Effective action and Schwinger-DeWitt technique in DGP brane models

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Abstract

We give blueprints of the Schwinger-DeWitt technique for the covariant curvature expansion of quantum effective action for the DGP type models in curved spacetime.

1 Introduction

Modified theories of gravity in the form of braneworld models can in principle account for the phenomenon of dark energy as well as for nontrivial compactifications of multi-dimensional string models. It becomes increasingly more obvious that one should include in such models the analysis of quantum effects beyond the tree-level approximation [1]. This is the only way to reach an ultimate conclusion on the resolution of such problems as the presence of ghosts [2] and low strong-coupling scale [3]. Quantum effects in brane models are also important for the stabilization of extra dimensions [4], fixing the cross-over scale in the Brans-Dicke modification of the DGP model [5] and in the recently suggested mechanism of the cosmological acceleration generated by the four-dimensional conformal anomaly [6].

A general framework for treating quantum effective actions in brane models (or, more generally, models with timelike and spacelike boundaries) was recently suggested in [7, 8, 9, 10]. The main peculiarity of these models is that due to quantum field fluctuations on the branes the field propagator is subject to generalized Neumann boundary conditions involving normal and tangential derivatives on the brane/boundary surfaces. This presents both technical and conceptual difficulties, because such boundary conditions are much harder to handle than the simple Dirichlet ones. The method of [9] provides a systematic reduction of the generalized Neumann boundary conditions to Dirichlet conditions. As a byproduct it disentangles from the quantum effective action the contribution of the surface modes mediating the brane-to-brane propagation, which play a very important role in the zero-mode localization mechanism of the Randall-Sundrum type [11]. The purpose of this work is to make the next step — to extend a well-known Schwinger-DeWitt technique [12, 13, 14, 15] to the calculation of this contribution in the DGP model in a weakly curved spacetime in the form of the *covariant* curvature expansion.

Briefly the method of [9] looks as follows. The action of a (free field) brane model generally contains the bulk and the brane parts,

$$S[\phi] = \frac{1}{2}\int_{\mathbf{B}} d^{d+1}X\, G^{1/2}\phi(X)F(\nabla_X)\,\phi(X) + \frac{1}{2}\int_{\mathbf{b}} d^d x\, g^{1/2}\varphi(x)\,\kappa(\nabla_x)\,\varphi(x)\ , \tag{1}$$

where the $(d+1)$-dimensional bulk and the d-dimensional brane coordinates are labeled respectively by $X = X^A$ and $x = x^\mu$, and the boundary values of bulk fields $\phi(X)$ on the

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brane/boundary $\mathbf{b} = \partial\mathbf{B}$ are denoted by $\varphi(x)$,

$$\phi(X)\Big|_{\mathbf{b}} = \varphi(x), \tag{2}$$

G and g are the determinants of the bulk G_{AB} and $g_{\mu\nu}$ metrics respectively.

The kernel of the bulk Lagrangian is given by the second order differential operator $F(\nabla_X)$, whose covariant derivatives ∇_X are integrated by parts in such a way that they form bilinear combinations of first order derivatives acting on two different fields. Integration by parts in the bulk gives nontrivial surface terms on the brane/boundary. In particular, this operation results in the Wronskian relation for generic test functions $\phi_{1,2}(X)$,

$$\int_{\mathbf{B}} d^{d+1}X\, G^{1/2} \Big(\phi_1 \overrightarrow{F}(\nabla_X)\phi_2 - \phi_1 \overleftarrow{F}(\nabla_X)\,\phi_2\Big) = -\int_{\partial\mathbf{B}} d^d x\, g^{1/2} \Big(\phi_1 \overrightarrow{W}\phi_2 - \phi_1 \overleftarrow{W}\phi_2\Big). \tag{3}$$

Arrows everywhere here indicate the direction of action of derivatives either on ϕ_1 or ϕ_2.

The brane part of the action contains as a kernel some local operator $\kappa(\nabla)$, $\nabla \equiv \nabla_x$. Its order in derivatives depends on the model in question. In the Randall-Sundrum model [11], for example, it is for certain gauges just an ultralocal multiplication operator generated by the tension term on the brane. In the Dvali-Gabadadze-Porrati (DGP) model [16] this is a second order operator induced by the brane Einstein term on the brane, $\kappa(\nabla) \sim \nabla\nabla/m$, where m is the DGP scale which is of the order of magnitude of the horizon scale, being responsible for the cosmological acceleration [17]. In the context of the Born-Infeld action in D-brane string theory with vector gauge fields, $\kappa(\nabla)$ is a first-order operator [18].

In all these cases the variational procedure for the action (1) with dynamical (not fixed) fields on the boundary $\varphi(x)$ naturally leads to generalized Neumann boundary conditions of the form

$$\Big(\overrightarrow{W}(\nabla_X) + \kappa(\nabla)\Big)\,\phi\,\Big|_{\mathbf{b}} = 0, \tag{4}$$

which uniquely specify the propagator of quantum fields and, therefore, a complete Feynman diagrammatic technique for the system in question. The method of [9] allows one to systematically reduce this diagrammatic technique to the one subject to the Dirichlet boundary conditions $\phi|_{\mathbf{b}} = 0$. The main additional ingredient of this reduction procedure is the brane operator $\boldsymbol{F}^{\,\rm brane}(x,x')$ which is constructed from the Dirichlet Green's function $G_D(X,X')$ of the operator $F(\nabla)$ in the bulk,

$$\boldsymbol{F}^{\,\rm brane}(x,x') = -\overrightarrow{W}(\nabla_X)\, G_D(X,X')\, \overleftarrow{W}(\nabla_{X'})\Big|_{X=e(x),\,X'=e(x')} + \kappa(\nabla)\,\delta(x,x')\;. \tag{5}$$

This expression expresses the fact that the kernel of the Dirichlet Green's function is being acted upon both arguments by the Wronskian operators with a subsequent restriction to the brane, with $X = e(x)$ denoting the brane embedding function.

As shown in [9], this operator determines the brane-to-brane propagation of the physical modes in the system with the classical action (1) (its inverse is the brane-to-brane propagator) and additively contributes to its full one-loop effective action according to

$$\boldsymbol{\varGamma}_{1-{\rm loop}} \equiv \frac{1}{2}\,{\rm Tr}^{(d+1)}_N \ln F = \frac{1}{2}\,{\rm Tr}^{(d+1)}_D \ln F + \frac{1}{2}\,{\rm Tr}^{(d)} \ln \boldsymbol{F}^{\,\rm brane}, \tag{6}$$

where ${\rm Tr}^{(d+1)}_{D,N}$ denotes functional traces of the bulk theory subject to Dirichlet and Neumann boundary conditions, respectively, while ${\rm Tr}^{(d)}$ is a functional trace in the boundary d-dimensional theory. The full quantum effective action of this model is obviously given by the functional determinant of the operator $F(\nabla_X)$ subject to the generalized Neumann boundary conditions (5), and the above equation reduces its calculation to that of the Dirichlet boundary conditions plus the contribution of the brane-to-brane propagation.

Here we apply (6) to a simple model of a scalar field which mimics in particular the properties of the brane-induced gravity models and the DGP model [16]. This is the $(d+1)$-dimensional massive scalar field $\phi(X) = \phi(x,y)$ with mass M living in the *curved* half-space $y \geq 0$ with the additional d-dimensional kinetic term for $\varphi(x) \equiv \phi(x,0)$ localized at the brane (boundary) at $y = 0$,

$$S[\phi] = \frac{1}{2} \int\limits_{y\geq 0} d^{d+1}X\, G^{1/2} \Big((\nabla_X \phi(X))^2 + M^2 \phi^2(X) \Big) + \frac{1}{4m} \int d^d x\, g^{1/2}\, (\nabla_x \varphi(x))^2. \qquad (7)$$

Here and in what follows we work in a Euclidean (positive-signature) spacetime. Therefore, this action corresponds to the following choice of $F(\nabla_X)$ in terms of $(d+1)$-dimensional and d-dimensional covariant D'Alembertians (Laplacians)

$$F(\nabla_X) = M^2 - \Box^{(d+1)} = M^2 - G^{AB} \nabla_A \nabla_B, . \qquad (8)$$

In the normal Gaussian coordinates its Wronskian operator is given by $W = -\partial_y$ — the normal derivative with respect to outward-pointing normal to the brane, and the boundary operator $\kappa(\nabla)$ equals

$$\kappa(\nabla) = -\frac{1}{2m} \Box, \quad \Box = \Box^{(d)} \equiv g^{\mu\nu} \nabla_\mu \nabla_\nu, \qquad (9)$$

where the dimensional parameter m mimics the role of the DGP scale [16]. Thus, the generalized Neumann boundary conditions in this model involve second-order derivatives tangential to the brane,

$$\Big(\partial_y + \frac{1}{2m} \Box \Big) \phi \Big|_{\mathbf{b}} = 0, \qquad (10)$$

cf. (4) with $W = -\partial_y$ and $\kappa = -\Box/2m$.

As was shown [10], the flat space brane-to-brane operator for such a model has the form of the pseudodifferential operator with the flat-space $\Box$,

$$\boldsymbol{F}^{\,\mathrm{brane}}(\nabla) = \frac{1}{2m} \left(-\Box + 2m\sqrt{M^2 - \Box} \right). \qquad (11)$$

In the massless case of the DGP model [16], $M = 0$, this operator is known to mediate the gravitational interaction on the brane, interpolating between the four-dimensional Newtonian law at intermediate distances and the five-dimensional law at the horizon scale $\sim 1/m$ [3].

Here we generalize this construction to a curved spacetime and expand the brane-to-brane operator and its effective action in covariant curvature series. This is the expansion in powers of the bulk curvature symbolically denoted below as R, extrinsic curvature of the brane denoted by K and their covariant derivatives — all taken at the location of the brane. The expansion starts with the approximation (11) based on the *full covariant* d'Alembertian on the brane. We present a systematic technique of calculating curvature corrections to (11) and rewrite their nonlocal operator coefficients — functions of the covariant $\Box$ — in the form of the generalized (weighted) proper time representation.

The success of the conventional Schwinger-DeWitt method is based on the fact that the one-loop effective action of the operator, say $M^2 - \Box$, has a proper time representation

$$\frac{1}{2} \mathrm{Tr} \ln \Big(M^2 - \Box \Big) = -\frac{1}{2} \int_0^\infty \frac{ds}{s}\, e^{-s M^2}\, \mathrm{Tr}\, e^{s\Box}. \qquad (12)$$

In view of the well-known small time expansion for the heat kernel [12, 13],

$$e^{s\Box}\delta(x,x') = \frac{1}{(4\pi s)^{d/2}} D^{1/2}(x,x')\, e^{-\sigma(x,x')/2s} \sum_{n=0}^{\infty} s^n\, a_n(x,x'), \tag{13}$$

($\sigma(x,x')$ is the geodetic world function, $D(x,x')$ is the associated Van Vleck determinant and $a_n(x,x')$ are the Schwinger-DeWitt or Gilkey-Seely coefficients) the curvature expansion eventually reduces to the calculation of the coincidence limits of $a_n(x,x')$ and a trivial proper time integration resulting in the inverse mass expansion

$$\frac{1}{2}\,\mathrm{Tr}\,\ln\,\Big(M^2 - \Box\Big) = -\frac{1}{2}\frac{1}{(4\pi)^{d/2}} \sum_{n=0}^{\infty} \frac{\Gamma(n-d/2)}{M^{2n-d}} \int dx\, g^{1/2}\, a_n(x,x). \tag{14}$$

As we will show below, the calculation of the brane effective action differs from the conventional Schwinger-DeWitt case in that the proper time integral (12) contains in the integrand a certain extra weight function $w(s)$ and instead of just $\mathrm{Tr}\, e^{s\Box}$ one has to calculate the trace of the heat kernel acted upon by a certain local differential operator $\mathrm{Tr}\,\big(W(\nabla)e^{s\Box}\big)$. This again reduces to the calculation of the coincidence limits — this time of the multiple covariant derivatives of $a_n(x,x')$, $\nabla_{\mu_1}...\nabla_{\mu_n} a_n(x,x')|_{x'=x}$ — the task easily doable within a conventional DeWitt recurrence procedure for $a_n(x,x')$.

2 Perturbation theory for the bulk Green's function and brane effective action

In normal Gaussian coordinates the covariant bulk d'Alembertian decomposes as $\Box^{(d+1)}_X = \partial_y^2 + \Box(y) + ...$, where ellipses denote depending on spin terms at most linear in derivatives[1] and $\Box(y)$ is a covariant d'Alembertian on the slice of constant coordinate y. Therefore the full bulk operator takes the form

$$F(\nabla) = M^2 - \Box^{(d+1)}_X + P(X) = M^2 - \Box - \partial_y^2 - V(X\,|\,\partial_y,\nabla) \equiv F^0 - V, \tag{15}$$

in which all nontrivial y-dependence is isolated as a perturbation term $V(X\,|\,\partial_y,\nabla) \equiv V(y,\partial_y)$ — a first-order differential operator in y, proportional to the extrinsic and bulk curvatures, and of second order in brane derivatives ∇ which we do not explicitly indicate here by assuming that they are encoded in the operator structure of $V(y,\partial_y)$. In particular, it includes the difference $\Box(0) - \Box(y) \equiv \Box - \Box(y)$ expandable in Taylor series in y.

The kernel of the bulk Green's function can formally be written as a y-dependent nonlocal operator acting on the d-dimensional brane — some non-polynomial function of the brane covariant derivative

$$G_D(X,X') = G_D(y,y'\,|\,\nabla)\,\delta(x,x'). \tag{16}$$

The perturbation expansion for $G_D(y,y'\,|\,\nabla)$ is usual

$$G_D = G_D^0 + G_D^0 V G_D^0 + ... = G_D^0 \sum_{n=0}^{\infty} \big(G_D^0\, V\big)^n, \tag{17}$$

where G_D^0 is the propagator for operator F^0 obeying Dirichlet boundary conditions and the composition law includes the integration over the bulk coordinates, like for example in the first subleading term

$$G_D^0 V G_D^0(y,y') = \int_0^{\infty} dy''\, G_D^0(y,y'')V(y'',\partial_{y''})G_D^0(y'',y'). \tag{18}$$

[1] This term for a general spin structurally has the form $K\nabla_X + K^2 + (\nabla K) + R$ where K is the extrinsic curvature of $y = \mathrm{const}$ slices and R is the bulk curvature.

The lowest order Green's function in the half-space of the DGP model setting — the Green's function of $F^0 = M^2 - \Box - \partial_y^2$ subject to Dirichlet conditions on the brane $y = 0$ and at infinity — reads as follows

$$G_D^0(y, y') = \frac{e^{-|\,y-y'|\,\sqrt{M^2-\Box}} - e^{-(y+y')\,\sqrt{M^2-\Box}}}{2\sqrt{M^2-\Box}}. \tag{19}$$

We want to stress that here we assume the exact (curved) d-dimensional d'Alembertian $\Box$ depending on the induced metric of the brane $g_{\mu\nu}(x)$. This means that in the lowest order approximation the underlying spacetime is not flat, but rather has a nontrivial but constant in y metric of constant y slices. Correspondingly in the zeroth order we have

$$\Big[\,\overrightarrow{W}\,G_D(y, y')\overleftarrow{W}\,\Big]^{\,0}_{\,y=y'=0} = \overrightarrow{\partial_y}\;G_D^0(y, y')\;\overleftarrow{\partial_y}\,\Big|_{\,y=y'=0} = -\sqrt{M^2-\Box}. \tag{20}$$

The perturbation of the bulk operator can be expanded in Taylor series in y, so that it reads

$$V(y, \partial_y) = \sum_{k=0}^{\infty} y^k\, V_k(\partial_y), \tag{21}$$

where $V_k(\partial_y) = V_k(\partial_y|\nabla)$ is a set of y-independent *local d-dimensional covariant* operators of second order in ∇_x and first order in ∂_y.

On substitution of (20) and (21) into (17) exactly calculable integrals over y result in a nonlocal series in inverse powers of $\sqrt{M^2-\Box}$, and the perturbation expansion takes the form

$$\Big[\,\overrightarrow{W}\,G_D(y, y')\overleftarrow{W}\,\Big]_{\,y=y'=0} = -\sqrt{M^2-\Box} + \sum_{k=0}^{\infty} U_k(\nabla)\frac{1}{(M^2-\Box)^{k/2}}, \tag{22}$$

where $U_k(\nabla)$ is a set of local covariant differential operators acting on the brane[2]. The dimensionality of each $U_k(\nabla)$ is the inverse length to the power $k+1$, which is composed of the dimensionalities of bulk and extrinsic curvatures and covariant derivatives all taken on the brane at $y = 0$.

With $\kappa(\nabla) = -\Box/2m$ the brane-to-brane operator reads

$$2m\boldsymbol{F}^{\,\rm brane}(\nabla) = -\Box + 2m\sqrt{M^2-\Box} - 2m\sum_{k=0}^{\infty} U_k(\nabla)\frac{1}{(M^2-\Box)^{k/2}}. \tag{23}$$

Then we consider the perturbation series for the functional trace of its logarithm in powers of the full U_k-series. After reexpansion in powers of two sets of nonlocal propagators $1/\sqrt{M^2-\Box}$ and $1/(-\Box + 2m\sqrt{M^2-\Box})$ the brane effective action finally takes the form

$$\begin{aligned}\frac{1}{2}\,{\rm Tr}\ln \boldsymbol{F}^{\,\rm brane} &= \frac{1}{2}{\rm Tr}\ln\Big(-\Box + 2m\sqrt{M^2-\Box}\Big)\\ &+ \sum_{k\geq 0,\,l\geq 1} {\rm Tr}\; W_{kl}(\nabla)\frac{1}{(M^2-\Box)^{k/2}}\,\frac{1}{(-\Box + 2m\sqrt{M^2-\Box}\,)^l}\end{aligned} \tag{24}$$

with a new set of local covariant differential operators $W_{kl}(\nabla)$ acting on the brane. The dimensionality of $W_{kl}(\nabla)$ is $k + 2l$ in units of inverse length. One should also remember that each power of $1/(-\Box + 2m\sqrt{M^2-\Box})$ is accompanied by one power of m in the numerator, so that structurally

$$W_{kl}(\nabla) \sim m^l\, \nabla^a\, R^b\, K^c, \tag{25}$$

where the integer overall powers of the covariant derivatives ∇, bulk curvatures R and extrinsic curvatures K are constrained by the relation $a + 2b + c = k + l$.

[2]Strictly speaking each k-th order of this series arises in the form of the following nonlocal chain of square root "propagators", $\frac{1}{(M^2-\Box)^{l_1/2}}\,U_1\,\frac{1}{(M^2-\Box)^{l_2/2}}\,U_2 ... U_{p-1}\,\frac{1}{(M^2-\Box)^{l_p/2}}$, $l_1 + l_2 + ... + l_p = k$, with differential operators U_i as its vertices, but all these propagators can be systematically commuted either to the uppermost right or left by the price of extra commutator terms of the same structure.

3 Generalized proper time method

Our goal now is to find the proper time representation of nonlocal operators in Eq.(24) in the form of the exponentiated $\Box$. A systematic way to do this consists in the following factorization of the brane-to-brane operator as

$$2m\boldsymbol{F}_0^{\rm brane}(\nabla) = -\Box + 2m\sqrt{M^2-\Box} = \big(\sqrt{M^2-\Box} - m_+\big)\big(\sqrt{M^2-\Box} - m_-\big). \tag{26}$$

Here the masses $m_\pm$ are the roots of the relevant quadratic equation, $x^2 + 2mx - M^2 = 0$, $x = \sqrt{M^2-\Box}$,

$$m_\pm = -m \pm \sqrt{M^2+m^2}, \qquad m_- < -M < 0 < m_+ < M, \tag{27}$$

which determine the poles of the propagator of $\boldsymbol{F}_0^{\rm brane}(\nabla)$ in spacetime with the Lorentzian signature[3]

$$\Box_+ = M^2 - m_+^2 > 0, \tag{29}$$
$$\Box_- = |M^2 - m_-^2|\, e^{3i\pi} < 0. \tag{30}$$

The factorization (26) allows one to rewrite the l-th power of the brane-to-brane propagator in (24) as

$$\frac{1}{(-\Box + 2m\sqrt{M^2-\Box})^l} = \frac{1}{(\sqrt{M^2-\Box} - m_+)^l}\,\frac{1}{(\sqrt{M^2-\Box} - m_-)^l} \tag{31}$$

and then decompose the resulting fraction into the sum of simple fractions for which one has explicit proper time representations. These representations begin with the following relations [19]

$$\frac{1}{(M^2-\Box)^{k/2}} = \frac{1}{\Gamma(k/2)}\int\limits_0^\infty ds\; s^{k/2-1}\, e^{\,s(\Box - M^2)}, \tag{32}$$

$$\frac{1}{\sqrt{M^2-\Box} - m} = \int\limits_0^\infty ds\; e^{\,s(\Box-M^2)}\left(\frac{1}{\sqrt{\pi s}} + m\, w(-m\sqrt{s})\right), \quad m < M, \tag{33}$$

$$\frac{1}{\sqrt{M^2-\Box}\,(\sqrt{M^2-\Box} - m)} = \int\limits_0^\infty ds\; e^{\,s(\Box-M^2)}\, w(-m\sqrt{s}), \;\; m < M, \tag{34}$$

which generate (by differentiating with respect to m, $\Box$, M and linear recombining the results) the list of fractions with all possible powers of the factors $\sqrt{M^2-\Box}$ and $\sqrt{M^2-\Box} - m_\pm$ in their denominators. Here the weight function $w(s)$ is given in terms of the error function [19] $\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x dy\,\exp(-y^2)$ and has the following ultraviolet and infrared asymptotics

$$w(x) \equiv e^{x^2}\Big(1 - \mathrm{erf}(x)\Big) \to \begin{cases} 1\,, & x \to 0, \\ \frac{1}{x\sqrt{\pi}}\,, & x \to +\infty, \\ 2\,e^{x^2}\,, & x \to -\infty \end{cases} \tag{35}$$

[3] The pole at $\Box_-$ is formally tachyonic, but it is located on the unphysical sheet of the Riemann surface for the propagator in the complex plane of $\Box$ [20] (which is indicated in (30) by the nontrivial phase). Moreover, its residue is identically vanishing in view of $m_- < 0$. Therefore this pole does not correspond to a real particle. For $M \neq 0$ only $\Box_+$ gives rise to a particle with the decreasing mass as $M \to 0$, $M^2 - m_+^2 \to 0$, which also disappears in the DGP limit because the pole residue also vanishes at $M = 0$. In this limit only the continuum spectrum of massive intermediate states survives forming the spectral representation for the DGP propagator [21]

$$\frac{1}{-\Box + 2m\sqrt{-\Box}} = \frac{4m}{\pi}\int_0^\infty \frac{d\mu}{\mu^2 + 4m^2}\,\frac{1}{\mu^2 - \Box}. \tag{28}$$

The last two proper time integrals above are defined only for $m < M$ (for any negative m and for $m < M$ if m is positive), because in view of these asymptotics they are convergent at infinity only in this range. Interestingly, the forbidden domain corresponds to the real tachyon, because for $m > M > 0$ the pole $\Box = M^2 - m^2$ belongs to the physical sheet of the propagator, and its residue is nonvanishing.

From (33)-(34) it immediately follows that the zeroth order brane-to-brane propagator and its one-loop functional determinant read

$$\frac{1}{-\Box + 2m\sqrt{M^2-\Box}} = \int\limits_0^\infty ds\, e^{s\,(\Box - M^2)}\, \frac{m_+ w(-m_+\sqrt{s}) - m_- w(-m_-\sqrt{s})}{m_+ - m_-}, \tag{36}$$

$$\mathrm{Tr}\,\ln\Big(-\Box + 2m\sqrt{M^2-\Box}\Big) = -\frac{1}{2}\,\mathrm{Tr}\int\limits_0^\infty \frac{ds}{s}\, e^{s\,(\Box - M^2)}\Big(w(-m_+\sqrt{s}) + w(-m_-\sqrt{s})\Big), \tag{37}$$

For the DGP model case with $M^2 = 0$ and $m_+ = 0$, $m_- = -2\,m$ these representations simplify to the equations derived in [10]

$$\frac{1}{-\Box + 2m\sqrt{-\Box}} = \int\limits_0^\infty ds\, e^{s\Box}\, w(\,2m\sqrt{s}), \tag{38}$$

$$\mathrm{Tr}\,\ln\Big(-\Box + 2m\sqrt{-\Box}\Big) = -\mathrm{Tr}\int\limits_0^\infty \frac{ds}{s}\, e^{s\Box}\, \frac{1 + w(\,2m\sqrt{s})}{2}. \tag{39}$$

The interpretation of the weight contribution here is very transparent. It interpolates between the ultraviolet and infrared domains where the brane operator and its logarithm have qualitatively different behaviors. In the domain of a small proper time $m\sqrt{s} \ll 1$ it approximates the brane operator by a large $-\Box \gg m^2$ (hence the overall weight $(1+w)/2 \simeq 1$), whereas in the infrared domain $m\sqrt{s} \gg 1$ it approximates the operator by $2m\sqrt{-\Box}$ (hence the weight is $(1+w)/2 \simeq 1/2$ corresponding to $\ln\sqrt{-\Box} = (1/2)\ln(-\Box)$).

By decomposing the nonlocal fractions of (24) into the sum of simple fractions and using the weighted proper time representations (32)-(34) and their derivatives with respect to mass parameters $m_\pm$ we obtain the following expression

$$\frac{1}{(M^2-\Box)^{k/2}}\,\frac{1}{(-\Box + 2m\sqrt{M^2-\Box}\,)^l} = \int\limits_0^\infty \frac{ds}{s}\, e^{-s(M^2-\Box)}\, w_{kl}(s,m,M), \tag{40}$$

with some weight function $w_{kl}(s,m,M)$.[4]

Using (37) and (40) we finally obtain for the perturbative expansion (24)

$$\frac{1}{2}\,\mathrm{Tr}\ln \boldsymbol{F}^{\,\rm brane} = -\frac{1}{2}\int\limits_0^\infty \frac{ds}{s}\, e^{-sM^2}\, \frac{w(-m_+\sqrt{s}) + w(-m_-\sqrt{s})}{2}\, \mathrm{Tr}\, e^{\,s\Box}$$
$$+ \sum_{k\geq 0,\, l\geq 1}\int\limits_0^\infty \frac{ds}{s}\, e^{-s\,M^2}\, w_{kl}(s,m,M)\ \mathrm{Tr}\Big[W_{kl}(\nabla)\, e^{\,s\Box}\Big]. \tag{41}$$

[4] Alternatively this weight function can of course be obtained as a Mellin transform of the function of $\Box$ in the left hand side, but this simple fraction decomposition method gives a more regular and systematic way to achieve the needed goal.

This formally solves the problem of constructing the Schwinger-DeWitt expansion for the brane effective action, because as it was expected all the remaining calculations reduce to the conventional calculation of the coincidence limits of the Schwinger-DeWitt coefficients and their covariant derivatives in

$$\mathrm{Tr}\Big[W_{kl}(\nabla)\,e^{\,s\,\Box}\Big]=\frac{1}{(4\pi s)^{d/2}}\int d^dx\,g^{1/2}\sum_{n=0}^{\infty}s^n\,\mathrm{tr}\,W_{kl}\Big(\nabla^x_\mu-\frac{\sigma_\mu(x,x')}{2s}\Big)\,\hat{a}_n(x,x')\,\Big|_{x'=x}. \qquad (42)$$

Remember that every $W_{kl}(\nabla)$ is a finite order covariant differential operator with the coefficients built of the powers of the bulk curvature, extrinsic curvature of the brane and their covariant derivatives. Here lengthening of the derivatives in $W_{kl}(\nabla)$ originates from commuting them with the the exponential factor $\exp(-\sigma(x,x')/2s)$ contained in the kernel of $\exp(s\Box)$, $\sigma_\mu(x,x')\equiv\nabla^x_\mu\sigma(x,x')$. This of course brings to life world function coincidence limits $\nabla_{\mu_1}...\nabla_{\mu_p}\sigma(x,x')|_{x'=x}$ also easily calculable by the DeWitt recurrence procedure [12, 13].

It is important that the expansion (41) is efficient for the purpose of obtaining the asymptotic $1/M$-expansion. Indeed, in view of the weight function asymptotics (35) the $w(-m_-\sqrt{s})$-parts of the overall $w_{kl}(s,m,M)$ (cf. Eq.(36)) with $m_-<0$ are suppressed at $s\to\infty$ by e^{-sM^2} and, therefore, generate after the integration over s the needed $1/M^2$-series. In the $w(-m_+\sqrt{s})$-parts the integrand behaves as $e^{-s(M^2-m_+^2)}$, and generates the $1/(M^2-m_+^2)\sim 1/2mM$-series also appropriate for the $M\to\infty$ limit, though converging slower than the $1/M^2$ one. This is the expansion in inverse squared masses of the real particle associated with the pole (29). Unfortunately, the powers of $1/M$ are accompanied by those of $1/m$, which comprises in the DGP model the problem of low strong-coupling scale [3] for small DGP crossover scale m.

4 Conclusions

This is obvious that the Schwinger-DeWitt technique in brane models is much more complicated than in models without spacetime boundaries. It does not reduce to a simple bookkeeping of local surface terms like the one for simple boundary conditions reviewed in [15]. Nevertheless it looks complete and self-contained, because it provides in a systematic way a manifestly covariant calculational procedure for a wide class of boundary conditions including tangential derivatives (in fact of any order). On the other hand, the calculational strategy of the above type is thus far nothing but a set of blueprints for the Schwinger-DeWitt technique in brane models, because there is still a large set of issues and possible generalizations to be resolved in concrete problems.

One important generalization is a physically most interesting limit of the vanishing bulk mass M^2. Local curvature expansion is perfect and nonsingular for nonvanishing M^2 and applicable in the range of curvatures and magnitudes of spacetime derivatives $(R,\,K^2,\nabla K)\ll M^2$, $\nabla\nabla R\ll M^4$, etc. However, for $M^2\to 0$ it obviously breaks down, because the proper time integrals start diverging at the upper limit. These infrared divergences can be avoided by a nonlocal curvature expansion of the heat kernel of [22]. Up to the cubic order in curvatures this expansion explicitly exists for $\mathrm{Tr}\,e^{s\Box}$ [23], but for the structure involving a local differential operator $\mathrm{Tr}\,W(\nabla)e^{s\Box}$ it still has to be developed.

Another important generalization is the extension of these calculations to the cases when already the lowest order approximation involves a curved spacetime background (i.e. dS or AdS bulk geometry, deSitter rather than flat brane, etc.). The success of the above technique is obviously based on the exact knowledge of the y-dependence in the lowest order Green's function in the bulk and the possibility to perform exactly (or asymptotically for large M^2) the integration over y. All these generalizations and open issues are currently under study.

To summarize, we developed a new scheme of calculating quantum effective action for the braneworld DGP-type system in curved spacetime. This scheme gives a systematic curvature expansion by means of a manifestly covariant technique. Combined with the method of fixing the background covariant gauge for diffeomorphism invariance developed in [8, 24] this gives the universal background field method of the Schwinger-DeWitt type in gravitational brane systems.

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