A singularity in dimensional regularization

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Abstract

A dimensionally regularized integral with singular properties is considered.

Dimensional regularization [1] is at present the only regularization which is suitable for practical multi-loop calculations in the Standard $SU(3) \times SU(2) \times U(1)$ Model of strong and electroweak interactions, see e.g. [2]. Hence to study its features is of importance. In the present paper we consider a dimensionally regularized two-loop integral which value depends on the order of calculation's steps.

The integral is

$$I = \int \frac{d^D p d^D k}{(p+k)^{2\alpha}(p+q)^{2\beta}p^{2\gamma}}, \quad \alpha+\beta+\gamma = D, \quad \alpha \neq D/2, \tag{1}$$

where $D = 4-2\epsilon$ is the dimension of the Euclidean momentum space, ϵ being the regularization parameter.

Let us first perform integration over the momentum k. One can use the known property of dimensional regularization to nullify massless vacuum integrals (massless tadpoles)

$$\int \frac{d^D k}{(p+k)^{2\alpha}} = 0, \quad \alpha \neq D/2. \tag{2}$$

Hence one obtains the value $I = 0$.

In the case $\alpha = D/2$ the integral (2) is proportional to $\delta(\alpha - D/2)$ [3].

One can change the order of integrations in (1) and integrate first over p. Here we use the so called uniqueness relation for the triangle diagram [4]

$$\int \frac{d^D p}{(p+k)^{2\alpha}(p+q)^{2\beta}p^{2\gamma}} = \pi^{D/2} \tag{3}$$

$$\frac{\Gamma(D/2-\alpha)\Gamma(D/2-\beta)\Gamma(\alpha+\beta-D/2)}{\Gamma(D-\alpha-\beta)\Gamma(\alpha)\Gamma(\beta)} \frac{1}{(k-q)^{2(\alpha+\beta-D/2)}k^{2(D/2-\beta)}q^{2(D/2-\alpha)}},$$

$$\alpha+\beta+\gamma = D.$$

This relation is obtained by the inversion of momenta $p_\mu = p'_\mu/p'^2$ (and the same for k and q) after which the integrand has only two propagators and integration is easily performed. If $\alpha+\beta+\gamma \neq D$ then the expression for the triangle diagram (3) is much more complicated [5].

Now we can perform integration of the expression (3) over k to obtain the second value for the integral (1)

$$I = \pi^D \frac{\Gamma(D/2-\alpha)\Gamma(\alpha-D/2)}{\Gamma(\alpha)\Gamma(D-\alpha)}. \tag{4}$$

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The dependence of the value of I on the order of integrations over momenta k and p appears on the surface $\alpha+\beta+\gamma=D$. For $\alpha+\beta+\gamma\neq D$ one obtains $I=0$ in both cases. Let us show this. We apply the Feynman reprezentation to obtain

$$I=\frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)}\int_0^1 dx\; x\int_0^1 dy\;(xy)^{\alpha-1}\left[x(1-y)\right]^{\beta-1}(1-x)^{\gamma-1} \tag{5}$$

$$\int\frac{d^Dp d^Dk}{[xy(p+k)^2+x(1-y)(p+q)^2+(1-x)p^2]^{\alpha+\beta+\gamma}}.$$

Performing integration over p one gets

$$I=\pi^{D/2}\frac{\Gamma(\alpha+\beta+\gamma-D/2)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)}\int_0^1 dx\; x\int_0^1 dy\;(xy)^{\alpha-1}\left[x(1-y)\right]^{\beta-1}(1-x)^{\gamma-1} \tag{6}$$

$$\int\frac{d^Dk}{\left[(xyk+x(1-y)q)^2-xyk^2-x(1-y)q^2\right]^{\alpha+\beta+\gamma-D/2}}.$$

Integration over k gives

$$I=\pi^D\frac{\Gamma(\alpha+\beta+\gamma-D)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)}\int_0^1 dx\; x^{D/2-\gamma-1}(1-x)^{D-\alpha-\beta-1} \tag{7}$$

$$\int_0^1 dy\; y^{\alpha-1-D/2}(1-y)^{D-\alpha-\gamma-1}(1-xy)^{\alpha+\beta+\gamma-3D/2}\frac{1}{q^{2(\alpha+\beta+\gamma-D)}}.$$

Performing now integrations over y and x we obtain

$$I=\pi^D\frac{\Gamma(\alpha+\beta+\gamma-D)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)}\frac{\Gamma(\alpha-D/2)\Gamma(D-\alpha-\gamma)}{\Gamma(D/2-\gamma)}\frac{1}{q^{2(\alpha+\beta+\gamma-D)}} \tag{8}$$

$$\int_0^1 dx\; x^{D/2-\gamma-1}(1-x)^{D-\alpha-\beta-1}\;{}_2F_1(3D/2-\alpha-\beta-\gamma,\alpha-D/2;D/2-\gamma;x)=$$

$$\pi^D\frac{\Gamma(\alpha+\beta+\gamma-D)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)}\frac{\Gamma(D-\alpha-\gamma)\Gamma(D-\alpha-\beta)}{\Gamma(3D/2-\alpha-\beta-\gamma)}\frac{1}{q^{2(\alpha+\beta+\gamma-D)}}$$

$$\sum_{n=0}^{\infty}\frac{\Gamma(\alpha-D/2+n)}{n!},$$

here ${}_2F_1$ is the Gauss hypergeometric function.

The sum is known

$$\sum_{n=0}^{\infty}\frac{\Gamma(\alpha-D/2+n)}{n!}=\Gamma(\alpha-D/2)\delta_K(\alpha-D/2), \tag{9}$$

where δ_K is the Kronecker delta-function: $\delta_K(x)=0, x\neq 0; \delta_K(0)=1$.

Thus one obtains I=0 for $\alpha\neq D/2, \alpha+\beta+\gamma\neq D$ independently on the order of integrations.

There are known cases where dimensional regularization does not regularize some integrals. For example, see [6], the following integral

$$\int\frac{d^Dk}{(k^2-m^2)(k^2-pk)qk} \tag{10}$$

is not defined. But it does not lead to a contradiction.

In the case of the integral I for $\alpha+\beta+\gamma=D$ comcidered in the present paper one has another situation. One gets different finite results depending on the order of calculational steps. Hence the rules of dimensional regularization are inconsistent in this case.

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