The (T)DBM equation as a twisted loop Toda system: Generalizations and soliton solutions

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Abstract

The (Tzitzéica)–Dodd–Bullough–Mikhailov equation is a simplest particular case of the twisted loop Toda systems associated with general linear groups. We present novel soliton solutions of these equations constructed by the method of rational dressing.

1 Introduction

To the best of our knowledge, the famous equation

$$\partial_+ \partial_- \phi = -m^2 \left[\exp(-2\phi) - \exp(\phi) \right] \tag{1.1}$$

for an unknown function of two independent variables $\phi(z^+, z^-)$, where ∂_+ and ∂_- denote the respective derivatives over z^+ and z^- and m is a nonzero constant, was formulated for the first time in 1908 by G. Tzitzéica in the context of the geometry of hyperbolic surfaces embedded in $\mathbb{R}^3$ [1]. Later on, it was rediscovered and considered by many authors in various branches of mathematical and theoretical physics. In particular, and most interesting to us, it was a subject of investigation within the theory of nonlinear completely integrable systems [2, 3].[1] While investigating completely integrable systems, one is mainly based on two reasons. First, such systems are useful for developing methods of solving nonlinear partial differential equations. Second, they have the so-called soliton solutions, with properties being rather attractive from the point of view of possible applications.

The two-dimensional loop Toda systems give highly interesting examples of completely integrable nonlinear equations, see, e.g., [4, 5]. Different methods for constructing soliton solutions to such systems were recently considered in [6]. Specifically, multi-soliton solutions of abelian *untwisted* loop Toda equations associated with the

[1] Due to great importance of the results of these authors in soliton theory, the equation under consideration is usually called the Dodd–Bullough–Mikhailov equation. Now, to give G. Tzitzéica his due on the occasion of the 100 years jubilee of the equation, we call it the (Tzitzéica)–Dodd–Bullough–Mikhailov equation.

general linear groups were explicitly constructed by the method of Hirota [7, 8, 9, 10, 11, 12] and the rational dressing method [13, 3], and a direct relationship between these approaches was established. Note also that sometimes it is reasonable to use a combination of complementary methods, as was, for example in [14, 15].

We present the results of our investigation of abelian loop Toda equations associated with the complex general linear groups started earlier in [6]. Here we consider abelian *twisted* loop Toda equations. The famous (Tzitzéica)–Dodd–Bullough–Mikhailov equation arises here as a simplest particular case of the twisted loop Toda systems. We apply the rational dressing method, a version of the inverse scattering method, developing it specifically for these classes of nonlinear equations, and construct for them new soliton solutions.

2 Formulating loop Toda equations

In this section, mainly following the monographs [4, 5] and our papers [16, 17], we recall basic notions and introduce notations to be used below. We start our consideration with a Lie group $\mathcal{G}$ whose Lie algebra $\mathfrak{G}$ is endowed with a $\mathbb{Z}$-gradation,

$$\mathfrak{G} = \bigoplus_{k \in \mathbb{Z}} \mathfrak{G}_k, \qquad [\mathfrak{G}_k, \mathfrak{G}_l] \subset \mathfrak{G}_{k+l},$$

and denote by L such a positive integer that the grading subspaces $\mathfrak{G}_k$, where $0 < |k| < L$, are trivial. We denote by $\mathcal{G}_0$ the closed Lie subgroup of $\mathcal{G}$ corresponding to the zero-grade Lie subalgebra $\mathfrak{G}_0$. Then, the Toda equation associated with $\mathcal{G}$ is an equation for a mapping $\varXi$ of the Euclidean plane $\mathbb{R}^2$ to $\mathcal{G}_0$, explicitly of the form

$$\partial_+(\varXi^{-1}\partial_-\varXi) = [\mathcal{F}_-, \varXi^{-1}\mathcal{F}_+\varXi], \tag{2.1}$$

where $\mathcal{F}_-$ and $\mathcal{F}_+$ are some fixed mappings of $\mathbb{R}^2$ to $\mathfrak{G}_{-L}$ and $\mathfrak{G}_{+L}$, respectively, satisfying the relations

$$\partial_+\mathcal{F}_- = 0, \qquad \partial_-\mathcal{F}_+ = 0. \tag{2.2}$$

Here we use the customary notation $\partial_- = \partial/\partial z^-$, $\partial_+ = \partial/\partial z^+$ for the partial derivatives over the standard coordinates on $\mathbb{R}^2$. Certainly, to obtain a nontrivial Toda equation we have to assume that the subspaces $\mathfrak{G}_{-L}$ and $\mathfrak{G}_{+L}$ are nontrivial.

When the Lie group $\mathcal{G}_0$ is abelian, the corresponding Toda equation is said to be *abelian*, otherwise we deal with a *non-abelian Toda equation*.

We see that a Toda equation is specified by a choice of a $\mathbb{Z}$-gradation of the Lie algebra $\mathfrak{G}$ of $\mathcal{G}$ and mappings $\mathcal{F}_-$, $\mathcal{F}_+$ satisfying the conditions (2.2). Therefore, to classify the Toda equations associated with a Lie group $\mathcal{G}$ we should classify $\mathbb{Z}$-gradations of the Lie algebra $\mathfrak{G}$ of $\mathcal{G}$ up to isomorphisms.

It is essential for our purposes that the Toda equation (2.1) together with the relations (2.2) are equivalent to the zero-curvature condition for a flat connection on the trivial fiber bundle $\mathbb{R}^2 \times \mathcal{G} \to \mathbb{R}^2$. Indeed, writing the zero-curvature condition as the equation

$$\partial_-\mathcal{O}_+ - \partial_+\mathcal{O}_- + [\mathcal{O}_-, \mathcal{O}_+] = 0 \tag{2.3}$$

for the $\mathfrak{G}$-valued components of the flat connection under consideration, imposing the grading conditions

$$\mathcal{O}_- = \mathcal{O}_{-0} + \mathcal{O}_{-L}, \qquad \mathcal{O}_+ = \mathcal{O}_{+0} + \mathcal{O}_{+L},$$

and destroying the residual gauge invariance by the condition

$$\mathcal{O}_{+0} = 0,$$

we bring the connection components to the form

$$\mathcal{O}_- = \Xi^{-1}\partial_-\Xi + \mathcal{F}_-, \qquad \mathcal{O}_+ = \Xi^{-1}\mathcal{F}_+\Xi, \tag{2.4}$$

and then derive the equation (2.1) and the relations (2.2) directly from the zero-curvature condition (2.3), as well as vice versa [18, 5, 19].

It follows from the equality (2.3) that there is a mapping Φ of $\mathbb{R}^2$ to $\mathcal{G}$ such that

$$\Phi^{-1}\partial_-\Phi = \mathcal{O}_-, \qquad \Phi^{-1}\partial_+\Phi = \mathcal{O}_+. \tag{2.5}$$

We say in this situation that the connection with the components $\mathcal{O}_-$ and $\mathcal{O}_+$ is generated by the mapping Φ.

We consider the case where $\mathcal{G}$ is a twisted loop group of a complex classical Lie group G which is defined as follows. Let a be an automorphism of G satisfying the relation $a^M = \mathrm{id}_G$ for some positive integer M.[2] The twisted loop group $\mathcal{L}_{a,M}(G)$ is formed by the mappings χ of the unit circle S^1 to G satisfying the equality

$$\chi(\epsilon_M \bar{p}) = a(\chi(\bar{p})),$$

where $\epsilon_M = \mathrm{e}^{2\pi\mathrm{i}/M}$ is the Mth principal root of unity. We think the circle S^1 as consisting of complex numbers of modulus one. The group law in $\mathcal{L}_{a,M}(G)$ is defined pointwise. The Lie algebra of $\mathcal{L}_{a,M}(G)$ is the twisted loop Lie algebra $\mathcal{L}_{A,M}(\mathfrak{g})$, where $\mathfrak{g}$ is the Lie algebra of G and A is the automorphism of $\mathfrak{g}$ corresponding to the automorphism a of G. The Lie algebra $\mathcal{L}_{A,M}(\mathfrak{g})$ is formed by the mappings ξ of S^1 to $\mathfrak{g}$ satisfying the equality

$$\xi(\epsilon_M \bar{p}) = A(\xi(\bar{p}))$$

with the Lie algebra operation defined pointwise. Note that the relation $A^M = \mathrm{id}_{\mathfrak{g}}$ is satisfied automatically.

In the paper [20] we classified a wide class of the so-called integrable $\mathbb{Z}$-gradations with finite-dimensional grading subspaces of the twisted loop Lie algebras of the finite-dimensional complex simple Lie algebras. Namely, we showed that any such $\mathbb{Z}$-gradation of a loop Lie algebra $\mathcal{L}_{A,M}(\mathfrak{g})$ is conjugated by an isomorphism to the standard $\mathbb{Z}$-gradation of another loop Lie algebra $\mathcal{L}_{A',M'}(\mathfrak{g})$, where the automorphisms A and A' differ by an inner automorphism of $\mathfrak{g}$.

Recall that for the *standard $\mathbb{Z}$-gradation* of the Lie algebra $\mathcal{L}_{A,M}(\mathfrak{g})$ the grading subspaces are

$$\mathcal{L}_{A,M}(\mathfrak{g})_k = \{\xi = \lambda^k x \in \mathcal{L}_{A,M}(\mathfrak{g}) \mid x \in \mathfrak{g},\ A(x) = \epsilon_M^k x\},$$

where by λ we denote the restriction of the standard coordinate on $\mathbb{C}$ to S^1.

It is well known that twisted loop Lie algebras defined by automorphisms which differ by an inner automorphism are isomorphic, and really different twisted loop Lie algebras can be labeled by the elements of the corresponding outer automorphism group. In particular, if A is an inner automorphism, the loop Lie algebra $\mathcal{L}_{A,M}(\mathfrak{g})$ is isomorphic to an untwisted loop Lie algebra $\mathcal{L}(\mathfrak{g}) = \mathcal{L}_{\mathrm{id}_{\mathfrak{g}},1}(\mathfrak{g})$. Therefore, in this case a

[2] Here M is not necessarily the order of the automorphism a, but can be its arbitrary multiple.

Toda equation associated with $\mathcal{L}_{a,M}(G)$ and specified by some choice of a $\mathbb{Z}$-gradation of $\mathcal{L}_{A,M}(\mathfrak{g})$ is equivalent to a Toda equation associated with $\mathcal{L}(G) = \mathcal{L}_{\mathrm{id}_G,1}(G)$ and specified by the corresponding choice of a $\mathbb{Z}$-gradation of $\mathcal{L}(\mathfrak{g})$.

Thus, to describe Toda equations associated with the loop groups $\mathcal{L}_{a,M}(G)$, where a is an inner automorphism of G, it suffices to describe the Toda equations associated with the untwisted loop groups $\mathcal{L}(G)$. However, due to simplicity of the standard $\mathbb{Z}$-gradation, to study Toda equations it is more convenient, instead of using one untwisted loop group $\mathcal{L}(G)$ and different $\mathbb{Z}$-gradations of $\mathcal{L}(\mathfrak{g})$, to use different twisted loop groups $\mathcal{L}_{a,M}(G)$ and the standard $\mathbb{Z}$-gradation of $\mathcal{L}_{A,M}(\mathfrak{g})$. Similarly, to describe the Toda equations associated with the loop groups $\mathcal{L}_{a,M}(G)$, where a is an outer automorphism of G satisfying the relation $a^M = \mathrm{id}_G$, it suffices to use only the standard $\mathbb{Z}$-gradation of the loop Lie algebras $\mathcal{L}_{A,M}(\mathfrak{g})$. Having all this in mind and slightly abusing terminology, we say that when a is an outer automorphism of G then a Toda equation associated with $\mathcal{L}_{a,M}(G)$ is a *twisted loop Toda equation associated with* G, and when a is an inner automorphism of G then a Toda equation associated with $\mathcal{L}_{a,M}(G)$ is an *untwisted loop Toda equation associated with* G.[3]

The group $\mathcal{L}_{a,M}(G)$ and its Lie algebra $\mathcal{L}_{A,M}(\mathfrak{g})$ are infinite-dimensional manifolds. Nevertheless, using the so-called exponential law [21, 22], it is possible to write the zero-curvature representation of the Toda equations associated with $\mathcal{L}_{a,M}(G)$ in terms of finite-dimensional manifolds. The essence of this useful law can be expressed by the canonical identification $C^\infty(\mathcal{M}, C^\infty(\mathcal{N}, \mathcal{P})) = C^\infty(\mathcal{M} \times \mathcal{N}, \mathcal{P})$, where $\mathcal{M}$, $\mathcal{N}$ and $\mathcal{P}$ are finite-dimensional manifolds, among which $\mathcal{N}$ is compact.

The connection components $\mathcal{O}_-$ and $\mathcal{O}_+$ entering the equality (2.3) are mappings of $\mathbb{R}^2$ to the loop Lie algebra $\mathcal{L}_{A,M}(\mathfrak{g})$. We denote the mappings of $\mathbb{R}^2 \times S^1$ to $\mathfrak{g}$, corresponding to $\mathcal{O}_-$ and $\mathcal{O}_+$ in accordance with the exponential law, by ω_- and ω_+, and call them also the connection components. The mapping Φ generating the connection under consideration is a mapping of $\mathbb{R}^2$ to $\mathcal{L}_{a,M}(G)$. Denoting the respective mapping of $\mathbb{R}^2 \times S^1$ to G by φ we can write

$$\varphi^{-1}\partial_-\varphi = \omega_-, \qquad \varphi^{-1}\partial_+\varphi = \omega_+, \tag{2.6}$$

which is equivalent to the expressions (2.5). Having in mind that the mapping φ uniquely determines the connection generating mapping Φ, we say that the mapping φ also generates the connection under consideration. We introduce, according to the exponential law, the smooth mapping γ of $\mathbb{R}^2 \times S^1$ to G respective to Ξ, and smooth mappings of $\mathbb{R}^2 \times S^1$ to $\mathfrak{g}$ respective to $\mathcal{F}_-$ and $\mathcal{F}_+$.

Now, explicitly describing the grading subspaces of the standard $\mathbb{Z}$-gradation of the loop Lie algebra $\mathcal{L}_{A,M}(\mathfrak{g})$, we find that the subalgebra $\mathcal{L}_{A,M}(\mathfrak{g})_0$ is isomorphic to the subalgebra $\mathfrak{g}_{[0]_M}$ of $\mathfrak{g}$, and the Lie group $\mathcal{L}_{a,M}(G)_0$ is isomorphic to the connected Lie subgroup G_0 of G corresponding to the Lie algebra $\mathfrak{g}_{[0]_M}$. Here $\mathfrak{g}_{[k]_M}$ mean the grading subspaces of the $\mathbb{Z}_M$-gradation of $\mathfrak{g}$ induced by the automorphism A, and $[k]_M$ denotes the element of the ring $\mathbb{Z}_M$ corresponding to the integer k. For the connection components ω_- and ω_+ we can write the expressions

$$\omega_- = \gamma^{-1}\partial_-\gamma + \lambda^{-L}c_-, \qquad \omega_+ = \lambda^L\gamma^{-1}c_+\gamma, \tag{2.7}$$

which are equivalent to the equalities (2.4). Here c_- and c_+ are mappings of $\mathbb{R}^2$ to

[3] It is common to omit the word 'untwisted'.

$\mathfrak{g}_{-[L]_M}$ and $\mathfrak{g}_{+[L]_M}$ respectively. Hence, the Toda equation (2.1) can be written as

$$\partial_+(\gamma^{-1}\partial_-\gamma) = [c_-, \gamma^{-1}c_+\gamma], \tag{2.8}$$

and the conditions (2.2) imply that

$$\partial_+ c_- = 0, \qquad \partial_- c_+ = 0. \tag{2.9}$$

We call an equation of the form (2.8) also a loop Toda equation.

Our classification of loop Toda equations is based on a convenient block-matrix representation of the grading subspaces [16, 17] we have implemented. Each element x of the complex classical Lie algebra $\mathfrak{g}$ under consideration is treated as a $p \times p$ block matrix $(x_{\alpha\beta})$, where $x_{\alpha\beta}$ is an $n_\alpha \times n_\beta$ matrix. The sum of the positive integers n_α is the size n of the matrices representing the elements of $\mathfrak{g}$. For the case of Toda equations associated with the loop groups $\mathcal{L}_{a,M}(\mathrm{GL}_n(\mathbb{C}))$, where a is an inner automorphism of $\mathrm{GL}_n(\mathbb{C})$, the integers n_α are arbitrary. For the other cases they should satisfy some restrictions dictated by the structure of the Lie algebra $\mathfrak{g}$.

The mapping γ has the block-diagonal form

$$\gamma = \begin{pmatrix} \varGamma_1 & & & \\ & \varGamma_2 & & \\ & & \ddots & \\ & & & \varGamma_p \end{pmatrix}. \tag{2.10}$$

For each $\alpha = 1, \dots, p$ the mapping $\varGamma_\alpha$ is a mapping of $\mathbb{R}^2$ to the Lie group $\mathrm{GL}_{n_\alpha}(\mathbb{C})$. For the case of Toda equations associated with the loop groups $\mathcal{L}_{a,M}(\mathrm{GL}_n(\mathbb{C}))$, where a is an inner automorphism of $\mathrm{GL}_n(\mathbb{C})$, the mappings $\varGamma_\alpha$ are arbitrary. For the other cases they satisfy some additional restrictions.

The mapping c_+ has the following block-matrix structure:

$$c_+ = \begin{pmatrix} 0 & C_{+1} & & & \\ & 0 & \ddots & & \\ & & \ddots & \ddots & \\ & & & 0 & C_{+(p-1)} \\ C_{+0} & & & & 0 \end{pmatrix}, \tag{2.11}$$

where for each $\alpha = 1, \dots, p-1$ the mapping $C_{+\alpha}$ is a mapping of $\mathbb{R}^2$ to the space of $n_\alpha \times n_{\alpha+1}$ complex matrices, and C_{+0} is a mapping of $\mathbb{R}^2$ to the space of $n_p \times n_1$ complex matrices. The mapping c_- has a similar block-matrix structure:

$$c_- = \begin{pmatrix} 0 & & & & C_{-0} \\ C_{-1} & 0 & & & \\ & \ddots & \ddots & & \\ & & \ddots & 0 & \\ & & & C_{-(p-1)} & 0 \end{pmatrix}, \tag{2.12}$$

where for each $\alpha = 1, \dots, p-1$ the mapping $C_{-\alpha}$ is a mapping of $\mathbb{R}^2$ to the space of $n_{\alpha+1} \times n_\alpha$ complex matrices, and C_{-0} is a mapping of $\mathbb{R}^2$ to the space of $n_1 \times n_p$ complex matrices. The conditions (2.9) imply

$$\partial_+ C_{-\alpha} = 0, \qquad \partial_- C_{+\alpha} = 0, \qquad \alpha = 0, 1, \dots, p-1. \tag{2.13}$$

For the case of Toda equations associated with the loop groups $\mathcal{L}_{a,M}(\mathrm{GL}_n(\mathbb{C}))$, where a is an inner automorphism of $\mathrm{GL}_n(\mathbb{C})$, the mappings $C_{\pm\alpha}$ are arbitrary. For the other cases they should satisfy some additional restrictions.

The Toda equation (2.8) is equivalent to the following system of equations for the mappings Γ_α:

$$
\begin{aligned}
\partial_+ \left(\Gamma_1^{-1}\partial_-\Gamma_1\right) &= -\Gamma_1^{-1}C_{+1}\Gamma_2 C_{-1} + C_{-0}\Gamma_p^{-1}C_{+0}\Gamma_1, \\
\partial_+ \left(\Gamma_2^{-1}\partial_-\Gamma_2\right) &= -\Gamma_2^{-1}C_{+2}\Gamma_3 C_{-2} + C_{-1}\Gamma_1^{-1}C_{+1}\Gamma_2, \\
&\ \vdots \\
\partial_+ \left(\Gamma_{p-1}^{-1}\partial_-\Gamma_{p-1}\right) &= -\Gamma_{p-1}^{-1}C_{+(p-1)}\Gamma_p C_{-(p-1)} + C_{-(p-2)}\Gamma_{p-2}^{-1}C_{+(p-2)}\Gamma_{p-1}, \\
\partial_+ \left(\Gamma_p^{-1}\partial_-\Gamma_p\right) &= -\Gamma_p^{-1}C_{+0}\Gamma_1 C_{-0} + C_{-(p-1)}\Gamma_{p-1}^{-1}C_{+(p-1)}\Gamma_p.
\end{aligned}
\tag{2.14}
$$

It appears that in the case under consideration without any loss of generality we can assume that the positive integer L, entering the construction of Toda equations, is equal to 1. Note also that if any of the mappings $C_{+\alpha}$ or $C_{-\alpha}$ is a zero mapping, then the equations (2.14) are equivalent to a Toda equation associated with a finite-dimensional group or to a set of two such equations.

3 Abelian loop Toda systems with complex general linear groups

It can be shown that there are three types of abelian loop Toda equations associated with the groups $\mathrm{GL}_n(\mathbb{C})$, see, for example the paper [6].

3.1 First type: untwisted loop Toda systems

The abelian Toda equations of the first type arise when the automorphism A is defined by the equality

$$A(x) = hxh^{-1}, \qquad x \in \mathfrak{gl}_n(\mathbb{C}),$$

where h is a diagonal matrix with the diagonal matrix elements

$$h_{kk} = \epsilon_n^{n-k+1}, \qquad k = 1, \dots, n. \tag{3.1}$$

The corresponding automorphism a of $\mathrm{GL}_n(\mathbb{C})$ is defined by the equality

$$a(g) = hgh^{-1}, \qquad g \in \mathrm{GL}_n(\mathbb{C}),$$

where again h is a diagonal matrix determined by the relation (3.1). Here the integer M is equal to n, and A is an inner automorphism which generates a $\mathbb{Z}_n$-gradation of $\mathfrak{gl}_n(\mathbb{C})$. The block-matrix structure related to this gradation is the matrix structure itself. In other words, all blocks are of size one by one. The mappings Γ_α are mappings of $\mathbb{R}^2$ to the Lie group $\mathrm{GL}_1(\mathbb{C})$ which is isomorphic to the Lie group $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$. The mappings $C_{\pm\alpha}$ are just complex functions on $\mathbb{R}^2$. The Toda equations under consideration have the form (2.14) with $p = n$.

We have shown in the paper [6] that if the functions $C_{-\alpha}$ and $C_{+\alpha}$ have no zeros then the Toda equations (2.14) are equivalent to the same equations, but where $C_{-\alpha} = C_{-}$ and $C_{+\alpha} = C_{+}$ for some functions C_{-} and C_{+} which have no zeros. If these functions are real, then with the help of an appropriate change of the coordinates z^{-} and z^{+} we can come to the Toda equations with $C_{\pm\alpha}$ equal to a nonzero constant m. This system of equations gives the Toda equations associated with untwisted loop groups of general linear groups. In the paper [6] we investigated the soliton solutions of the above Toda equations obtained by two different approaches, the Hirota's and rational dressing methods, and established explicit relationships between these methods.

3.2 Second type: twisted loop Toda systems, odd-dimensional case

The abelian Toda equations of the other two types arise when we use outer automorphisms of $\mathfrak{gl}_n(\mathbb{C})$. For the equations of the second type n is odd, and for the equations of the third type n is even.

Consider first the case of an odd $n = 2s - 1$, $s \geq 2$. In this case an abelian Toda equation arises when the automorphism A is defined by the equality

$$A(x) = -h(B^{-1}\,{}^t x B)h^{-1}, \tag{3.2}$$

where ${}^t x$ means the transpose of x, h is a diagonal matrix with the diagonal matrix elements

$$h_{kk} = \epsilon_{2n}^{n-k+1} = \epsilon_{4s-2}^{2s-k},$$

and B is an $n \times n$ matrix of the form

$$B = \left(\begin{array}{c|cccccc} 1 & & & & & & \\ \hline & & & & & & 1 \\ & & & & & \iddots & \\ & & & & 1 & & \\ & & & -1 & & & \\ & & \iddots & & & & \\ & -1 & & & & & \end{array}\right).$$

The corresponding group automorphism a is defined as

$$a(g) = h(B^{-1}\,{}^t g^{-1} B)h^{-1}. \tag{3.3}$$

The order M of the automorphism A is $2N = 4s - 2$ and the integer p is $2s - 1$. The mapping γ is a diagonal matrix of the form (2.10), where the mappings $\mathit{\Gamma}_\alpha$ are mappings of $\mathbb{R}^2$ to $\mathbb{C}^\times$, subject to the constraints

$$\mathit{\Gamma}_1 = 1, \qquad \mathit{\Gamma}_{2s-\alpha+1} = \mathit{\Gamma}_\alpha^{-1}, \quad \alpha = 2, \ldots, s.$$

The mappings $C_{\pm\alpha}$ in the relations (2.11) and (2.12) are complex functions satisfying the equality

$$C_{\pm 0} = C_{\pm 1}, \tag{3.4}$$

and for $s > 2$ the equalities

$$C_{\pm(2s-\alpha)} = -C_{\pm\alpha}, \quad \alpha = 2, \ldots, s-1. \tag{3.5}$$

Let us choose the mappings Γ_α, $\alpha = 2, \ldots, s$, as a complete set of mappings parameterizing the mapping γ. Taking into account the equalities (3.4) and (3.5) we come to a set of $s-1$ independent equations equivalent to the Toda equation under consideration. As well as in the untwisted case, under appropriate conditions the Toda equations under consideration are equivalent to the same equations, but where

$$C_{\pm 0} = C_{\pm 1} = C_{\pm s} = m, \tag{3.6}$$

and

$$C_{\pm\alpha} = -C_{\pm(2s-\alpha)} = m, \qquad \alpha = 2, \ldots, s-1. \tag{3.7}$$

Explicitly, we have the equations

$$\begin{aligned} \partial_+(\Gamma_2^{-1}\partial_-\Gamma_2) &= -m^2(\Gamma_2^{-1}\Gamma_3 - \Gamma_2), \\ \partial_+(\Gamma_3^{-1}\partial_-\Gamma_3) &= -m^2(\Gamma_3^{-1}\Gamma_4 - \Gamma_2^{-1}\Gamma_3), \\ &\vdots \\ \partial_+(\Gamma_{s-1}^{-1}\partial_-\Gamma_{s-1}) &= -m^2(\Gamma_{s-1}^{-1}\Gamma_s - \Gamma_{s-2}^{-1}\Gamma_{s-1}), \\ \partial_+(\Gamma_s^{-1}\partial_-\Gamma_s) &= -m^2(\Gamma_s^{-2} - \Gamma_{s-1}^{-1}\Gamma_s), \end{aligned} \tag{3.8}$$

where m is again a nonzero constant, see also the papers [3, 23].

For $s = 2$ denoting Γ_2 by Γ we have the equation

$$\partial_+(\Gamma^{-1}\partial_-\Gamma) = -m^2(\Gamma^{-2} - \Gamma).$$

Putting $\Gamma = \exp(F)$ we obtain

$$\partial_+\partial_- F = -m^2[\exp(-2F) - \exp(F)].$$

This is exactly the (Tzitzéica)–Dodd–Bullough–Mikhailov equation [1, 2, 3].

3.3 Third type: twisted loop Toda systems, even-dimensional case

In the case of an even $n = 2s$, $s \geq 2$, to come to an abelian Toda equation we should use again the Lie algebra automorphism A and the corresponding group automorphism a defined by the relations (3.2) and (3.3), respectively, where now

$$B = \left(\begin{array}{cc|cccccc} & 1 & & & & & & \\ 1 & & & & & & & \\ \hline & & & & & & & 1 \\ & & & & & & \cdot^{\cdot^{\cdot}} & \\ & & & & & 1 & & \\ & & & & -1 & & & \\ & & & \cdot^{\cdot^{\cdot}} & & & & \\ & & -1 & & & & & \end{array}\right)$$

and h is a diagonal matrix with the diagonal matrix elements

$$h_{11} = \epsilon_{2n-2}^{n-1} = \epsilon_{4s-2}^{2s-1} = -1, \qquad h_{ii} = \epsilon_{2n-2}^{n-i+1} = \epsilon_{4s-2}^{2s-i+1}, \quad i = 2, \ldots, n.$$

The order M of the automorphism A is again $2N = 4s-2$, and the number p characterizing the block structure is equal to $n-1 = 2s-1$, $n_1 = 2$, and $n_\alpha = 1$ for $\alpha = 2, \ldots, 2s-1$.

The mapping $\varGamma_1$ is a mapping of $\mathbb{R}^2$ to the Lie group $\mathrm{SO}_2(\mathbb{C})$ which is isomorphic to $\mathbb{C}^\times$. Actually $\varGamma_1$ is a 2×2 complex matrix-valued function satisfying the relation

$$J_2^{-1}\,{}^t\varGamma_1 J_2 = \varGamma_1^{-1},$$

where

$$J_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

It is easy to show that $\varGamma_1$ has the form

$$\varGamma_1 = \begin{pmatrix} (\varGamma_1)_{11} & 0 \\ 0 & (\varGamma_1)_{11}^{-1} \end{pmatrix},$$

where $(\varGamma_1)_{11}$ is a mapping of $\mathbb{R}^2$ to $\mathbb{C}^\times$. The mappings $\varGamma_\alpha$, $\alpha = 2, \ldots, 2s-1$, are mappings of $\mathbb{R}^2$ to $\mathbb{C}^\times$ satisfying the relations

$$\varGamma_{2s-\alpha+1} = \varGamma_\alpha^{-1}.$$

The mappings C_{-1}, C_{+0} are complex 1×2 matrix-valued functions, the mappings C_{-0}, C_{+1} are complex 2×1 matrix-valued functions. Here we have

$$C_{-0} = J_2^{-1}\,{}^tC_{-1}, \qquad C_{+0} = {}^tC_{+1} J_2. \tag{3.9}$$

The mappings $C_{\pm\alpha}$, $\alpha = 2, \ldots, p-1 = 2s-2$, are just complex functions, satisfying for $s > 2$ the equalities

$$C_{\pm(2s-\alpha)} = -C_{\pm\alpha}, \qquad \alpha = 2, \ldots, s-1. \tag{3.10}$$

The mappings $(\varGamma_1)_{11}$ and $\varGamma_\alpha$, $\alpha = 2, \ldots, s$, form a complete set of mappings parameterizing the mapping γ. Taking into account the equalities (3.9) and (3.10) we come to a set of s independent equations equivalent to the Toda equation under consideration. As well as for the first two types, under appropriate conditions these equations can be reduced to equations with

$$C_{-\alpha} = m, \qquad C_{+\alpha} = m, \qquad \alpha = 2, \ldots, s, \tag{3.11}$$

and

$$(C_{-1})_{11} = (C_{-1})_{12} = m/\sqrt{2}, \qquad (C_{+1})_{11} = (C_{+1})_{21} = m/\sqrt{2}, \tag{3.12}$$

where m is a nonzero constant. Thus, we come to the equations

$$\begin{aligned}
\partial_+(\varGamma_1^{-1}\partial_-\varGamma_1) &= -\frac{m^2}{2}(\varGamma_1^{-1} - \varGamma_1)\varGamma_2, \\
\partial_+(\varGamma_2^{-1}\partial_-\varGamma_2) &= -m^2\varGamma_2^{-1}\varGamma_3 + \frac{m^2}{2}(\varGamma_1^{-1} + \varGamma_1)\varGamma_2, \\
\partial_+(\varGamma_3^{-1}\partial_-\varGamma_3) &= -m^2(\varGamma_3^{-1}\varGamma_4 - \varGamma_2^{-1}\varGamma_3), \\
&\ \vdots \\
\partial_+(\varGamma_{s-1}^{-1}\partial_-\varGamma_{s-1}) &= -m^2(\varGamma_{s-1}^{-1}\varGamma_s - \varGamma_{s-2}^{-1}\varGamma_{s-1}), \\
\partial_+(\varGamma_s^{-1}\partial_-\varGamma_s) &= -m^2(\varGamma_s^{-2} - \varGamma_{s-1}^{-1}\varGamma_s),
\end{aligned} \tag{3.13}$$

where, with a slight abuse of notation, we have denoted $(\varGamma_1)_{11}$ by $\varGamma_1$.

We also note that all three systems of Toda equations described above can be represented in standard forms with explicit indication of the Cartan matrices of the corresponding affine Lie algebras of the types $A^{(1)}_{n-1}$, $A^{(2)}_{2s-2}$ and $A^{(2)}_{2s-1}$, respectively, see, for example, the paper [6].

4 The method of rational dressing

In this section we apply the method of rational dressing to construct solutions of the abelian Toda systems associated with the loop groups of the complex general linear groups. Here we solve the abelian Toda equations of the second and third types which have the forms (3.8) and (3.13)respectively. In fact, some preliminary relations of the rational dressing formalism can be introduced on a common basis in application to the both types of abelian Toda systems.

Because in the cases under consideration the matrices c_- and c_+ are commuting, it is obvious that

$$\gamma = I_n, \tag{4.1}$$

where I_n is the $n \times n$ unit matrix, is a solution to the Toda equation (2.8). Denote a mapping of $\mathbb{R}^2 \times S^1$ to $\mathrm{GL}_n(\mathbb{C})$, which generates the corresponding connection, by φ. Using the equalities (2.6) and (2.7) and remembering that in our case $L = 1$, we write

$$\varphi^{-1}\partial_-\varphi = \lambda^{-1}c_-, \qquad \varphi^{-1}\partial_+\varphi = \lambda c_+, \tag{4.2}$$

where the matrices c_+ and c_- having generally the forms (2.11) and (2.12), are specified by the relations (3.6), (3.7) for the Toda equations of the second type, and by the relations (3.11), (3.12) for the Toda equations of the third type.

To construct some other solutions to the Toda equations we will look for a mapping ψ, such that the mapping

$$\varphi' = \varphi\psi \tag{4.3}$$

would generate a connection satisfying the grading condition

$$\omega_- = \omega_{-0} + \omega_{-1}, \qquad \omega_+ = \omega_{+0} + \omega_{+1} \tag{4.4}$$

and the gauge-fixing constraint

$$\omega_{+0} = 0. \tag{4.5}$$

For any $\bar{m} \in \mathbb{R}^2$ the mapping $\tilde{\psi}_m$ defined by the equality $\tilde{\psi}_m(\bar{p}) = \psi(\bar{m}, \bar{p})$, $\bar{p} \in S^1$, is a smooth mapping of S^1 to $\mathrm{GL}_n(\mathbb{C})$. We treat S^1 as a subset of the complex plane which, in turn, will be treated as a subset of the Riemann sphere. Assume that it is possible to extend analytically each mapping $\tilde{\psi}_m$ to all of the Riemann sphere. As the result we obtain a mapping of the direct product of $\mathbb{R}^2$ and the Riemann sphere to $\mathrm{GL}_n(\mathbb{C})$, which we also denote by ψ. Suppose that for any $\bar{m} \in \mathbb{R}^2$ the analytic extension of $\tilde{\psi}_m$ results in a rational mapping regular at the points 0 and ∞, hence the name *rational dressing*. Below, for each point $\bar{p}$ of the Riemann sphere we denote by ψ_p the mapping of $\mathbb{R}^2$ to $\mathrm{GL}_n(\mathbb{C})$ defined by the equality $\psi_p(\bar{m}) = \psi(\bar{m}, \bar{p})$.

Since we deal with the Toda equations described in Sections 3.2 and 3.3, for any $\bar{m} \in \mathbb{R}^2$ and $\bar{p} \in S^1$ we should have

$$\psi(\bar{m}, \epsilon_{2N}\bar{p}) = hB^{-1}\,{}^t\psi^{-1}(\bar{m}, \bar{p})Bh^{-1}, \tag{4.6}$$

where h is a block-diagonal matrix described by the relation

$$h_{\alpha\beta} = \epsilon_{2N}^{N-\alpha+1} I_{n_\alpha} \delta_{\alpha\beta}, \qquad \alpha, \beta = 1, \ldots, p,$$

with $n_1 = 1$ for the Toda equations of the second type, and $n_1 = 2$ for the Toda equations of the third type, while for all other indices $\alpha = 2, \ldots, p$ we always have $n_\alpha = 1$. Note that $h_{11} = -I_{n_1}$. Here we also use the notation

$$B = \left(\begin{array}{c|ccccc} J_{n_1} & & & & & \\ \hline & & & & & 1 \\ & & & & \cdot^{\cdot^{\cdot}} & \\ & & & 1 & & \\ & & -1 & & & \\ & \cdot^{\cdot^{\cdot}} & & & & \\ & -1 & & & & \end{array}\right)$$

common for the both cases. The equality (4.6) means that for any $\bar m \in \mathbb{R}^2$ two rational mappings coincide on S^1, therefore, they must coincide on the entire Riemann sphere.

We define a linear mapping $\hat a$ acting on a mapping χ of the direct product of $\mathbb{R}^2$ and the Riemann sphere to the algebra $\mathrm{Mat}_n(\mathbb{C})$ of $n \times n$ complex matrices as[4]

$$\hat a\chi(\bar m, \bar p) = hB^{-1t}\chi^{-1}(\bar m, \epsilon_{2N}^{-1}\bar p)Bh^{-1}.$$

The relation (4.6) is equivalent to the equality $\hat a\psi = \psi$. To construct rational mappings satisfying this relation we will use the following procedure. First, we construct a family of mappings ψ satisfying the relation $\hat a^2\psi = \psi$, and then select from it the mappings satisfying the equality $\hat a\psi = \psi$.

It is easy to see that the mapping

$$\psi = \sum_{k=1}^{N} \hat a^{2k}\chi \tag{4.7}$$

satisfies the relation $\hat a^2\psi = \psi$. It is worth to note that $\hat a^{2N}\chi = \chi$. We start with a rational mapping χ regular at the points 0 and ∞ and having poles at r different nonzero points μ_i, $i = 1, \ldots, r$. More specifically, we consider a mapping χ of the form

$$\chi = \left(I_n + N\sum_{i=1}^{r}\frac{\lambda}{\lambda - \mu_i}P_i\right)\chi_0,$$

where P_i are some smooth mappings of $\mathbb{R}^2$ to the algebra $\mathrm{Mat}_n(\mathbb{C})$ and χ_0 is a mapping of $\mathbb{R}^2$ to the Lie subgroup of $\mathrm{GL}_n(\mathbb{C})$ formed by the elements $g \in \mathrm{GL}_n(\mathbb{C})$ satisfying the equality

$$h^2gh^{-2} = g. \tag{4.8}$$

With account of the equality

$$\hat a^2\chi(\bar m, \bar p) = h^2\chi(\bar m, \epsilon_N^{-1}\bar p)h^{-2}$$

the averaging procedure (4.7) leads to the mapping

$$\psi = \left(I_n + \sum_{i=1}^{r}\sum_{k=1}^{N}\frac{\lambda}{\lambda - \epsilon_{2N}^{2k}\mu_i}h^{2k}P_ih^{-2k}\right)\psi_0, \tag{4.9}$$

[4]Note that below χ is a mapping to the Lie group $\mathrm{GL}_n(\mathbb{C})$, although to justify the relation (4.7) it is convenient to think $\mathrm{GL}_n(\mathbb{C})$ as a subset of $\mathrm{Mat}_n(\mathbb{C})$.

where $\psi_0 = N\chi_0$. We assume that $\mu_i^{2N} \neq \mu_j^{2N}$ for all $i \neq j$.

Denote by ψ^{-1} the mapping of $\mathbb{R}^2 \times S^1$ to $\mathrm{GL}_n(\mathbb{C})$ defined by the relation

$$\psi^{-1}(\bar{m}, \bar{p}) = (\psi(\bar{m}, \bar{p}))^{-1}.$$

Suppose that for any fixed $\bar{m} \in \mathbb{R}^2$ the mapping $\tilde{\psi}_m^{-1}$ of S^1 to $\mathrm{GL}_n(\mathbb{C})$, defined by the equality $\tilde{\psi}_m^{-1}(\bar{p}) = \psi^{-1}(\bar{m}, \bar{p})$, can be extended analytically to a mapping of the Riemann sphere to $\mathrm{GL}_n(\mathbb{C})$, which we also denote by ψ^{-1}, and as the result we obtain a rational mapping of the same structure as the mapping ψ,

$$\psi^{-1} = \psi_0^{-1} \left(I_n + \sum_{i=1}^{r} \sum_{k=1}^{N} \frac{\lambda}{\lambda - \epsilon_{2N}^{2k} \nu_i} h^{2k} Q_i h^{-2k} \right), \tag{4.10}$$

with the pole positions satisfying the conditions $\nu_i \neq 0$, $\nu_i^{2N} \neq \nu_j^{2N}$ for all $i \neq j$, and additionally $\nu_i^N \neq \mu_j^N$ for any i and j. [5]

The mappings ψ and ψ^{-1} given by the equalities (4.9) and (4.10), respectively, satisfy the relations $\hat{a}^2\psi = \psi$ and $\hat{a}^2\psi^{-1} = \psi^{-1}$. To satisfy the relations $\hat{a}\psi = \psi$ and $\hat{a}\psi^{-1} = \psi^{-1}$ we have to assume that the pole positions of the mappings ψ and ψ^{-1} are necessarily connected as

$$\nu_i = \mu_i / \epsilon_{2N}, \qquad i = 1, \ldots, r,$$

and the matrices P_i and Q_i are related as

$$Q_i = h^{-1} B^{-1} {}^t\!P_i B h, \qquad i = 1, \ldots, r, \tag{4.11}$$

By definition, the equality

$$\psi^{-1}\psi = I_n$$

is valid at all points of the direct product of $\mathbb{R}^2$ and S^1. Since $\psi^{-1}\psi$ is a rational mapping, the above equality is valid at all points of the direct product of $\mathbb{R}^2$ and the Riemann sphere. Hence, the residues of $\psi^{-1}\psi$ at the points $\nu_i = \mu_i/\epsilon_{2N}$ and μ_i should be equal to zero. Explicitly we have

$$h^{-1}B^{-1}{}^t\!P_i B h \left(I_n + \sum_{j=1}^{r} \sum_{k=1}^{N} \frac{\mu_i/\epsilon_{2N}}{\mu_i/\epsilon_{2N} - \epsilon_{2N}^{2k} \mu_j} h^{2k} P_j h^{-2k} \right) = 0, \tag{4.12}$$

$$\left(I_n + \sum_{j=1}^{r} \sum_{k=1}^{N} \frac{\mu_i}{\mu_i - \epsilon_{2N}^{2k-1} \mu_j} h^{2k-1} B^{-1} {}^t\!P_j B h^{-2k+1} \right) P_i = 0. \tag{4.13}$$

We will discuss later how to satisfy these relations, and now let us consider what connection is generated by the mapping φ' defined by (4.3) with the mapping ψ possessing the properties described above.

Using the equality (4.3) and the relations (4.2), we obtain for the components of the connection generated by φ' the expressions

$$\omega_- = \psi^{-1}\partial_-\psi + \lambda^{-1}\psi^{-1}c_-\psi, \tag{4.14}$$

$$\omega_+ = \psi^{-1}\partial_+\psi + \lambda\psi^{-1}c_+\psi. \tag{4.15}$$

[5] Actually, as it will be clear, for the extended mappings ψ and ψ^{-1} we have $\psi^{-1}\psi = I_n$. This justifies the notation used.

We see that the component ω_- is a rational mapping which has simple poles at the points μ_i, $\nu_i = \mu_i/\epsilon_{2N}$ and zero.[6] Similarly, the component ω_+ is a rational mapping which has simple poles at the points μ_i, $\nu_i = \mu_i/\epsilon_{2N}$ and infinity. We are looking for a connection which satisfies the grading condition (4.4) and the gauge-fixing condition (4.5). The grading condition in our case is the requirement that for each point of $\mathbb{R}^2$ the component ω_- is rational and has the only simple pole at zero, while the component ω_+ is rational and has the only simple pole at infinity. Hence, we demand that the residues of ω_- and ω_+ at the points μ_i and $\nu_i = \mu_i/\epsilon_{2N}$ should vanish.

The residues of ω_- and ω_+ at the points $\nu_i = \mu_i/\epsilon_{2N}$ are equal to zero if and only if

$$(\partial_- Q_i - \epsilon_{2N}\mu_i^{-1} Q_i c_-)\left(I_n + \sum_{j=1}^{r}\sum_{k=1}^{N} \frac{\mu_i/\epsilon_{2N}}{\mu_i/\epsilon_{2N} - \epsilon_{2N}^{2k}\mu_j} h^{2k} P_j h^{-2k}\right) = 0, \tag{4.16}$$

$$(\partial_+ Q_i - \epsilon_{2N}^{-1}\mu_i Q_i c_+)\left(I_n + \sum_{j=1}^{r}\sum_{k=1}^{N} \frac{\mu_i/\epsilon_{2N}}{\mu_i/\epsilon_{2N} - \epsilon_{2N}^{2k}\mu_j} h^{2k} P_j h^{-2k}\right) = 0, \tag{4.17}$$

respectively, with the equality (4.11) to be taken into account. Similarly, the requirement of vanishing of the residues at the points μ_i gives the relations

$$\left(I_n + \sum_{j=1}^{r}\sum_{k=1}^{N} \frac{\mu_i}{\mu_i - \epsilon_{2N}^{2k-1}\mu_j} h^{2k-1} B^{-1}{}^t P_j B h^{-2k+1}\right)(\partial_- P_i + \mu_i^{-1} c_- P_i) = 0, \tag{4.18}$$

$$\left(I_n + \sum_{j=1}^{r}\sum_{k=1}^{N} \frac{\mu_i}{\mu_i - \epsilon_{2N}^{2k-1}\mu_j} h^{2k-1} B^{-1}{}^t P_j B h^{-2k+1}\right)(\partial_+ P_i + \mu_i c_+ P_i) = 0. \tag{4.19}$$

To obtain the relations (4.16)–(4.19) we made use of the equalities (4.12), (4.13).

Suppose that we have succeeded in satisfying the relations (4.12), (4.13) and (4.16)–(4.19). In such a case from the equalities (4.14) and (4.15) it follows that the connection under consideration satisfies the grading condition.

It follows from the equality (4.15) that

$$\omega_+(\bar{m}, 0) = \psi_0^{-1}(\bar{m})\partial_+\psi_0(\bar{m}).$$

Taking into account that $\omega_{+0}(\bar{m}) = \omega_+(\bar{m}, 0)$, we conclude that the gauge-fixing constraint $\omega_{+0} = 0$ is equivalent to the relation

$$\partial_+\psi_0 = 0. \tag{4.20}$$

Assuming that this relation is satisfied, we come to a connection satisfying both the grading condition and the gauge-fixing condition.

Recall that if a flat connection ω satisfies the grading and gauge-fixing conditions, then there exist a mapping γ from $\mathbb{R}^2$ to G and mappings c_- and c_+ of $\mathbb{R}^2$ to $\mathfrak{g}_{-1}$ and $\mathfrak{g}_{+1}$, respectively, such that the representation (2.7) for the components ω_- and ω_+ is valid. In general, the mappings c_- and c_+ parameterizing the connection components may be different from the mappings c_- and c_+ which determine the mapping φ. Let

[6] Here and below discussing the holomorphic properties of mappings and functions we assume that the point of the space $\mathbb{R}^2$ is arbitrary but fixed.

us denote the mappings corresponding to the connection under consideration by γ', c'_- and c'_+. Thus, we have

$$\psi^{-1}\partial_-\psi + \lambda^{-1}\psi^{-1}c_-\psi = \gamma'^{-1}\partial_-\gamma' + \lambda^{-1}c'_-, \tag{4.21}$$

$$\psi^{-1}\partial_+\psi + \lambda\psi^{-1}c_+\psi = \lambda\gamma'^{-1}c'_+\gamma'. \tag{4.22}$$

Note that ψ_∞ is a mapping of $\mathbb{R}^2$ to the Lie subgroup of $\mathrm{GL}_n(\mathbb{C})$ defined by the relation (4.8). We recall that this subgroup coincides with G_0 and denote ψ_∞ by γ. From the relation (4.21) we obtain the equality

$$\gamma'^{-1}\partial_-\gamma' = \gamma^{-1}\partial_-\gamma.$$

The same relation (4.21) gives

$$\psi_0^{-1}c_-\psi_0 = c'_-.$$

Impose the condition $\psi_0 = I_n$, which is consistent with the condition (4.20). Here we have

$$c'_- = c_-.$$

Finally, from the equality (4.22) we obtain

$$\gamma'^{-1}c'_+\gamma' = \gamma^{-1}c_+\gamma.$$

We see that if we impose the condition $\psi_0 = I_n$, then the components of the connection under consideration have the form (2.7) where $\gamma = \psi_\infty$.

Thus, to find solutions to the Toda equations under consideration, we can use the following procedure. We fix $2r$ complex numbers μ_i and ν_i and find matrix-valued functions P_i and Q_i satisfying the relations (4.12), (4.13) and (4.16)–(4.19). With the help of the relations (4.9), (4.10), assuming that

$$\psi_0 = I_n,$$

we construct the mappings ψ and ψ^{-1}. Then, the mapping

$$\gamma = \psi_\infty \tag{4.23}$$

satisfies the Toda equation (2.8).

Let us return to the relations (4.12), (4.13). It is easy to see that they are equivalent, and so, we will use the relation (4.13) for further calculations. We can show that, if we suppose that the matrix P_i has the maximum rank, then we get the trivial solution of the Toda equation given by (4.1). Hence, we will assume that P_i is not of maximum rank. The simplest case here is given by matrices of rank one which can be represented as

$$P_i = u_i{}^t w_i,$$

where u and w are n-dimensional column vectors. This representation allows writing the relations (4.13) as

$$u_i + \sum_{j=1}^{r}\sum_{k=1}^{N} \frac{\mu_i}{\mu_i - \epsilon_{2N}^{2k-1}\mu_j} h^{2k-1}B^{-1}w_j({}^t u_j B h^{-2k+1} u_i) = 0. \tag{4.24}$$

Using the identity

$$\sum_{k=1}^{N} \frac{z\epsilon_{2N}^{-2kj}}{z-\epsilon_{2N}^{2k}} = N\frac{z^{N-|j|_N}}{z^N-1},$$

where $|j|_N$ is the residue of division of j by N, we can rewrite the equality (4.24) in terms of the components of u_i as follows:

$$^t u_{i,1} J_{n_1} + N \sum_{j=1}^{r} (R_1)_{ij} {}^t w_{j,1} = 0,$$

where $u_{i,1}$ and $w_{i,1}$ gather first n_1 components of the corresponding n-dimensional column vectors, so these are in fact n_1-dimensional column vectors,[7]

$$u_{i,N+2-k} - N \sum_{j=1}^{r} (R_k)_{ij} w_{j,k} = 0, \qquad k = 2, \ldots, s,$$

and

$$u_{i,N+2-k} + N \sum_{j=1}^{r} (R_k)_{ij} w_{j,k} = 0, \qquad k = s+1, \ldots, p = N.$$

Here the $r \times r$ matrices R_1 and R_k are defined as

$$\begin{aligned}(R_1)_{ij} &= \frac{1}{\mu_i^N + \mu_j^N} \Bigg(\mu_i^N ({}^t u_{i,1} J_{n_1} u_{j,1}) - \sum_{\ell=2}^{s} \mu_i^{N-|\ell-1|_N} \mu_j^{|\ell-1|_N} (u_{i,N+2-\ell} u_{j,\ell}) \\ &\qquad + \sum_{\ell=s+1}^{N} \mu_i^{N-|\ell-1|_N} \mu_j^{|\ell-1|_N} (u_{i,N+2-\ell} u_{j,\ell}) \Bigg), \qquad (4.25)\end{aligned}$$

$$\begin{aligned}(R_k)_{ij} &= \frac{1}{\mu_i^N + \mu_j^N} \Bigg(-\mu_i^{N-|1-k|_N} \mu_j^{|1-k|_N} ({}^t u_{i,1} J_{n_1} u_{j,1}) \\ &\qquad + \sum_{\ell=2}^{k-1} \mu_i^{N-|\ell-k|_N} \mu_j^{|\ell-k|_N} (u_{i,N+2-\ell} u_{j,\ell}) - \sum_{\ell=k}^{s} \mu_i^{N-|\ell-k|_N} \mu_j^{|\ell-k|_N} (u_{i,N+2-\ell} u_{j,\ell}) \\ &\qquad + \sum_{\ell=s+1}^{N} \mu_i^{N-|\ell-k|_N} \mu_j^{|\ell-k|_N} (u_{i,N+2-\ell} u_{j,\ell}) \Bigg) \qquad (4.26)\end{aligned}$$

for $k = 2, \ldots, s$, and

$$\begin{aligned}(R_k)_{ij} &= \frac{1}{\mu_i^N + \mu_j^N} \Bigg(-\mu_i^{N-|1-k|_N} \mu_j^{|1-k|_N} ({}^t u_{i,1} J_{n_1} u_{j,1}) \\ &\qquad + \sum_{\ell=2}^{s} \mu_i^{N-|\ell-k|_N} \mu_j^{|\ell-k|_N} (u_{i,N+2-\ell} u_{j,\ell}) - \sum_{\ell=s+1}^{k-1} \mu_i^{N-|\ell-k|_N} \mu_j^{|\ell-k|_N} (u_{i,N+2-\ell} u_{j,\ell}) \\ &\qquad + \sum_{\ell=k}^{N} \mu_i^{N-|\ell-k|_N} \mu_j^{|\ell-k|_N} (u_{i,N+2-\ell} u_{j,\ell}) \Bigg) \qquad (4.27)\end{aligned}$$

for $k = s+1, \ldots, N$. Recall that for all cases considered here $N = p = 2s - 1$.

[7] We remember that either $n_1 = 1$ or $n_1 = 2$.

We use the equations (4.25), (4.26) and (4.27) to express the vectors w_i via the vectors u_i,

$$ {}^t w_{i,1} = -\frac{1}{N} \sum_{j=1}^{r} (R_1^{-1})_{ij} {}^t u_{j,1} J_{n_1}, \qquad w_{i,k} = \frac{1}{N} \sum_{j=1}^{r} (R_k^{-1})_{ij} u_{j,N+2-k} $$

for $k = 2, \ldots, s$, and

$$ w_{i,k} = -\frac{1}{N} \sum_{j=1}^{r} (R_k^{-1})_{ij} u_{j,N+2-k} $$

for $k = s+1, \ldots, N = p$. As the result, having expressed the matrices P_i and Q_i in terms of the components of the vectors u_i, we find a solution of the relations (4.12) and (4.13).

Further, it follows from the equality (4.24) that, to fulfill also the relations (4.16)–(4.19), it is sufficient to satisfy the equations

$$ \partial_- u_i = -\mu_i^{-1} c_- u_i, \qquad \partial_+ u_i = -\mu_i c_+ u_i. $$

The general solution to these equations is given formally by the expression

$$ u_i(z^-, z^+) = \exp(-\mu_i^{-1} c_- z^- - \mu_i c_+ z^+) u_i^0, \tag{4.28} $$

where $u_i^0 = u_i(0,0)$. We will make explicit this formal solution when later constructing soliton solutions.

Thus, we see that it is possible to satisfy the relations (4.12), (4.13) and (4.16)–(4.19). This gives us solutions of the Toda equation (2.8), and so, to the equations (3.8) and (3.13) by specifying the above formal expression of u_i for the two corresponding cases. Let us show that they can be written in a simple determinant form.

Using the equalities (4.23) and (4.9), we get

$$ \gamma = \psi_\infty = I_n + \sum_{i=1}^{r} \sum_{k=1}^{N} h^{2k} P_i h^{-2k}. $$

For the matrix elements of γ this gives the expression

$$ \gamma_{k\ell} = \delta_{k\ell} \left(1 + N \sum_{i=1}^{r} (P_i)_{kk} \right) = \delta_{k\ell} \Gamma_k. $$

Hence, we have

$$ \Gamma_1 = I_{n_1} - \sum_{i,j=1}^{r} u_{i,1} (R_1^{-1})_{ij} {}^t u_{j,1} J_{n_1}, \qquad \Gamma_\alpha = 1 + \sum_{i,j=1}^{r} u_{i,\alpha} (R_\alpha^{-1})_{ij} u_{j,2s+1-\alpha}, $$

where $\alpha = 2, \ldots, s$, and

$$ \Gamma_\alpha = 1 - \sum_{i,j=1}^{r} u_{i,\alpha} (R_\alpha^{-1})_{ij} u_{j,2s+1-\alpha}, \qquad \alpha = s+1, \ldots, 2s-1. $$

We assume for convenience that the functions $u_{i,\alpha}$ are defined for arbitrary integral values of α so that

$$ u_{i,2s-1+\alpha} = u_{i,\alpha}. $$

By definition the matrices R_α are periodic in the index α. It appears that it is more appropriate to use quasi-periodic quantities $\widetilde{u}_\alpha$ and $\widetilde{R}_\alpha$ defined as

$$\widetilde{u}_\alpha = M^\alpha u_\alpha, \qquad \widetilde{R}_1 = M R_1 M^{2s}, \qquad \widetilde{R}_\alpha = M^{2s+1-\alpha} R_\alpha M^\alpha,$$

where $\alpha = 2, \dots, 2s-1$; here M is a diagonal $r \times r$ matrix given by

$$M_{ij} = \mu_i \delta_{ij}.$$

For these quantities we have quasi-periodicity conditions

$$\widetilde{u}_{2s-1+\alpha} = M^{2s-1} \widetilde{u}_\alpha, \qquad \widetilde{R}_{2s} = M^{2s-1} \widetilde{R}_1 M^{2s-1}, \qquad \widetilde{R}_{2s-1+\alpha} = M^{-2s+1} \widetilde{R}_\alpha M^{2s-1}.$$

The expression of the matrix elements of the matrices $\widetilde{R}_\alpha$ through the quasi-periodic quantities $\widetilde{u}_{i\alpha}$ has a remarkably simple form. We have for $\alpha = 1$

$$(\widetilde{R}_1)_{ij} = \frac{1}{\mu_i^{2s-1} + \mu_j^{2s-1}} \left(\mu_i^{2s-1} ({}^t\widetilde{u}_{i,1} J_{n_1} \widetilde{u}_{j,1}) \mu_j^{2s-1} - \mu_j^{2s-1} \sum_{\beta=2}^{s} \widetilde{u}_{i,2s+1-\beta} \widetilde{u}_{j,\beta} \right.$$
$$\left. + \mu_j^{2s-1} \sum_{\beta=s+1}^{2s-1} \widetilde{u}_{i,2s+1-\beta} \widetilde{u}_{j,\beta} \right).$$

Further, we have for $\alpha = 2, \dots, s$

$$(\widetilde{R}_\alpha)_{ij} = \frac{1}{\mu_i^{2s-1} + \mu_j^{2s-1}} \left(-\mu_i^{2s-1} ({}^t\widetilde{u}_{i,1} J_{n_1} \widetilde{u}_{j,1}) \mu_j^{2s-1} + \mu_j^{2s-1} \sum_{\beta=2}^{\alpha-1} \widetilde{u}_{i,2s+1-\beta} \widetilde{u}_{j,\beta} \right.$$
$$\left. - \mu_i^{2s-1} \sum_{\beta=\alpha}^{s} \widetilde{u}_{i,2s+1-\beta} \widetilde{u}_{j,\beta} + \mu_i^{2s-1} \sum_{\beta=s+1}^{2s-1} \widetilde{u}_{i,2s+1-\beta} \widetilde{u}_{j,\beta} \right),$$

and for $\alpha = s+1, \dots, 2s-1$

$$(\widetilde{R}_\alpha)_{ij} = \frac{1}{\mu_i^{2s-1} + \mu_j^{2s-1}} \left(-\mu_i^{2s-1} ({}^t\widetilde{u}_{i,1} J_{n_1} \widetilde{u}_{j,1}) \mu_j^{2s-1} + \mu_j^{2s-1} \sum_{\beta=2}^{s} \widetilde{u}_{i,2s+1-\beta} \widetilde{u}_{j,\beta} \right.$$
$$\left. - \mu_j^{2s-1} \sum_{\beta=s+1}^{\alpha-1} \widetilde{u}_{i,2s+1-\beta} \widetilde{u}_{j,\beta} + \mu_i^{2s-1} \sum_{\beta=\alpha}^{2s-1} \widetilde{u}_{i,2s+1-\beta} \widetilde{u}_{j,\beta} \right).$$

Here we used the identity $|-k|_N = N - 1 - |k-1|_N$. The quasi-periodic functions have the following useful properties:

$$(\widetilde{R}_{\alpha+1})_{ij} = (\widetilde{R}_\alpha)_{ij} + \widetilde{u}_{i,2s+1-\alpha} \widetilde{u}_{j,\alpha}, \qquad \alpha = 2, \dots, s, \tag{4.29}$$
$$(\widetilde{R}_{\alpha+1})_{ij} = (\widetilde{R}_\alpha)_{ij} - \widetilde{u}_{i,2s+1-\alpha} \widetilde{u}_{j,\alpha}, \qquad \alpha = s+1, \dots, 2s-1, \tag{4.30}$$

and

$$(\widetilde{R}_1)_{ij} = -(\widetilde{R}_2)_{ji}, \qquad (\widetilde{R}_\alpha)_{ij} = (\widetilde{R}_{2s+2-\alpha})_{ji}, \qquad \alpha = 2, \dots, 2s-1. \tag{4.31}$$

In terms of the quasi-periodic quantities, for the $n_1 \times n_1$ matrix-valued function $\mathit{\Gamma}_1$ and for the functions $\mathit{\Gamma}_\alpha$ we have

$$\mathit{\Gamma}_1 = I_{n_1} - \sum_{i,j=1}^{r} \mu_i^{2s-1} \widetilde{u}_{i,1} (\widetilde{R}_1^{-1})_{ij} {}^t\widetilde{u}_{j,1} J_{n_1}, \qquad \mathit{\Gamma}_\alpha = 1 + \sum_{i,j=1}^{r} \widetilde{u}_{i,\alpha} (\widetilde{R}_\alpha^{-1})_{ij} \widetilde{u}_{j,2s+1-\alpha},$$

for $\alpha = 2, \dots, s$, and

$$\Gamma_\alpha = 1 - \sum_{i,j=1}^{r} \widetilde{u}_{i,\alpha} (\widetilde{R}_\alpha^{-1})_{ij} \widetilde{u}_{j,2s+1-\alpha},$$

for $\alpha = s+1, \dots, 2s-1$. The expressions for the functions Γ_α for $\alpha > 1$ can be written as

$$\Gamma_\alpha = 1 + {}^t\widetilde{u}_\alpha \widetilde{R}_\alpha^{-1} \widetilde{u}_{2s+1-\alpha}, \qquad \alpha = 2, \dots, s,$$

and

$$\Gamma_\alpha = 1 - {}^t\widetilde{u}_\alpha \widetilde{R}_\alpha^{-1} \widetilde{u}_{2s+1-\alpha}, \qquad \alpha = s+1, \dots, 2s-1.$$

Here $\widetilde{R}_\alpha$ is an $r \times r$ matrix and $\widetilde{u}_\alpha$ is an r-dimensional column vector. We remember that in the cases under consideration we should have

$$J_{n_1}^{-1} {}^t\Gamma_1 J_{n_1} = \Gamma_1^{-1}, \qquad \Gamma_{2s+1-\alpha} = \Gamma_\alpha^{-1}, \qquad \alpha = 2, \dots, 2s-1. \tag{4.32}$$

To verify these relations we use that

$$\gamma^{-1} = \psi_\infty^{-1} = I_n + \sum_{i=1}^{r} \sum_{k=1}^{2s-1} h^{2k} (h^{-1} B^{-1} {}^tP_i B h) h^{-2k},$$

therefore we find the following expression of Γ_1^{-1} in terms of quasi-periodic quantities,

$$\Gamma_1^{-1} = I_{n_1} - \sum_{i,j=1}^{r} \widetilde{u}_{i,1} \mu_j^{2s-1} (\widetilde{R}_1^{-1})_{ji} {}^t\widetilde{u}_{j,1} J_{n_1}.$$

Comparing now this expression with what we have for Γ_1 above, we conclude that the first relation of equations (4.32) is satisfied.

The expressions just given above allow writing a remarkable determinant representation for the functions Γ_α. It can be shown that

$$\Gamma_\alpha = \frac{\det(\widetilde{R}_\alpha + \widetilde{u}_{2s+1-\alpha} {}^t\widetilde{u}_\alpha)}{\det \widetilde{R}_\alpha}, \qquad \alpha = 2, \dots, s,$$

and

$$\Gamma_\alpha = \frac{\det(\widetilde{R}_\alpha - \widetilde{u}_{2s+1-\alpha} {}^t\widetilde{u}_\alpha)}{\det \widetilde{R}_\alpha}, \qquad \alpha = s+1, \dots, 2s-1.$$

Using the properties (4.29) and (4.30) we can see

$$\Gamma_\alpha = \frac{\det \widetilde{R}_{\alpha+1}}{\det \widetilde{R}_\alpha}, \qquad \alpha = 2, \dots, s, s+1, \dots, 2s-1.$$

For these functions we can also easily demonstrate that

$$\Gamma_{2s+1-\alpha} = \frac{\det \widetilde{R}_{2s+2-\alpha}}{\det \widetilde{R}_{2s+1-\alpha}} = \frac{\det({}^t\widetilde{R}_\alpha)}{\det({}^t\widetilde{R}_{\alpha+1})} = \frac{\det \widetilde{R}_\alpha}{\det \widetilde{R}_{\alpha+1}} = \Gamma_\alpha^{-1},$$

using for this purpose the relations (4.30). Hence all equations (4.32) are fulfilled.

We remember also that for the case $n_1 = 1$ corresponding to the second type of abelian twisted loop Toda equations considered here ($n = p = 2s - 1$), we have $I_1 = J_1 = 1$, and so, we can write for the function Γ_1 the expression

$$\Gamma_1 = 1 - {}^t\widetilde{u}_1 M^{2s-1} \widetilde{R}_1^{-1} \widetilde{u}_1,$$

where $\widetilde{u}_1$ is also an r-dimensional column vector. It can be shown that

$$\Gamma_1 = \frac{\det(\widetilde{R}_1 - \widetilde{u}_1 {}^t u_1 M^{2s-1})}{\det \widetilde{R}_1},$$

We obtain from the expressions of $\widetilde{R}_1$ and $\widetilde{R}_2$ directly that

$$(\widetilde{R}_1)_{ij}\mu_j^{-N} = {}^t\widetilde{u}_{i,1} J_{n_1} \widetilde{u}_{j,1} + \mu_i^{-N}(\widetilde{R}_2)_{ij},$$

and so, for $n_1 = 1$ we can write

$$M^{-2s+1}\widetilde{R}_2 + \widetilde{u}_1 {}^t\widetilde{u}_1 = \widetilde{R}_1 M^{-2s+1}.$$

Using this relation in the above expression for Γ_1 as the ratio of determinants, we easily derive

$$\Gamma_1 = \frac{\det \widetilde{R}_2}{\det \widetilde{R}_1}. \tag{4.33}$$

But we have from the equality (4.31) that

$$\widetilde{R}_1 = -{}^t\widetilde{R}_2,$$

and so, for $n_1 = 1$ the expression (4.33) gives

$$\Gamma_1 = (-1)^r.$$

5 Soliton solutions

5.1 Odd-dimensional case

Here we consider the case of $n = p = N = 2s - 1$. It means also that we have $n_1 = 1$. The eigenvectors of the matrices ${}^tc_-$, ${}^tc_+$, c_- and c_+ are n-dimensional column vectors Ψ_ρ, $\rho = 1, \dots, 2s-1$, satisfying the relations

$${}^tc_-\Psi_\rho = m\epsilon_{2N}^{s+2\rho}\Psi_\rho, \quad {}^tc_+\Psi_\rho = m\epsilon_{2N}^{-s-2\rho}\Psi_\rho, \quad c_-\Psi_\rho = m\epsilon_{2N}^{-s-2\rho}\Psi_\rho, \quad c_+\Psi_\rho = m\epsilon_{2N}^{s+2\rho}\Psi_\rho,$$

where the $2s-1$ components of Ψ_ρ are defined as

$$\begin{aligned} (\Psi_\rho)_\alpha &= \epsilon_{2N}^{\alpha(s+2\rho)}, & \alpha &= 1, \dots, s, \\ (\Psi_\rho)_\alpha &= (-1)^{\alpha-s-1}\epsilon_{2N}^{\alpha(s+2\rho)}, & \alpha &= s+1, \dots, 2s-1. \end{aligned}$$

Consequently, we can give a concrete expression to the formal solution (4.28) as

$$u_{i,\alpha} = \sum_{\rho=1}^{2s-1} c_{i\rho}\epsilon_{2N}^{\alpha(s+2\rho)} \mathrm{e}^{-Z_\rho(\mu_i)}, \qquad \alpha = 1, \dots, s$$

$$u_{i,\alpha} = \sum_{\rho=1}^{2s-1} c_{i\rho}(-1)^{\alpha-s-1}\epsilon_{2N}^{\alpha(s+2\rho)} \mathrm{e}^{-Z_\rho(\mu_i)}, \qquad \alpha = s+1, \dots, 2s-1,$$

where $c_{i\rho}$ are arbitrary constants and we have introduced the notation

$$Z_\rho(\mu_i) = m\,(\epsilon_{2N}^{-s-2\rho}\,\mu_i^{-1} z^- + \epsilon_{2N}^{s+2\rho}\,\mu_i z^+).$$

Then, after some calculation using, in particular, properties of ϵ_{2N}, we write for the matrix elements of $\widetilde{R}_\alpha$ for $\alpha \geq 2$:

$$(\widetilde{R}_\alpha)_{ij} = (-1)^\alpha \mu_i^{2s+1-\alpha}\,\mu_j^\alpha \sum_{\rho,\sigma=1}^{2s-1} c_{i\rho}\,c_{j\sigma}\,\frac{\epsilon_{2N}^{4\rho+1-2(\rho-\sigma)\alpha}}{1+\mu_j\mu_i^{-1}\epsilon_{2N}^{-2(\rho-\sigma)}}\,\mathrm{e}^{-Z_\rho(\mu_i)-Z_\sigma(\mu_j)}. \tag{5.1}$$

It is clear that to obtain nontrivial solutions to the Toda equations we should require that at least two coefficients $c_{i\rho}$ for any $i = 1,\dots,r$ are different from zero. In this, we construct solutions depending on only r combinations of independent variables z^- and z^+. We denote such nonzero constants by C_{J_i} and C_{K_i}. The expression for the matrix elements (5.1) takes then the form

$$(\widetilde{R}_\alpha)_{ij} = (-1)^\alpha\,\mu_i^{2s+1-\alpha}\,\epsilon_{2N}^{4J_i+1-2\alpha J_i}\,C_{J_i}\,\mathrm{e}^{-Z_{J_i}(\mu_i)}\,(\widetilde{R}'_\alpha)_{ij}\,\mu_j^\alpha\,C_{J_j}\,\epsilon_{2N}^{2\alpha J_j}\,\mathrm{e}^{-Z_{J_j}(\mu_j)},$$

where

$$\begin{aligned}(\widetilde{R}'_\alpha)_{ij} = {} & \frac{1}{1+\mu_j\mu_i^{-1}\epsilon_{2N}^{2(J_j-J_i)}} + \frac{C_{K_j}}{C_{J_j}}\frac{\epsilon_{2N}^{2(K_j-J_j)\alpha}}{1+\mu_j\mu_i^{-1}\epsilon_{2N}^{2(K_j-J_i)}}\mathrm{e}^{Z_{J_j}(\mu_j)-Z_{K_j}(\mu_j)} \\ & + \frac{C_{K_i}}{C_{J_i}}\frac{\epsilon_{2N}^{4(K_i-J_i)-2(K_i-J_i)\alpha}}{1+\mu_j\mu_i^{-1}\epsilon_{2N}^{2(J_j-K_i)}}\mathrm{e}^{Z_{J_i}(\mu_i)-Z_{K_i}(\mu_i)} \\ & + \frac{C_{K_i}C_{K_j}}{C_{J_i}C_{J_j}}\frac{\epsilon_{2N}^{4(K_i-J_i)-2(K_i-J_i+K_j-J_j)\alpha}}{1+\mu_j\mu_i^{-1}\epsilon_{2N}^{2(K_j-K_i)}}\mathrm{e}^{Z_{J_i}(\mu_i)-Z_{K_i}(\mu_i)+Z_{J_j}(\mu_j)-Z_{K_j}(\mu_j)}.\end{aligned}$$

It is easy to show that

$$\Gamma_\alpha = \frac{\det \widetilde{R}_{\alpha+1}}{\det \widetilde{R}_\alpha} = (-1)^r\frac{\det \widetilde{R}'_{\alpha+1}}{\det \widetilde{R}'_\alpha}.$$

Recalling also that $\Gamma_1 = (-1)^r$, we see that we can take $\widetilde{R}'_\alpha$ instead of $\widetilde{R}_\alpha$ to construct solutions of the Toda equations using for that the above determinant representation.

Defining a new set of parameters

$$\rho_i = J_i - K_i, \qquad \theta_{\rho_i} = \frac{\pi\rho_i}{2s-1}, \qquad \kappa_{\rho_i} = -\mathrm{i}(\epsilon_{2N}^{\rho_i} - \epsilon_{2N}^{-\rho_i}) = 2\sin\theta_{\rho_i},$$

$$\exp\delta_i = \frac{C_{K_i}}{C_{J_i}}, \qquad \zeta_i = \mathrm{i}\epsilon_{2N}^{s+J_i+K_i}\mu_i, \qquad f_i = \epsilon_{2N}^{\rho_i}\zeta_i, \qquad \widetilde{f}_i = \epsilon_{2N}^{-\rho_i}\zeta_i,$$

and introducing the notation

$$D_{ij}(f,g) = \frac{f_i}{f_i+g_j},$$

we can rewrite the expression for $\widetilde{R}'_\alpha$ as

$$\begin{aligned}(\widetilde{R}'_\alpha)_{ij} = {} & D_{ij}(f,f) + \epsilon_{2N}^{2\rho_i(\alpha-1)}\mathrm{e}^{Z_i(\zeta)+\delta_i-2\mathrm{i}\theta_{\rho_i}}D_{ij}(\widetilde{f},f) + D_{ij}(f,\widetilde{f})\mathrm{e}^{Z_j(\zeta)+\delta_j-2\mathrm{i}\theta_{\rho_j}}\epsilon_{2N}^{-2\rho_j(\alpha-1)} \\ & + \epsilon_{2N}^{2\rho_i(\alpha-1)}\mathrm{e}^{Z_i(\zeta)+\delta_i-2\mathrm{i}\theta_{\rho_i}}D_{ij}(\widetilde{f},\widetilde{f})\mathrm{e}^{Z_j(\zeta)+\delta_j-2\mathrm{i}\theta_{\rho_j}}\epsilon_{2N}^{-2\rho_j(\alpha-1)},\end{aligned} \tag{5.2}$$

where now the dependence on independent variables is given through

$$Z_i(\zeta) = m\kappa_{\rho_i}(\zeta_i^{-1}z^- + \zeta_i z^+).$$

In fact, it appears that it is appropriate to use the matrices $T_\alpha = D^{-1}(f,f)\,\widetilde{R}'_\alpha$ and write the solutions under construction as

$$\Gamma_\alpha = \frac{\det T_{\alpha+1}}{\det T_\alpha}.$$

The problem of constructing multi-soliton solutions for the Toda equations (3.8) is thus reduced to calculating the determinant of the $r \times r$ matrix T_α.

To obtain a one-soliton solution we set $r = 1$. In this case T_α are ordinary functions, and we easily find

$$T_{\alpha+1} = 1 + 2\frac{\cos(2\alpha-1)\theta_\rho}{\cos\theta_\rho}\mathrm{e}^{Z(\zeta)+\delta-2\mathrm{i}\theta_\rho} + \mathrm{e}^{2(Z(\zeta)+\delta-2\mathrm{i}\theta_\rho)}. \tag{5.3}$$

Setting $r = 2$ we work out the determinant of the respective 2×2 matrix explicitly given above and thus obtain for the two-soliton solution the expression

$$\begin{aligned}
\det T_{\alpha+1} \;=\;& 1 + 2\frac{\cos(2\alpha-1)\theta_{\rho_1}}{\cos\theta_{\rho_1}}\mathrm{e}^{\widetilde{Z}_1} + 2\frac{\cos(2\alpha-1)\theta_{\rho_2}}{\cos\theta_{\rho_2}}\mathrm{e}^{\widetilde{Z}_2} + \mathrm{e}^{2\widetilde{Z}_1} + \mathrm{e}^{2\widetilde{Z}_2}\\
&+ \left(2\eta_{12}^+\frac{\cos(2\alpha-1)(\theta_{\rho_1}-\theta_{\rho_2})}{\cos\theta_{\rho_1}\cos\theta_{\rho_2}} + 2\eta_{12}^-\frac{\cos(2\alpha-1)(\theta_{\rho_1}+\theta_{\rho_2})}{\cos\theta_{\rho_1}\cos\theta_{\rho_2}}\right)\mathrm{e}^{\widetilde{Z}_1+\widetilde{Z}_2}\\
&+2\eta_{12}^+\eta_{12}^-\left(\frac{\cos(2\alpha-1)\theta_{\rho_1}}{\cos\theta_{\rho_1}}\mathrm{e}^{\widetilde{Z}_1+2\widetilde{Z}_2} + \frac{\cos(2\alpha-1)\theta_{\rho_2}}{\cos\theta_{\rho_2}}\mathrm{e}^{2\widetilde{Z}_1+\widetilde{Z}_2}\right)\\
&+ \left(\eta_{12}^+\eta_{12}^-\right)^2\mathrm{e}^{2(\widetilde{Z}_1+\widetilde{Z}_2)},
\end{aligned} \tag{5.4}$$

with the 'soliton interaction factors'

$$\eta_{12}^+ = \frac{(\zeta_1\zeta_2^{-1}+\zeta_2\zeta_1^{-1}) + 2\cos(\theta_{\rho_1}+\theta_{\rho_2})}{(\zeta_1\zeta_2^{-1}+\zeta_2\zeta_1^{-1}) + 2\cos(\theta_{\rho_1}-\theta_{\rho_2})}, \quad \eta_{12}^- = \frac{(\zeta_1\zeta_2^{-1}+\zeta_2\zeta_1^{-1}) - 2\cos(\theta_{\rho_1}-\theta_{\rho_2})}{(\zeta_1\zeta_2^{-1}+\zeta_2\zeta_1^{-1}) - 2\cos(\theta_{\rho_1}+\theta_{\rho_2})},$$

and the constant parameters

$$\mathrm{e}^{\delta'_1} = \frac{(f_1+f_2)(\widetilde{f}_1-f_2)}{(f_1-f_2)(\widetilde{f}_1+f_2)}, \qquad \mathrm{e}^{\delta'_2} = \frac{(f_1+f_2)(f_1-\widetilde{f}_2)}{(f_1-f_2)(f_1+\widetilde{f}_2)}$$

giving rise to a shift in the exponents as

$$\widetilde{Z}_i = Z_i(\zeta) + \delta_i + \delta'_i - 2\mathrm{i}\theta_{\rho_i}.$$

It can also be shown that performing the corresponding change of variables as suggested in the paper [6], one can reach the same result along the lines of the Hirota's approach. Here, the quantities $\det T_{\alpha+1}$ constructed by means of the rational dressing formalism, will coincide with the Hirota's τ-functions τ_α, see the paper [6] where such correspondence was established for the untwisted case.

Now, considering $s = 2$, so that $N = 3$, we describe the Dodd–Bullough–Mikhailov equation from Section 3.2. Here we have $\Gamma_1 = (-1)^r$, $\Gamma_3 = \Gamma_2^{-1}$, and so, the mapping

γ is parameterized by the only nontrivial function Γ_2, denoted here by Γ. The corresponding soliton solutions can easily be derived from the relations (5.3) and (5.4) putting $\alpha = 2$ and $\alpha = 3$ in order and taking into account that $\theta_\rho = \pi\rho/3$. Remember here that $\rho = J - K$, where J and K take values 1, 2 or 3 only. In particular, it is easy to see that the one-soliton solution can be written as

$$\Gamma = \frac{1 - 4\mathrm{e}^{Z(\zeta)+\delta-2\mathrm{i}\theta_\rho} + \mathrm{e}^{2(Z(\zeta)+\delta-2\mathrm{i}\theta_\rho)}}{(1 + \mathrm{e}^{Z(\zeta)+\delta-2\mathrm{i}\theta_\rho})^2}.$$

For the two-soliton solution, we should respectively simplify the expression (5.4). The corresponding expressions reproduce the one- and two-soliton solutions of the Dodd–Bullough–Mikhailov equation obtained in the paper [15] by means of the Hirota's method.

5.2 Even-dimensional case

Here we consider the case of $n = 2s$, with $p = N = 2s - 1$. It means also that now we have $n_1 = 2$. The eigenvectors of the matrices ${}^t c_-$, ${}^t c_+$, c_- and c_+ are $2s$-dimensional column vectors Ψ_ρ, $\rho = 1, \ldots, 2s-1$, satisfying the relations

$${}^t c_- \Psi_\rho = m\epsilon_{2N}^{s+2\rho}\Psi_\rho, \quad {}^t c_+ \Psi_\rho = m\epsilon_{2N}^{-s-2\rho}\Psi_\rho, \quad c_- \Psi_\rho = m\epsilon_{2N}^{-s-2\rho}\Psi_\rho, \quad c_+ \Psi_\rho = m\epsilon_{2N}^{s+2\rho}\Psi_\rho,$$

where we define the $2s$ components of Ψ_ρ as

$$(\Psi_\rho)_0 = (\Psi_\rho)_1 = \epsilon_{2N}^{\alpha(s+2\rho)}, \qquad (\Psi_\rho)_\alpha = \sqrt{2}\epsilon_{2N}^{\alpha(s+2\rho)}, \qquad \alpha = 2, \ldots, s$$

and

$$(\Psi_\rho)_\alpha = (-1)^{\alpha-s-1}\sqrt{2}\epsilon_{2N}^{\alpha(s+2\rho)}, \qquad \alpha = s+1, \ldots, 2s-1.$$

Besides, respective to the only zero eigenvalue, c_-, c_+ and their transposed matrices have one and the same null-vector that can be defined as ${}^t\Psi_0 = (1, -1, 0, \ldots, 0)$.

Consequently, the solution (4.28) takes the form

$$(u_{i,1})_0 = c_{i0} + \sum_{\rho=1}^{2s-1} c_{i\rho}\epsilon_{2N}^{s+2\rho}\,\mathrm{e}^{-Z_\rho(\mu_i)}, \qquad (u_{i,1})_1 = -c_{i0} + \sum_{\rho=1}^{2s-1} c_{i\rho}\epsilon_{2N}^{s+2\rho}\,\mathrm{e}^{-Z_\rho(\mu_i)},$$

and

$$u_{i,\alpha} = \sum_{\rho=1}^{2s-1} c_{i\rho}\sqrt{2}\epsilon_{2N}^{\alpha(s+2\rho)}\,\mathrm{e}^{-Z_\rho(\mu_i)}, \qquad \alpha = 2, \ldots, s$$

$$u_{i,\alpha} = \sum_{\rho=1}^{2s-1} c_{i\rho}(-1)^{\alpha-s-1}\sqrt{2}\epsilon_{2N}^{\alpha(s+2\rho)}\,\mathrm{e}^{-Z_\rho(\mu_i)}, \qquad \alpha = s+1, \ldots, 2s-1,$$

where c_{i0} and $c_{i\rho}$ are arbitrary constants and, as usual, we have introduced the notation

$$Z_\rho(\mu_i) = m(\epsilon_{2N}^{-s-2\rho}\mu_i^{-1}z^- + \epsilon_{2N}^{s+2\rho}\mu_i z^+).$$

Note that $u_{i,1}$ is now a 2-dimensional column vector with the components $(u_{i,1})_0$ and $(u_{i,1})_1$ given in order.

For the quasi-periodic quantities $\widetilde{R}_\alpha$ introduced in Section 4 we obtain the expressions

$$
\begin{aligned}
(\widetilde{R}_1)_{ij} = &- \frac{2\mu_i^{2s}\mu_j^{2s}}{\mu_i^{2s-1}+\mu_j^{2s-1}}\, c_{i0}c_{j0} \\
&+ 2\mu_i\mu_j^{2s}\sum_{\rho,\sigma=1}^{2s-1} c_{i\rho}c_{j\sigma}\frac{\epsilon_{2N}^{2(s+\rho+\sigma)}}{1+\mu_j\mu_i^{-1}\epsilon_{2N}^{-2(\rho-\sigma)}}\,\mathrm{e}^{-Z_\rho(\mu_i)-Z_\sigma(\mu_j)}, \\
(\widetilde{R}_\alpha)_{ij} = &\frac{2\mu_i^{2s}\mu_j^{2s}}{\mu_i^{2s-1}+\mu_j^{2s-1}}\, c_{i0}c_{j0} \\
&+ 2(-1)^\alpha\mu_i^{2s+1-\alpha}\mu_j^{\alpha}\sum_{\rho,\sigma=1}^{2s-1} c_{i\rho}c_{j\sigma}\frac{\epsilon_{2N}^{4\rho+1-2(\rho-\sigma)\alpha}}{1+\mu_j\mu_i^{-1}\epsilon_{2N}^{-2(\rho-\sigma)}}\,\mathrm{e}^{-Z_\rho(\mu_i)-Z_\sigma(\mu_j)},
\end{aligned}
$$

that are to be used in the determinant representation derived earlier for constructing the soliton solutions.

It is easy to see that if we set here $c_{i0}=0$, then we come to the same solutions given for $n=2s-1$ by the relations (5.2)–(5.4), with the only unessential difference that for $n=2s$ the rational dressing gives $\mathit{\Gamma}_1=(-1)^r J_2^r$. Therefore, to obtain new solutions we consider that in what follows c_{i0} does not vanish.

To construct such new simplest soliton solutions we thus assume that for each value of the index i only one arbitrary constant $c_{i\rho}$, apart from the c_{i0}, is different from zero. To keep up with the notations used in the preceding section, we denote such nonvanishing coefficients by C_{I_i} and C_{0_i}. Then we can write for the above $r\times r$ matrices $\widetilde{R}_1$ and $\widetilde{R}_\alpha$

$$
(\widetilde{R}_1)_{ij} = -2\mu_i C_{0_i}(\widetilde{R}'_1)_{ij}C_{0_j}\mu_j^{2s}, \qquad (\widetilde{R}_\alpha)_{ij} = 2\mu_i C_{0_i}(\widetilde{R}'_\alpha)_{ij}C_{0_j}\mu_j^{2s},
$$

where $\widetilde{R}'_1$ and $\widetilde{R}'_\alpha$ can be represented as

$$
\begin{aligned}
(\widetilde{R}'_1)_{ij} &= D_{ij}(\zeta^{2s-1},\zeta^{2s-1}) - \mathrm{e}^{-Z'_i}D_{ij}(\zeta,\zeta)\,\mathrm{e}^{-Z'_j}, \\
(\widetilde{R}'_\alpha)_{ij} &= D_{ij}(\zeta^{2s-1},\zeta^{2s-1}) - (-1)^\alpha\zeta_i^{2s-\alpha}\,\mathrm{e}^{-Z'_i}D_{ij}(\zeta,\zeta)\,\mathrm{e}^{-Z'_j}\zeta_j^{\alpha-2s}.
\end{aligned}
$$

Here we use the same notation for the matrices $D(f,g)$ introduced in the preceding Section 5.1, and besides,

$$
Z'_i = Z_i(\zeta) - \delta_i - \mathrm{i}\theta_{s+2I_i}, \qquad Z_i(\zeta) = m(\zeta_i^{-1}z^- + \zeta_i z^+),
$$

with the set of parameters

$$
\zeta_i = \epsilon_{2N}^{s+2I_i}\mu_i, \qquad \mathrm{e}^{\delta_i} = \frac{C_{I_i}}{C_{0_i}}, \qquad \theta_{s+2I_i} = \frac{\pi(s+2I_i)}{2s-1}.
$$

We also rewrite the explicit forms of the components of the 2-dimensional column vector $\widetilde{u}_{i,1}$ in terms of the notations introduced above. We have

$$
(\widetilde{u}_{i,1})_0 = \mu_i C_{0_i}(1+\exp(-Z'_i)), \qquad (\widetilde{u}_{i,1})_1 = -\mu_i C_{0_i}(1-\exp(-Z'_i)).
$$

Hence, according to the general relations derived in Section 4, we can take the matrices $T_\alpha = D^{-1}(\zeta^{2s-1}, \zeta^{2s-1})\widetilde{R}'_\alpha$ instead of $\widetilde{R}_\alpha$ and write for the solutions of the Toda equations (3.13) the following expressions:

$$\Gamma_1 = I_2 + \sum_{i,j=1}^{r} v_i \, (\widetilde{R}_1'^{-1})_{ij} \, {}^t v_j \, J_2,$$

where v_i are 2-dimensional column vectors with the components

$$v_{i,0} = \frac{1}{\sqrt{2}}(1 + \exp(-Z'_i)), \qquad v_{i,1} = -\frac{1}{\sqrt{2}}(1 - \exp(-Z'_i)),$$

and

$$\Gamma_\alpha = \frac{\det T_{\alpha+1}}{\det T_\alpha}, \qquad \alpha = 2, \ldots, s.$$

To obtain a one-soliton solution of the type under consideration, we put $r = 1$, for which T_α are ordinary functions. It is easy to show that in this case we have

$$\Gamma_1 = \begin{pmatrix} 0 & \Gamma \\ \Gamma^{-1} & 0 \end{pmatrix}, \qquad \Gamma = \frac{1 + \exp(-Z')}{1 - \exp(-Z')},$$

and

$$\Gamma_\alpha = \frac{1 + (-1)^\alpha \exp(-2Z')}{1 - (-1)^\alpha \exp(-2Z')}, \qquad \alpha = 2, \ldots, s.$$

Note that apart from the relation $\Gamma_{2s+1-\alpha} = \Gamma_\alpha^{-1}$ here we also have $\Gamma_{\alpha+1} = \Gamma_\alpha^{-1}$. It is clear that to have a mapping γ belonging to G_0 we should take $\Gamma_1 J_2$ instead of the above Γ_1.

Setting $r = 2$ we work out the corresponding 2×2 matrices and thus obtain new two-soliton solutions to (3.13). The calculations lead to the expressions

$$\Gamma_1 = \begin{pmatrix} \Gamma & 0 \\ 0 & \Gamma^{-1} \end{pmatrix}, \qquad \Gamma = \frac{1 + \mathrm{e}^{-\widetilde{Z}_1} - \mathrm{e}^{-\widetilde{Z}_2} - \eta_{12}\,\mathrm{e}^{-(\widetilde{Z}_1 + \widetilde{Z}_2)}}{1 - \mathrm{e}^{-\widetilde{Z}_1} + \mathrm{e}^{-\widetilde{Z}_2} - \eta_{12}\,\mathrm{e}^{-(\widetilde{Z}_1 + \widetilde{Z}_2)}},$$

where the 'soliton interaction factor' is now

$$\eta_{12} = \frac{\zeta_1 - \zeta_2}{\zeta_1 + \zeta_2} \cdot \frac{\zeta_1^{2s-1} - \zeta_2^{2s-1}}{\zeta_1^{2s-1} + \zeta_2^{2s-1}},$$

and we have introduced a new parameter δ' defined by

$$\mathrm{e}^{\delta'} = \frac{\zeta_1^{2s-1} + \zeta_2^{2s-1}}{\zeta_1^{2s-1} - \zeta_2^{2s-1}}$$

and producing a shift in the exponents,

$$\widetilde{Z}_i = Z'_i - \delta' = Z_i(\zeta) - \delta_i - \delta' - \mathrm{i}\theta_{s+2I_i}.$$

We also have

$$\det T_{\alpha+1} = 1 + (-1)^\alpha \, (\mathrm{e}^{-2\widetilde{Z}_1} + \mathrm{e}^{-2\widetilde{Z}_2}) - 4\,(-1)^\alpha \, \frac{\zeta_1^\alpha \zeta_2^{2s-\alpha} + \zeta_2^\alpha \zeta_1^{2s-\alpha}}{(\zeta_1 + \zeta_2)(\zeta_1^{2s-1} + \zeta_2^{2s-1})} \, \mathrm{e}^{-(\widetilde{Z}_1 + \widetilde{Z}_2)} + \eta_{12}^2 \, \mathrm{e}^{-2(\widetilde{Z}_1 + \widetilde{Z}_2)}.$$

Note finally that under the permutation of the parameters ζ_1 and ζ_2 the function Γ transforms into Γ^{-1}, thus Γ_1 goes to Γ_1^{-1}, while Γ_α for the other values of α all stay invariant.

6 Conclusion

We have considered the abelian Toda systems associated with the loop groups of the complex general linear groups. Our construction was based on the classification of Toda equations associated with loop groups of complex classical Lie groups whose Lie algebras are endowed with *integrable* $\mathbb{Z}$*-gradations* with finite-dimensional grading subspaces (Nirov and Razumov, 2005–2007). Using the method of rational dressing, along the lines of [6], we have constructed soliton solutions to these equations in the twisted cases, that is, when the gradations are generated by outer automorphisms of the structure Lie algebras. Our consideration can be generalized to Toda systems connected with other loop groups, such as twisted and untwisted loop groups of the complex orthogonal and symplectic groups. It is worth noting that, as we have already observed here, although initially there were no group and algebra defining conditions in the case of general linear groups, certain restrictions were to be imposed on the mappings entering the loop Toda equation, so that the pole positions of the dressing meromorphic mappings and their inverse ones turn out to be related with each other, just due to the specific structure of the outer automorphism leading to the twisted cases. This circumstance made part of the formulae more intricate than in the untwisted general linear case considered in the preceding paper [6]. Actually, similar problems of coinciding pole positions arise also due to the specific group conditions. We will address to this problem and present our respective results in some future publications.

Note finally that Toda systems based on twisted affine Kac–Moody algebras were approached in [24] by means of vertex operators, by constructing twisted algebras from untwisted ones with the help of relevant folding techniques. And it should be an interesting problem to compare the rational dressing method also with this approach.

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