable as one traverses the horizon. In fact we can write

$$Edt = E\frac{dt}{dT}dT = -\frac{E}{T}dT \tag{22}$$

so that

$$\begin{aligned}
\beta E &= \mathrm{Im} \oint P_\mu dx^\mu \\
&= \mathrm{Im} \oint \left[\frac{E}{T}dT + p_r dr\right] \\
&= \mathrm{Im} \int_0^{2\pi} \left[\frac{E}{T}\frac{dT}{d\theta} + p_r\frac{dr}{d\theta}\right] d\theta \\
&= \mathrm{Im} \int_0^{2\pi} \left[2MiE + p_r\frac{dr}{d\theta}\right] d\theta \\
&= \mathrm{Im}\ [4\pi MiE + 4\pi MiE]
\end{aligned} \tag{23}$$

where we used the residue of the pole in p_r to get the second term. Thus $\beta = 8\pi M$ as expected. Notice that the form $P = P_\mu dx^\mu$ as given in (23) is closed in complex $t - r$ space, but not exact (since there are poles). Thus it is an element of the first homotopy group of the torus $\pi_1(S^1 \times S^1)$ and corresponds to traversing each of the generating loops once, giving the same contribution to the temperature from each one as we have shown in (23) above. One might expect that since we have formulated the expression for the temperature in terms of a 1-form, the result should be coordinate invariant. However, this is not so. The path in (23) is only closed in complex r and t space; the real part of the path is not closed by itself. This means that the Hawking temperature is not invariant under general coordinate transformations since any coordinate transformation which removes the pole on the horizon will result in a vanishing residue.

[1] This transformation is completely unnecessary for our result given in (17). It is only done to close the path in both variables and make a connection to the homotopy group of the torus.

3 Euclidean time

We can use these results to make a connection to the Euclidean time formalism of Gibbons, Hartle and Hawking [9, 10] by eliminating the r dependence of the path in terms of t alone. Restrict consideration to massless particles and let the parameter θ introduced in the previous section be chosen as the affine parameter so that

$$\begin{aligned} \beta E &= -\mathrm{Im} \oint p_t \frac{dt}{d\tau} d\tau + \mathrm{Im} \oint p_r \frac{dr}{d\tau} d\tau \\ &= \mathrm{Im} \oint \left(-g_{00}\dot{t}^2 + g_{11}\dot{r}^2 \right) d\tau \end{aligned} \tag{24}$$

The lagrangian (4) can be used to eliminate the $\dot{r}^2$ term

$$\begin{aligned} \beta E &= \mathrm{Im} \oint \left(-g_{00}\dot{t}^2 - g_{00}\dot{t}^2 - 2g_{01}\dot{r}\dot{t} \right) d\tau \\ &= 2E \ \mathrm{Im} \oint \frac{dt}{d\tau} d\tau = 2E \ \mathrm{Im} \ \Delta t \\ \Rightarrow \beta &= 8\pi M \end{aligned} \tag{25}$$

This says that twice the imaginary part of the change in t as you traverse the loop gives the Hawking temperature. As can be seen by our derivation of Δt in Kruskal-Szekeres coordinates, this is related to the result of Gibbons and Hawking [10] which says that the period of the Euclidean time as you traverse a loop about the horizon is the Hawking temperature.

4 Blackhole thermodynamics

A simple argument can be given to create a connection between our quasi-classical formula and the first law of blackhole thermodynamics. Black hole thermodynamics is a set of relations that are satisfied by black holes and are usually called thermodynamical because of their similarity to the usual laws of classical thermodynamics. The temperature is defined via the first law of black hole thermodynamics as

$$dS = \frac{dM}{T} \equiv \beta \, dM \tag{26}$$

where we are assuming a non-charged, non-rotating black hole. The entropy of the black hole is defined in terms of the horizon radius as $S = A/4 = \pi r^2|_{r=r_H}$ and so the first law can be written as

$$\beta = \frac{dS}{dM} = 2\pi r \frac{dr_H}{dM} \tag{27}$$

We will now assume a fluctuating mass (and therefore radius) and compute the resulting temperature [7, 8]. In particular, the change in the mass could be caused by out-falling matter, i.e. particle emission. The result will be a formula for the temperature which bears a close resemblance to our quasi-classical formula (17) above.

To see this connection explicitly consider again a counterclockwise oriented complex path surrounding the horizon radius and write r_H as a residue integral over this path. Since the Killing horizon is defined as the point where $g_{00} = 0$ we will write this, for Schwarzschild coordinates, as

$$r_H = \frac{i}{2\pi} \oint \frac{\sqrt{-g}}{g_{00}} dr \tag{28}$$

so that (27) becomes

$$\beta = i \frac{dr_H}{dM} \oint \frac{\sqrt{-g}}{g_{00}} dr \tag{29}$$

For a Schwarzschild black hole $r_H = 2M$ and so this immediately gives

$$\beta = 2i \oint \frac{\sqrt{-g}}{g_{00}} dr \tag{30}$$

which is just twice the spatial part of (17). Since the spatial and temporal parts contribute equally this gives the same result as that of our quasi-classical analysis.

5 Examples and potentially problematic cases

The formula (29) seems to be problematic when the metric determinant is zero, or when the horizon radius is at $r_H = 0$. In this section we present some explicit examples where these problems occur and present resolutions to them. We also apply the method to the cases of charged and rotating black holes.

Schwarzschild metric in isotropic coordinates: Strange things sometimes occur in the other methods of calculating the Hawking temperature in isotropic coordinates since the transformation changes the order of the pole at the horizon. On the other hand, our formula works nicely. The transformation to isotropic coordinates is given by

$$r' = r\left(1 + \frac{m}{2r}\right)^2$$

so that

$$g_{00} = -\frac{(2r-m)^2}{(2r+m)^2}$$

and

$$\sqrt{-g} = \frac{(2r-m)(2r+m)}{4r^2}$$

where we notice that $\sqrt{-g}$ is zero on the horizon. (29) gives

$$\beta = -\frac{i}{2} \oint \frac{(2r+m)^3}{4r^2(2r-m)} dr \tag{31}$$

and using a partial fraction decomposition

$$\beta = -\frac{i}{2} \oint \left(1 + \frac{8m}{2r-m} - \frac{2m}{r} - \frac{m^2}{4r^2}\right) dr \tag{32}$$

and circling the pole at $r = m/2$ (which is the zero of g_{00}) we get

$$\beta = 8\pi m \tag{33}$$

as expected.

Rindler space: The form of the Rindler metric for an accelerated observer in Minkowski space with $g_{00} = -(1 + 2ar)$ presents no problem. On the other hand, the form of the Rindler metric with $g_{00} = -\alpha^2 r^2$ has $\sqrt{-g} = \alpha r$. Here, $r_H = 0$ and so the determinant vanishes on the Horizon. Furthermore, the usual formula (29) can not work as it stands since dr_H/dM is not defined. However, if we set $dr_H/dM \equiv 1$ we get

$$\beta = -i \oint \frac{\alpha r}{\alpha^2 r^2} dr = \frac{2\pi}{\alpha} \tag{34}$$

which is the correct temperature. However, even though it gives the correct answer, this is a problematic case because our setting $dr_H/dM = 1$ is *ad hoc* and should be reasoned out more fully. Since $r_H = 0$ and $M = 0$ for Rindler space one may be able to view this as a limiting process whereby $r_H \to 0$ as $M \to 0$.

Reissner-Nordström black hole: The metric for the charged black hole has $g_{00} = -(1 - \frac{2m}{r} + \frac{q^2}{r^2}) = \frac{(r-r_+)(r-r_-)}{r^2}$ where $r_\pm = m \pm \sqrt{m^2 - q^2}$. Integrating around the outer horizon gives

$$\begin{aligned} \beta_+ &= -2i \oint \frac{z^2}{(r-r_+)(r-r_-)} dr = 4\pi \frac{r_+^2}{r_+ - r_-} \\ &= 2\pi \frac{2m(m+\sqrt{m^2-q^2}) - q^2}{\sqrt{m^2-q^2}} \end{aligned} \tag{35}$$

As a side comment, notice that if you integrate around both horizons, which leads to summing the residues, the result would be $\beta_+ + \beta_- = 8\pi m$. We also note that the temperature vanishes in the extremal $|q| = m$ limit.

Kerr black hole: The rotating black hole has $g_{00} = -\frac{\Delta - a^2 \sin^2\theta}{\Sigma}$ where $\Delta = (r-r_+)(r-r_-)$ and $\Sigma = r^2 + a^2\cos^2\theta$. Choose our path at $\theta = 0$ leaving

$$\begin{aligned} \beta_+ &= \frac{4\pi m(m+\sqrt{m^2-a^2})}{\sqrt{m^2-a^2}} \\ \beta_- &= \frac{4\pi m(-m+\sqrt{m^2-a^2})}{\sqrt{m^2-a^2}} \end{aligned} \tag{36}$$

and notice that $\beta_+ + \beta_- = 8\pi m$ in the same way as the Reissner-Nordström black hole and again the temperature vanishes in the extremal $|a| = m$ limit.

6 Conclusion

We have reviewed the quasi-classical 'tunneling' computation of Hawking radiation, showing that it leads to a simple expression (17) for the Hawking temperature in terms of a closed path in complex r space around the horizon. This can be written in terms of the integral of a 1-form around the path given by (23) if the time coordinate is transformed as in (21). We have discussed these results in the context of both black hole thermodynamics and the Euclidean time formalism.

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