

# Hypergeometric functions, their $\varepsilon$ expansions and Feynman diagrams

M. Yu. Kalmykov<sup>a,b</sup>, B.A. Kniehl<sup>a</sup>, B.F.L. Ward<sup>c</sup>, S.A. Yost<sup>d</sup> \*

<sup>a</sup> *II. Institut für Theoretische Physik, Universität Hamburg,*

*Luruper Chaussee 149, 22761 Hamburg, Germany*

<sup>b</sup> *Joint Institute for Nuclear Research, 141980 Dubna (Moscow Region), Russia*

<sup>c</sup> *Department of Physics, Baylor University, One Bear Place, Waco, TX 76798, USA*

<sup>d</sup> *Department of Physics, The Citadel, 171 Moultrie St., Charleston, SC 29409, USA*

## Abstract

We review the hypergeometric function approach to Feynman diagrams. Special consideration is given to the construction of the Laurent expansion. As an illustration, we describe a collection of physically important one-loop vertex diagrams for which this approach is useful.

**1. Introduction.** Recent interest in the mathematical structure of Feynman diagrams has been inspired by the persistently increasing accuracy of high-energy experiments and the advent of the LHC epoch. For stable numerical evaluation of diagrams, a knowledge of their analytical properties is necessary. We will review some of the progress in this area, focusing on the hypergeometric function representation of Feynman diagrams.

Forty-five years ago, Regge proposed [1] that any Feynman diagram can be understood in terms of a special class of hypergeometric functions satisfying some system of differential equations so that the singularity surface of the relevant hypergeometric function coincides with the surface of the Landau singularities [2] of the original Feynman diagram.<sup>1</sup> Based on Regge's conjecture, explicit systems of differential equations for particular types of diagrams have been constructed. For some examples, the hypergeometric representation for  $N$ -point one-loop diagrams has been derived in Ref. [4] via a series representation (Appell functions and Lauricella functions appear here), the system of differential equations and its solution in terms of Lappo-Danilevsky functions [5] has been constructed in Ref. [6], and the monodromy structure of some Feynman diagrams has been studied in Ref. [7].

A review of results derived up to the mid-1970's can be found in Ref. [8]. It was known at that time that each Feynman diagram is a function of the "Nilsson class." This means that the Feynman diagram is a multivalued analytical function in complex projective space  $\mathbb{C}\mathbb{P}^n$ . The singularities of this function are described by Landau's equation. Later, Kashiwara and Kawai showed [9] that any regularized Feynman integral satisfies some holonomic system of linear differential equations whose characteristic variety is confined to the extended Landau variety.

The modern technology for evaluating Feynman diagrams is based mainly on techniques which do not explicitly use properties of hypergeometric functions, but are based on relationships among the Feynman diagrams derived from their internal structure.<sup>2</sup> It was shown, for

---

\***e-mail:** kalmykov@theor.jinr.ru, kniehl@mail.desy.de, BFL\_Ward@baylor.edu, scott.yost@citadel.edu

<sup>1</sup>For a review of different approaches to the analysis of the singularities of Feynman diagrams see Ref. [3].

<sup>2</sup>By "internal structure," we mean any representation described in standard textbooks, such as Ref. [10].

example, that there are algebraic relations between dimensionally regularized [11] Feynman diagrams with different powers of propagator [12]. Tarasov showed in 1996 that similar algebraic relations could also be found relating different dimensions of the integral [13]. The Davydychev-Tarasov algorithm [13, 14] allows a Feynman diagram with arbitrary numerator to be transformed into a linear combination of diagrams of the original type with shifted powers of propagators and space-time dimension, multiplied by a linear combination of tensors constructed from the metric tensor and external momenta. This set of algebraic relations is analogous to *contiguous relations* for hypergeometric functions.<sup>3</sup>

Solving the algebraic relations among Feynman diagrams allows them to be expressed in terms of a restricted set called “master integrals.” Such a solution is completely equivalent to the differential reduction of hypergeometric functions [16, 17, 18]. The technique of describing Feynman diagrams by a system of differential equations was further extended in Ref. [19], where it was realized that the solution of the recurrence relations can be used to close the system of differential equations for any Feynman diagram. This led to useful techniques for evaluating diagrams [20, 21]. Most of the progress to date in this type of analysis has been for diagrams related to the “Fuchs” type of differential equation, with three regular singular points [22]<sup>4</sup>.

Since Feynman diagrams are often UV- or IR-divergent, it is important to also consider the construction of the Laurent expansion of dimensionally-regularized diagrams about integral values of the dimension (typically  $d = 4 - 2\varepsilon$ ). This is called an “ $\varepsilon$  expansion” of the diagram. For practical applications, we need the numerical values of the coefficients of this expansion. Purely numerical approaches are under development (*e.g.* Ref. [24]), but this is a complex problem for many realistic diagrams having UV and IR singularities and several mass scales.

The case of one-loop Feynman diagrams has been studied the most. The hypergeometric representations for  $N$ -point one-loop diagrams with arbitrary powers of propagators and an arbitrary space-time dimension have been derived for non-exceptional kinematics<sup>5</sup> by Davydychev in 1991 [25]. His approach is based on the Mellin-Barnes technique [26]. The results are expressible in terms of hypergeometric functions with one less variable than the number of kinematic invariants.

An alternative hypergeometric representation for one-loop diagrams has been derived recently in Ref. [28], using a difference equation in the space-time dimension. This approach has been applied only to a set of master integrals<sup>6</sup>, but, fortunately, an arbitrary  $N$ -point function can be reduced to the set of master integrals analytically [29, 30]. In Ref. [28], the one-loop  $N$ -point function was shown to be expressible in terms of hypergeometric functions of  $N - 1$  variables. One remarkable feature of the derived results is a one-to-one correspondence between arguments of the hypergeometric functions and Gram and Cayley determinants, which are two of the main characteristics of diagrams.

Beyond one loop, a general hypergeometric representation is available only for sunset-type diagrams with arbitrary kinematics [31], with a simpler representation for particular kinematics [32, 33]. In all other cases beyond one loop, master integrals have been expressed in terms of hypergeometric functions of type  ${}_pF_{p-1}$  [34].

The program of constructing the analytical coefficients of the  $\varepsilon$ -expansion is a more complicated matter. The finite parts of one-loop diagrams in  $d = 4$  dimension are expressible in terms

---

<sup>3</sup>The full set of contiguous relations for generalized hypergeometric functions  ${}_pF_q$  is found in Ref. [15].

<sup>4</sup>The analysis of some diagrams with four regular singularities was done recently in Ref. [23].

<sup>5</sup>“Non-exceptional kinematics” refers to the case where all masses and momenta are non-zero and not proportional to each other.

<sup>6</sup>The hypergeometric representations of one-loop master integrals of propagator and vertex type have been constructed in [26, 27].

of the Spence dilogarithm function [35]. However, only partial results for higher-order terms in the  $\varepsilon$ -expansion are known at one loop. The all-order  $\varepsilon$ -expansion of the one-loop propagator with an arbitrary values of masses and external momentum has been constructed [37] in terms of Nielsen polylogarithms [36]. The term linear in  $\varepsilon$  for the one-loop vertex diagram with non-exceptional kinematics has also been constructed in terms of Nielsen polylogarithms [38]. It was shown in Ref. [39] that the all-order  $\varepsilon$  expansion for the one-loop vertex with non-exceptional kinematics is expressible in terms of multiple polylogarithms of two variables [40].

Beyond these examples, the situation is less complete. The term linear in  $\varepsilon$  for the box diagram is still under construction. Some cases for particular masses<sup>7</sup> have been analyzed [41, 42]. Many physically interesting particular cases have been considered beyond one loop. Among these are the  $\varepsilon$  expansion of massless propagator diagrams [43] and the sunset diagram [44].

**2. Hypergeometric Functions.** Let us recall that there are several different ways to describe special functions: (i) as an integral of the Euler or Mellin-Barnes type; (ii) by a series whose coefficients satisfy certain recurrence relations; (iii) as a solution of a system of differential and/or difference equations (holonomic approach). These approaches and interrelations between them have been discussed in series of a papers [45]. In this section, we review some essential definitions relevant for each of these representations.

- **Integral representation:** An Euler integral has the form

$$\Phi(\vec{\alpha}, \vec{\beta}, P) = \int_{\Sigma} \Pi_i P_i(x_1, \dots, x_k)^{\beta_i} x_1^{\alpha_1} \dots x_k^{\alpha_k} dx_1 \dots dx_k, \quad (1)$$

where  $P_i$  is some Laurent polynomial with respect to variables  $x_1, \dots, x_k$ :  $P_i(x_1, \dots, x_k) = \sum c_{\omega_1 \dots \omega_k} x_1^{\omega_1} \dots x_k^{\omega_k}$ , with  $\omega_j \in \mathbb{Z}$ , and  $\alpha_i, \beta_j \in \mathbb{C}$ . We assume that the region  $\Sigma$  is chosen such that the integral exists.

A Mellin-Barnes integral has the form

$$\Phi(a_{js}, b_{kr}, c_i, d_j, \gamma, \vec{x}) = \int_{\gamma+i\mathbb{R}} dz_1 \dots dz_m \frac{\prod_{j=1}^p \Gamma(\sum_{s=1}^m a_{js} z_s + c_j)}{\prod_{k=1}^q \Gamma(\sum_{r=1}^m b_{kr} z_r + d_k)} x_1^{-z_1} \dots x_m^{-z_m}, \quad (2)$$

where  $a_{js}, b_{kr}, c_i, d_j \in \mathbb{R}$ ,  $\alpha_k \in \mathbb{C}$ , and  $\gamma$  is chosen such that the integral exists. This integral can be expressed in terms of a sum of the residues of the integrated expression.

- **Series representation:** We will take the Horn definition of the series representation. In accordance with this definition, a formal (Laurent) power series in  $r$  variables,

$$\Phi(\vec{x}) = \sum C(\vec{m}) \vec{x}^{\vec{m}} \equiv \sum_{m_1, m_2, \dots, m_r} C(m_1, m_2, \dots, m_r) x_1^{m_1} \dots x_r^{m_r}, \quad (3)$$

is called *hypergeometric* if for each  $i = 1, \dots, r$  the ratio  $C(\vec{m} + \vec{e}_i)/C(\vec{m})$  is a rational function<sup>8</sup> in the index of summation:  $\vec{m} = (m_1, \dots, m_r)$ , where  $\vec{e}_j = (0, \dots, 0, 1, 0, \dots, 0)$ , is unit vector with unity in the  $j^{\text{th}}$  place. Ore and Sato [46] found that the coefficients of such a series have the general form

$$C(\vec{m}) = \prod_{i=1}^r \lambda_i^{m_i} R(\vec{m}) \left( \prod_{j=1}^N \Gamma(\mu_j(\vec{m}) + \gamma_j + 1) \right)^{-1}, \quad (4)$$

<sup>7</sup>In Ref. [28], box diagrams have been written in terms of the Lauricella-Saran function  $F_S$  of three variables, and a one-fold integral representation was established for their all-order  $\varepsilon$  expansion. However, it is not proven that this representation coincides with multiple polylogarithms.

<sup>8</sup>A “rational function” is any function which can be written as the ratio of two polynomial functions.

where  $N \geq 0$ ,  $\lambda_j, \gamma_j \in \mathbb{C}$  are arbitrary complex numbers,  $\mu_j : \mathbb{Z}^r \rightarrow \mathbb{Z}$  are arbitrary linear maps, and  $R$  is an arbitrary rational function. The fact that all the  $\Gamma$  factors are in the denominator is inessential: using the relation  $\Gamma(z)\Gamma(1-z) = \pi/\sin(\pi z)$ , they can be converted to factors in the numerator. A series of this type is called a ‘‘Horn-type’’ hypergeometric series. In this case, the system of differential equations has the form

$$Q_j \left( \sum_{k=1}^r x_k \frac{\partial}{\partial x_k} \right) \frac{1}{x_j} \Phi(\vec{x}) = P_j \left( \sum_{k=1}^r x_k \frac{\partial}{\partial x_k} \right) \Phi(\vec{x}), \quad j = 1, \dots, r, \quad (5)$$

where  $P_j$  and  $Q_r$  are polynomials satisfying

$$\frac{C(\vec{m} + e_j)}{C(\vec{m})} = \frac{P_j(\vec{m})}{Q_j(\vec{m})}. \quad (6)$$

- **Holonomic representation:** A combination of differential and difference equations can be found to describe functions of the form

$$\Phi(\vec{z}, \vec{x}, W) = \sum_{k_1, \dots, k_r=0}^{\infty} \left( \prod_{a=1}^m \frac{1}{z_a + \sum_{b=1}^r W_{ab} k_b} \right) \prod_{j=1}^r \frac{x_j^{k_j}}{k_j!}, \quad (7)$$

where  $W$  is an  $r \times m$  matrix. In particular, this function satisfies the equations

$$\frac{\partial \Phi(\vec{z}, \vec{x}, W)}{\partial x_j} = \Phi(\vec{z} + \omega_j, x, W), \quad j = 1, \dots, r, \quad (8)$$

$$\frac{\partial}{\partial z_i} \left( z_i \Phi + \sum_{j=1}^r W_{ij} x_j \frac{\partial \Phi}{\partial x_j} \right) = 0, \quad i = 1, \dots, m, \quad (9)$$

where  $\omega_j$  is the  $j^{\text{th}}$  column of the matrix  $W$ .

**3. Construction of the all-order  $\varepsilon$  expansion of hypergeometric functions.** Recently, several theorems have been proven on the all-order  $\varepsilon$  expansion of hypergeometric functions about integer and/or rational values of parameters [33, 37, 47, 48, 49, 50, 51, 52]. For hypergeometric functions of one variable, all three of the representations (i)–(iii) described in the previous section are equivalent, but some properties of the function may be more evident in one representation than another.

In the Euler integral representation, the most important results are related to the construction of the all-order  $\varepsilon$  expansion of Gauss hypergeometric function with special values of parameters in terms of Nielsen polylogarithms [37]. There are several important master integrals expressible in terms of this type of hypergeometric function, including one-loop propagator-type diagrams with arbitrary values of mass and momentum [26], two-loop bubble diagrams with arbitrary values of masses, and one-loop massless vertex diagrams with three non-zero external momenta [53].

The series representation (ii) is an intensively studied approach. The first results of this type were derived in the context of the so-called ‘‘single-scale’’ diagrams [54] related to multiple harmonic sums. These results have been extended to the case of multiple (inverse) binomial sums [57] that correspond to the  $\varepsilon$ -expansion of hypergeometric functions with one unbalanced half-integer parameter and values of argument equal to  $1/4$ , or diagrams with two massive-particle cuts. Particularly impressive results involving series representations were derived in the framework of the nested-sum approach for hypergeometric functions with a balanced set of parameters in Refs. [47, 48],<sup>9</sup> and in framework of the generating-function approach for hypergeometric functions with one unbalanced set of parameters in Refs. [33, 51, 58, 59].

<sup>9</sup>Computer realizations of nested sums approach to expansion of hypergeometric functions are given in [55, 56].

An approach using the iterated solution of differential equations has been explored in Refs. [33, 49, 50, 52]. One of the advantages of the iterated-solution approach over the series approach is that it provides a more efficient way to calculate each order of the  $\varepsilon$  expansion, since it relates each new term to the previously derived terms, rather than having to work with an increasingly large collection of independent sums at each order. This technique includes two steps: (i) the differential-reduction algorithm (to reduce a generalized hypergeometric function to basic functions); (ii) iterative solution of the proper differential equation for the basic functions (equivalent to iterative algorithms for calculating the analytical coefficients of the  $\varepsilon$  expansion).

An important tool for constructing the iterative solution is the iterated integral defined as  $I(z; a_k, a_{k-1}, \dots, a_1) = \int_0^z \frac{dt}{t-a_k} I(t; a_{k-1}, \dots, a_1)$ , where we assume that all  $a_j \neq 0$ . A special case of this integral,

$$G_{m_k, m_{k-1}, \dots, m_1}(z; a_k, \dots, a_1) \equiv I(z; \underbrace{0, \dots, 0}_{m_k-1 \text{ times}}, a_k, \underbrace{0, \dots, 0}_{m_{k-1}-1 \text{ times}}, a_{k-1}, \dots, \underbrace{0, \dots, 0}_{m_1-1 \text{ times}}, a_1),$$

where all  $a_k \neq 0$ , is related to the multiple polylogarithm [40, 61]

$$\text{Li}_{k_1, k_2, \dots, k_n}(x_1, x_2, \dots, x_n) = \sum_{m_n > m_{n-1} > \dots > m_2 > m_1 > 0} \frac{x_1^{m_1} x_2^{m_2}}{m_1^{k_1} m_2^{k_2}} \times \dots \times \frac{x_n^{m_n}}{m_n^{k_n}} \quad (10)$$

by

$$G_{m_n, m_{n-1}, \dots, m_1}(z; x_n, x_{n-1}, \dots, x_1) = (-1)^n \text{Li}_{m_1, m_2, \dots, m_{n-1}, m_n} \left( \frac{x_2}{x_1}, \frac{x_3}{x_2}, \dots, \frac{x_n}{x_{n-1}}, \frac{z}{x_n} \right),$$

$$\text{Li}_{k_1, k_2, \dots, k_{n-1}, k_n}(y_1, y_2, \dots, y_{n-1}, y_n) = (-1)^n G_{k_n, k_{n-1}, \dots, k_2, k_1} \left( 1; \frac{1}{y_n}, \dots, \frac{1}{y_n \times \dots \times y_1} \right).$$

In Eq. (10),  $k = k_1 + k_2 + \dots + k_n$  is called the “weight” and  $n$  the “depth.” Multiple polylogarithms (10) are defined for  $|z_n| < 1$  and  $|z_i| \leq 1 (i = 1, \dots, n-1)$  and for  $|z_n| \leq 1$  if  $m_n \leq 2$ . We mention also that multiple polylogarithms form two Hopf algebras. One is related to the integral representation, and the other one to the series.

A particular case of the multiple polylogarithm is the “generalized polylogarithm” defined by

$$\text{Li}_{k_1, k_2, \dots, k_n}(z) = \sum_{m_n > m_{n-1} > \dots > m_1 > 0} \frac{z^{m_n}}{m_1^{k_1} m_2^{k_2} \dots m_n^{k_n}} = \text{Li}_{k_1, k_2, \dots, k_n}(1, 1, \dots, 1, z), \quad (11)$$

where  $|z| < 1$  when all  $k_i \geq 1$ , or  $|z| \leq 1$  when  $k_n \leq 2$ . Another particular case is a “multiple polylogarithm of a square root of unity,” defined as

$$\text{Li}_{\left( \begin{smallmatrix} \sigma_1, \sigma_2, \dots, \sigma_n \\ s_1, s_2, \dots, s_n \end{smallmatrix} \right)}(z) = \sum_{m_n > m_{n-1} > \dots > m_1 > 0} z^{m_n} \frac{\sigma_n^{m_n} \dots \sigma_1^{m_1}}{m_n^{s_n} \dots m_1^{s_1}}. \quad (12)$$

where  $\vec{s} = (s_1, \dots, s_n)$  and  $\vec{\sigma} = (\sigma_1, \dots, \sigma_n)$  are multi-indices and  $\sigma_k$  belongs to the set of the square roots of unity,  $\sigma_k = \pm 1$ . This particular case of multiple polylogarithms has been analyzed in detail by Remiddi and Vermaseren [62]<sup>10</sup>.

Special consideration is necessary when the last few arguments  $a_{k-j}, a_{k-j-1}, \dots, a_k$  in the integral  $I(z; a_1, \dots, a_k)$  are equal to zero, which is called the “trailing-zero” case. It is possible to factorize such a function into a product of a power of a logarithm and a multiple polylogarithm. An appropriate procedure for multiple polylogarithms of a square root of unity was

<sup>10</sup>As was pointed out by Goncharov [40], the iterated integral as a function of the variable  $z$  has been studied by Kummer, Poincare, and Lappo-Danilevsky, and was called a *hyperlogarithm*. Goncharov [40] analyzed it as a multivalued analytical function of  $a_1, \dots, a_k, z$ . From this point of view, only the functions considered in Ref. [63] are multiple polylogarithms of two variables.

described in Ref. [62] and extended to the case of multiple polylogarithms in Ref. [64]. For the numerical evaluation of multiple polylogarithms or its particular cases, see Ref. [64, 65]. Let us consider the Laurent expansion of a generalized hypergeometric functions of one variable  ${}_pF_{p-1}(\vec{A}; \vec{B}; z)$  with respect to its parameters. Such an expansion can be written as

$$\begin{aligned} & {}_pF_{p-1}(\vec{A}; \vec{B}; z) = {}_pF_{p-1}(\vec{A}_0; \vec{B}_0; z) \\ & + \sum_{m_i, l_j=1}^{\infty} \prod_{i=1}^p \prod_{j=1}^{p-1} \frac{(A_i - A_{0i})^{m_i}}{m_i!} \frac{(B_j - B_{0j})^{l_j}}{l_j!} \left( \frac{\partial}{\partial A_i} \right)^{m_i} \left( \frac{\partial}{\partial B_j} \right)^{l_j} {}_pF_{p-1}(\vec{A}; \vec{B}; z) \Bigg|_{\substack{A_i=A_{0i} \\ B_j=B_{0j}}} \\ & = {}_pF_{p-1}(\vec{A}_0; \vec{B}_0; z) + \sum_{m_i, l_j=1} \prod_{i=1}^p \prod_{j=1}^{p-1} (A_i - A_{0i})^{m_i} (B_j - B_{0j})^{l_j} L_{\vec{A}, \vec{B}}(z), \end{aligned} \quad (13)$$

where  ${}_pF_{p-1}(\vec{A}; \vec{B}; z)$  is a hypergeometric function defined by  ${}_pF_{p-1}(\vec{A}; \vec{B}; z) = \sum_{j=0}^{\infty} \frac{\prod_{i=1}^p (A_i)_j}{\prod_{k=1}^{p-1} (B_k)_j} \frac{z^j}{j!}$  and  $(A)_j$  is the Pochhammer symbol:  $(A)_j = \Gamma(A + j)/\Gamma(A)$ . Our goal is to completely describe the coefficients  $L_{\vec{A}, \vec{B}}(z)$  entering the r.h.s. of Eq. (13). To reach this goal, we must first characterize the complete set of parameters for which known special functions suffice to express the coefficients. Beyond this, we wish to identify the complete set of new functions which must be invented in order to express all of the coefficients in the Laurent expansion.

The first simplification comes from the well-known fact that any hypergeometric function  ${}_pF_{p-1}(\vec{A} + \vec{m}; \vec{B} + \vec{k}; z)$  may be expressed in terms of  $p$  other functions of the same type:

$$R_{p+1}(\vec{A}, \vec{B}, z) {}_pF_{p-1}(\vec{A} + \vec{m}; \vec{B} + \vec{k}; z) = \sum_{j=1}^p R_j(\vec{A}, \vec{B}, z) {}_pF_{p-1}(\vec{A} + \vec{e}_j; \vec{B} + \vec{E}_j; z), \quad (14)$$

where  $\vec{m}$ ,  $\vec{k}$ ,  $\vec{e}_k$ , and  $\vec{E}_k$  are lists of integers, and the  $R_k$  are polynomials in the parameters  $\vec{A}$ ,  $\vec{B}$ , and  $z$ . In particular, we can take the function and its first  $p-1$  derivatives as a basis for the reduction (see Ref. [16] for the details of this approach). Then Eq. (14) will take the form<sup>11</sup>

$$\tilde{R}_{p+1}(\vec{A}, \vec{B}, z) {}_pF_{p-1}(\vec{A} + \vec{m}; \vec{B} + \vec{k}; z) = \sum_{k=1}^p \tilde{R}_k(\vec{A}, \vec{B}, z) \left( \frac{d}{dz} \right)^{k-1} {}_pF_{p-1}(\vec{A}; \vec{B}; z), \quad (15)$$

with a new polynomial  $\tilde{R}_k$ . In this way, the problem of finding the Laurent expansion of the original hypergeometric function is reduced to the analysis of a set of basic functions and the Laurent expansion of a (formally) known polynomial.

As is well known, hypergeometric functions satisfy the differential equation<sup>12</sup>

$$\left[ z \prod_{i=1}^p \left( z \frac{d}{dz} + A_i \right) - z \frac{d}{dz} \prod_{k=1}^{p-1} \left( z \frac{d}{dz} + B_k - 1 \right) \right] {}_pF_{p-1}(\vec{A}; \vec{B}; z) = 0. \quad (16)$$

Due to the analyticity of the hypergeometric function  ${}_pF_{p-1}(\vec{A}; \vec{B}; z)$  with respect to its parameters  $A_i, B_k$ , the differential equation for the coefficients  $L_{\vec{A}, \vec{B}}(z)$  of the Laurent expansion could be directly derived from Eq. (16) without any reference to the series or integral representation. This was the main idea of the approach developed in Refs. [33, 49, 50, 52, 60]. An analysis of this system of equations and/or their explicit analytical solution gives us the analytical form of  $L_{\vec{A}, \vec{B}}(z)$ . It is convenient to introduce a new parametrization,  $A_i \rightarrow A_{0i} + a_i \varepsilon$ ,  $B_j \rightarrow B_{0j} + b_j \varepsilon$ , where  $\varepsilon$  is some small number, so that the Laurent expansion (13) takes the form of an “ $\varepsilon$  expansion,”

$${}_pF_{p-1}(\vec{A} + \vec{a}\varepsilon; \vec{B} + \vec{b}\varepsilon; z) = {}_pF_{p-1}(\vec{A}; \vec{B}; z) + \sum_{k=1}^{\infty} \varepsilon^k L_{\vec{a}, \vec{b}, k}(z) \equiv \sum_{k=0}^{\infty} \varepsilon^k L_{\vec{a}, \vec{b}, k}(z),$$

<sup>11</sup>For simplicity, we will assume that no difference  $B_k - A_j$  is a positive integer.

<sup>12</sup>This equation follows from Eqs. (5) – (6), where  $P(j) = \prod_{k=1}^p (A_k + j)$  and  $Q(j) = (j+1) \prod_{k=1}^{p-1} (B_k + j)$ .

where  $L_{\vec{a}, \vec{b}, 0}(z) = {}_pF_{p-1}(\vec{A}; \vec{B}; z)$ . The differential operator can also be expanded in powers of  $\varepsilon$ :

$$D^{(p)} = \left[ \prod_{i=1}^p (\theta + A_i + a_i \varepsilon) - \frac{1}{z} \theta \prod_{k=1}^{p-1} (\theta + B_k - 1 + b_k \varepsilon) \right] = \sum_{j=0}^p \varepsilon^j D_j^{(p-j)}(\vec{A}, \vec{B}, \vec{a}, \vec{b}, z), \quad (17)$$

where  $\theta = z \frac{d}{dz}$ , the upper index gives the order of the differential operator,  $D_p^{(0)} = \prod_{k=1}^p a_k$ , and

$$\begin{aligned} D_0^{(p)} &= \prod_{i=1}^p (\theta + A_i) - \frac{1}{z} \theta \prod_{k=1}^{p-1} (\theta + B_k - 1) \\ &= \left\{ -(1-z) \frac{d}{dz} + \sum_{k=1}^p A_k - \frac{1}{z} \sum_{j=1}^{p-1} (B_j - 1) \right\} \theta^{p-1} + \sum_{j=1}^{p-1} \left[ X_j(\vec{A}, \vec{B}) - \frac{1}{z} Y_j(\vec{A}, \vec{B}) \right] \theta^{p-1-j}, \end{aligned} \quad (18)$$

where  $X_j(\vec{A}, \vec{B})$  and  $Y_j(\vec{A}, \vec{B})$  are polynomials. Combining all of the expansions together, we obtain a system of equations  $\sum_{r=0}^{\infty} \varepsilon^r \sum_{j=0}^p D_j^{(p-j)} L_{\vec{a}, \vec{b}, r-j}(z) = 0$ , which could be split into following system (each order of  $\varepsilon$ ):  $(\varepsilon^0) D_0^{(p)} L_{\vec{a}, \vec{b}, 0}(z) = 0$ ;  $(\varepsilon^k, 1 \leq k \leq p) \sum_{r=0}^k D_k^{(p-k)} L_{\vec{a}, \vec{b}, k-r}(z) = 0$ ;  $(\varepsilon^k, k \geq p+1) \sum_{j=0}^p D_j^{(p-j)} L_{\vec{a}, \vec{b}, k-j}(z) = 0$ . Further simplification comes from the explicit forms of  $D_k^{(p-k)}$  and the polynomials  $X_j(\vec{A}, \vec{B}), Y_j(\vec{A}, \vec{B})$  in Eq. (19). For example, for integer values of parameters, we can put  $A_k = 0, B_j = 1$ , so that all of the  $X_j(\vec{A}, \vec{B})$  and  $Y_j(\vec{A}, \vec{B})$  are equal to zero. Further details can be found in our papers, Refs. [33, 50, 51, 52, 60].

Here, we will mention some of the existing results. <sup>13</sup>

- If  $I_1, I_2, I_3$  are arbitrary integers, the Laurent expansions of the Gauss hypergeometric functions

$$\begin{aligned} &{}_2F_1(I_1 + a\varepsilon, I_2 + b\varepsilon; I_3 + \frac{p}{q} + c\varepsilon; z), \quad {}_2F_1(I_1 + \frac{p}{q} + a\varepsilon, I_2 + \frac{p}{q} + b\varepsilon; I_3 + \frac{p}{q} + c\varepsilon; z), \\ &{}_2F_1(I_1 + \frac{p}{q} + a\varepsilon, I_2 + b\varepsilon; I_3 + c\varepsilon; z), \quad {}_2F_1(I_1 + \frac{p}{q} + a\varepsilon, I_2 + b\varepsilon; I_3 + \frac{p}{q} + c\varepsilon; z) \end{aligned}$$

are expressible in terms of multiple polylogarithms of arguments being powers of  $q$ -roots of unity and a new variable, that is an algebraic function of  $z$ , with coefficients that are ratios of polynomials.

- If  $\vec{A}, \vec{B}$  are lists of integers and  $I, p, q$  are integers, the Laurent expansions of the generalized hypergeometric functions

$${}_pF_{p-1}(\vec{A} + \vec{a}\varepsilon, \frac{p}{q} + I; \vec{B} + \vec{b}\varepsilon; z), \quad {}_pF_{p-1}(\vec{A} + \vec{a}\varepsilon; \vec{B} + \vec{b}\varepsilon, \frac{p}{q} + I; z)$$

are expressible in terms of multiple polylogarithms of arguments that are powers of  $q$ -roots of unity and a new variable that is an algebraic function of  $z$ , with coefficients that are ratios of polynomials.

- If  $\vec{A}, \vec{B}$  are lists of integers, the Laurent expansion of the generalized hypergeometric function

$${}_pF_{p-1}(\vec{A} + \vec{a}\varepsilon; \vec{B} + \vec{b}\varepsilon; z)$$

are expressible via generalized polylogarithms (11).

We should also mention the following case [48] in which the  $\varepsilon$  expansion has been constructed via the nested sum approach:

If  $p, q, I_k$  are any integers and  $\vec{A}, \vec{B}$  are lists of integers, the generalized hypergeometric function

$${}_pF_{p-1}(\{\frac{p}{q} + \vec{A} + \vec{a}\varepsilon\}_r, \vec{I}_1 + \vec{c}\varepsilon; \{\frac{p}{q} + \vec{B} + \vec{b}\varepsilon\}_r, \vec{I}_2 + \vec{d}\varepsilon; z)$$

---

<sup>13</sup>In the following, we will assume that  $a, b, c$  are an arbitrary numbers and  $\varepsilon$  is a small parameter.

is expressible in terms of multiple polylogarithms of arguments that are powers of  $q$ -roots of unity and the new variable  $z^{1/q}$ , with coefficients that are ratios of polynomials. A hypergeometric function of this form is said to have a *zero-balance* set of parameters.

We will now demonstrate some algebraic relations between functions generated by the  $\varepsilon$  expansion of hypergeometric functions with special sets of parameters. Let us consider the analytic continuation of the generalized hypergeometric function  ${}_{p+1}F_p$  with respect to the transformation  $z \rightarrow 1/z$  [34]:

$$\left( \prod_{j=1}^p \frac{1}{\Gamma(b_j)} \right) {}_{p+1}F_p \left( \begin{matrix} a_1, a_2, \dots, a_{p+1} \\ b_1, b_2, \dots, b_p \end{matrix} \middle| z \right) = \sum_{k=1}^{p+1} \frac{\prod_{j=1, j \neq k}^{p+1} \Gamma(a_j - a_k)}{\left( \prod_{j=1, j \neq k}^{p+1} \Gamma(a_j) \right) \left( \prod_{j=1}^p \Gamma(b_j - a_k) \right)} \times (-z)^{-a_k} {}_{p+2}F_{p+1} \left( \begin{matrix} 1, a_k, 1+a_k-b_1, 1+a_k-b_2, \dots, 1+a_k-b_p \\ 1+a_k-a_1, 1+a_k-a_2, \dots, 1+a_k-a_{p+1} \end{matrix} \middle| \frac{1}{z} \right), \quad (19)$$

where none of the differences between pairs of parameters  $a_j - a_k$  is an integer.

On the r.h.s. of Eq. (19), we actually have a hypergeometric function  ${}_{p+1}F_p$ , since one of the parameters is always equal to unity. If we make the replacements

$$a_j \rightarrow \frac{r}{q} + a_j \varepsilon, \quad b_j \rightarrow \frac{r}{q} + b_j \varepsilon$$

in Eq. (19), we obtain the relation

$${}_{p+1}F_p \left( \begin{matrix} \left\{ \frac{r}{q} + a_j \varepsilon \right\}_{p+1} \\ \left\{ \frac{r}{q} + b_j \varepsilon \right\}_p \end{matrix} \middle| z \right) = \sum_{s=1}^p c_s {}_{p+1}F_p \left( \begin{matrix} \frac{r}{q} + \tilde{c} \varepsilon, \{1 + \tilde{a}_j \varepsilon\}_p \\ \{1 + \tilde{b}_j \varepsilon\}_p \end{matrix} \middle| \frac{1}{z} \right), \quad (20)$$

where the  $c_r$  are constants. Another relation follows if we choose in Eq. (19) the following set of parameters:

$$a_j \rightarrow a_j \varepsilon, \quad j = 1, \dots, p+1, \quad b_k \rightarrow b_k \varepsilon, \quad k = 1, \dots, p-1, \quad b_p = \frac{r}{q} + b_p \varepsilon.$$

Then we have

$${}_{p+1}F_p \left( \begin{matrix} \{a_j \varepsilon\}_{p+1} \\ \{b_j \varepsilon\}_{p-1}, \frac{r}{q} + b_p \varepsilon \end{matrix} \middle| z \right) = \sum_{s=1}^p \tilde{c}_s {}_{p+1}F_p \left( \begin{matrix} 1 - \frac{r}{q} - \tilde{c} \varepsilon, \{1 + \tilde{a}_j \varepsilon\}_p \\ \{1 + \tilde{b}_j \varepsilon\}_p \end{matrix} \middle| \frac{1}{z} \right), \quad (21)$$

where the  $\tilde{c}$  are constants. In this way, we find a proof of the following result:

**Lemma:** *When none of the difference between two upper parameters is an integer, and the differences between any lower and upper parameters are positive integers, the coefficients of the  $\varepsilon$  expansion of the hypergeometric functions*

$${}_{p+1}F_p \left( \begin{matrix} \vec{A} + \frac{r}{q} + \vec{a} \varepsilon \\ \vec{B} + \frac{r}{q} + \vec{b} \varepsilon \end{matrix} \middle| z \right), \quad {}_{p+1}F_p \left( \begin{matrix} \vec{A} + \vec{a} \varepsilon \\ \vec{B} + \vec{b} \varepsilon, I + \frac{r}{q} + c \varepsilon \end{matrix} \middle| z \right), \quad {}_{p+1}F_p \left( \begin{matrix} I + \frac{r}{q} + c \varepsilon, \vec{A} + \vec{a} \varepsilon \\ \vec{B} + \vec{b} \varepsilon \end{matrix} \middle| z \right),$$

where  $\vec{A}, \vec{B}, \vec{a}, \vec{b}, c$  and  $I$  are all integers, are related to each other.

Note that none of the functions of this lemma belongs to the zero-balance case.

**4. One-loop vertex as hypergeometric function.** Let us consider now the one-loop vertex diagram. We recall that any one-loop vertex diagram with the arbitrary masses, external momenta and power of propagators can be reduced by recurrence relations to a vertex-type master integral (with all powers of propagators being equal to unity) or, in the case of zero Gram and/or Cayley determinants, to a linear combination of propagator-type diagrams [29]. In the case of non-zero Gram and/or Cayley determinants, the one-loop master integrals are expressible in terms of linear combinations of two Gauss hypergeometric functions and the Appell function  $F_1$  [27, 28].

Our starting point is the hypergeometric representation for one-loop diagrams with three



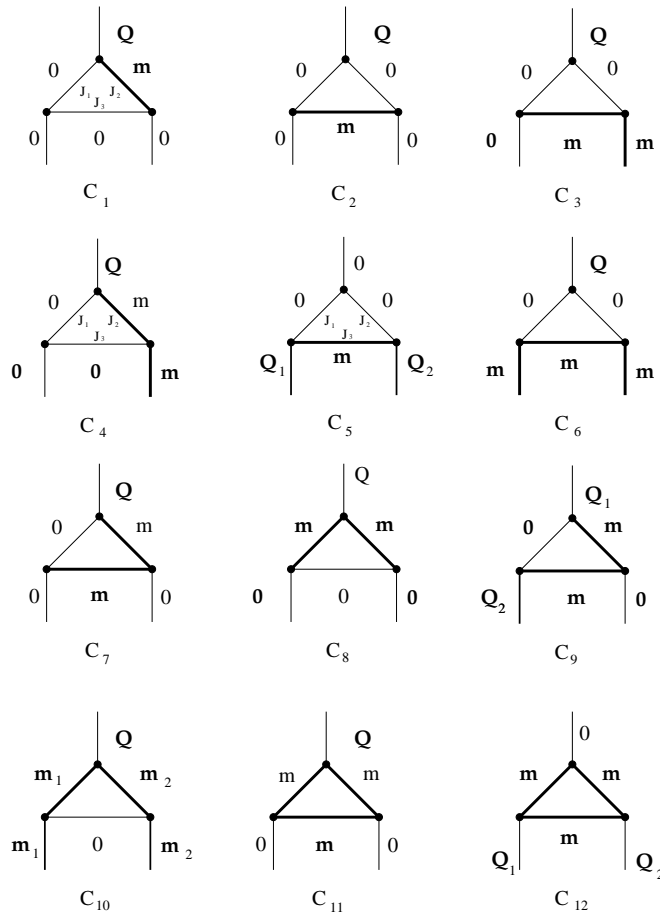


Figure 1: One-loop vertex-type diagrams expressible in terms of generalized hypergeometric functions. Bold and thin lines correspond to massive and massless propagators, respectively.

arbitrary external momenta and one massive line or two or three massive lines with an equal masses, derived in Ref. [26]. Let us consider a one-loop vertex-type diagram, as shown in Fig. 1. Using properties of functions of several variables [34, 67], these diagrams can be expressed in terms of hypergeometric functions of one variable<sup>14</sup>, whose  $\varepsilon$  expansions up to weight 4 are presented in Ref. [56, 59, 66]<sup>15</sup> and available via the web [70]. We recall that up to weight 4, the  $\varepsilon$  expansions of all master integrals collected here are expressible in terms of Nielsen polylogarithms only. The hypergeometric representations have been derived also in [68] for  $C_1$  and  $C_2$ , in [28, 67] for  $C_6$  and in [26] for  $C_{11}$ . Up to the finite part, some of these diagrams have been studied in [69]. For certain diagrams ( $C_4, C_6, C_9, C_{10}, C_{11}$ ), the  $\varepsilon$  expansion of the first several coefficients was given in Ref. [42] in terms of multiple polylogarithms of two variables. We use the notations  $j_{123} = j_1 + j_2 + j_3$ ,  $j_{mn} = j_m + j_n$  below.

We will conclude with a review of the results for special cases:

- The massless triangle diagram with one massless external on-shell momentum is expressible in terms of two Gauss hypergeometric functions. This result follows directly from a relation in Ref. [26]. The Cayley determinant vanishes in this case.

<sup>14</sup>We are indebted to A. Davydychev for assistance in that analysis.

<sup>15</sup>This is enough for the calculation of two-loop corrections.

- The analytical result for diagram  $C_1$  with arbitrary powers of the propagators is expressible in terms of a Gauss hypergeometric function with one integer upper parameter:

$$\frac{C_1}{i^{1-n}\pi^{n/2}} = (-m^2)^{n/2-j_{123}} \frac{\Gamma(j_{123}-\frac{n}{2}) \Gamma(\frac{n}{2}-j_{13})}{\Gamma(\frac{n}{2}) \Gamma(j_2)} {}_2F_1\left(j_{123}-\frac{n}{2}, j_1 \mid \frac{Q^2}{m^2}\right).$$

The differential reduction will result in one Gauss hypergeometric function. The Cayley determinant vanishes for  $C_1$ .

- The diagram  $C_2$  with arbitrary powers of propagators is expressible in terms of two hypergeometric functions  ${}_3F_2$ . In this case, both the Gram and Cayley determinants are nonzero, and the master integral is

$$\begin{aligned} \frac{C_2}{i\pi^{n/2}} = & -(m^2)^{\frac{n}{2}-3} \left\{ \frac{\Gamma(3-\frac{n}{2}) \Gamma(\frac{n}{2}-2)}{\Gamma(\frac{n}{2})} {}_2F_1\left(\frac{1, 1}{\frac{n}{2}} \mid -\frac{Q^2}{m^2}\right) \right. \\ & \left. + \left(-\frac{Q^2}{m^2}\right)^{\frac{n}{2}-2} \frac{\Gamma^2(\frac{n}{2}-1) \Gamma(2-\frac{n}{2})}{\Gamma(n-2)} {}_2F_1\left(\frac{1, \frac{n}{2}-1}{n-2} \mid -\frac{Q^2}{m^2}\right) \right\}. \end{aligned}$$

- For diagram  $C_3$ , the result for arbitrary powers of propagators is expressible in terms of the function  ${}_3F_2$ . Both the Gram and Cayley determinants are nonzero, and the master integral is a combination of two Gauss hypergeometric functions:

$$\begin{aligned} \frac{C_3}{i\pi^{n/2}} = & -(m^2)^{\frac{n}{2}-3} \frac{\Gamma(\frac{n}{2}-2)}{\Gamma(n-3)} \left\{ \frac{\Gamma(n-4)}{\Gamma(\frac{n}{2}-1)} {}_2F_1\left(\frac{1, 1}{5-n} \mid \frac{Q^2}{m^2}\right) \right. \\ & \left. + \left(-\frac{Q^2}{m^2}\right)^{\frac{n}{2}-2} \frac{\Gamma(\frac{n}{2}-1) \Gamma(2-\frac{n}{2})}{\Gamma(3-\frac{n}{2})} {}_2F_1\left(\frac{1, \frac{n}{2}-1}{3-\frac{n}{2}} \mid \frac{Q^2}{m^2}\right) \right\}. \end{aligned}$$

- The diagram  $C_4$  with arbitrary powers of propagators is expressible in terms of a Gauss hypergeometric function with one integer parameter:

$$\frac{C_4}{i^{1-n}\pi^{n/2}} = \frac{\Gamma(j_{123}-\frac{n}{2}) \Gamma(\frac{n}{2}-j_{13}) \Gamma(n-j_{12}-2j_3)}{(-m^2)^{j_{123}-\frac{n}{2}} \Gamma(n-j_{123}) \Gamma(\frac{n}{2}-j_3) \Gamma(j_2)} {}_2F_1\left(j_{123}-\frac{n}{2}, j_1 \mid \frac{Q^2}{m^2}\right).$$

- For arbitrary powers of propagators, the diagram  $C_5$  is expressible in terms of the Appell function  $F_1$ :

$$\frac{C_5}{i^{1-n}\pi^{n/2}} = (-m^2)^{\frac{n}{2}-j_{123}} \frac{\Gamma(j_{123}-\frac{n}{2}) \Gamma(\frac{n}{2}-j_{12})}{\Gamma(j_3) \Gamma(\frac{n}{2})} F_1\left(j_{123}-\frac{n}{2}, j_1, j_2; \frac{n}{2} \mid \frac{Q_1^2}{m^2}, \frac{Q_2^2}{m^2}\right).$$

When the squared external momenta are equal,  $Q_1^2 = Q_2^2 = Q^2$ , it reduces to the Gauss hypergeometric function:

$$\frac{C_5}{i^{1-n}\pi^{n/2}} \Big|_{Q_1^2=Q_2^2=Q^2} = (-m^2)^{\frac{n}{2}-j_{123}} \frac{\Gamma(j_{123}-\frac{n}{2}) \Gamma(\frac{n}{2}-j_{12})}{\Gamma(j_3) \Gamma(\frac{n}{2})} {}_2F_1\left(j_{123}-\frac{n}{2}, j_{12} \mid \frac{Q^2}{m^2}\right).$$

For  $Q_1^2 = Q_2^2$ , the Gram determinant is zero, and when  $Q_1^2 = Q_2^2 = m^2$ , the Cayley determinant is also zero.

- For  $C_6$ , both the Gram and Cayley determinants are nonzero, and

$$\begin{aligned} \frac{C_6}{i\pi^{n/2}} = & -(m^2)^{\frac{n}{2}-3} \left\{ \frac{\Gamma(3-\frac{n}{2}) \Gamma(n-5)}{\Gamma(n-3)} {}_2F_1\left(\frac{1, 1}{\frac{7-n}{2}} \mid \frac{Q^2}{4m^2}\right) \right. \\ & \left. + \left(-\frac{Q^2}{m^2}\right)^{\frac{n}{2}-2} \frac{\Gamma^2(\frac{n}{2}-1) \Gamma(2-\frac{n}{2})}{\Gamma(n-2)} \left(\frac{3-n}{2}\right) {}_2F_1\left(\frac{1, \frac{n}{2}-1}{\frac{3}{2}} \mid \frac{Q^2}{4m^2}\right) \right\}. \end{aligned}$$

- The diagram  $C_7$  with arbitrary powers of propagators is expressible in terms of the function  ${}_3F_2$ . For this diagram, both the Gram and Cayley determinants are nonzero, and the master integral is

$$\frac{C_7}{i\pi^{n/2}} = -(m^2)^{\frac{n}{2}-3} \frac{\Gamma(\frac{n}{2}-1) \Gamma(3-\frac{n}{2})}{\Gamma(\frac{n}{2})} {}_3F_2 \left( \begin{matrix} 1, 1, 3-\frac{n}{2} \\ \frac{n}{2}, 2 \end{matrix} \middle| \frac{Q^2}{m^2} \right).$$

- The diagram  $C_8$  with arbitrary powers of propagators is expressible in terms of the function  ${}_4F_3$ . For this diagram, both the Gram and Cayley determinants are nonzero. The master integral is

$$\frac{C_8}{i\pi^{n/2}} = -(m^2)^{\frac{n}{2}-3} \frac{\Gamma(\frac{n}{2}-1) \Gamma(3-\frac{n}{2})}{\Gamma(\frac{n}{2})} {}_3F_2 \left( \begin{matrix} 1, 3-\frac{n}{2}, \frac{n}{2}-1 \\ \frac{n}{2}, \frac{3}{2} \end{matrix} \middle| \frac{Q^2}{4m^2} \right).$$

- For  $C_9$ , both the Gram and Cayley determinants are nonzero.

$$\begin{aligned} \frac{C_9}{i\pi^{n/2}} &= -(m^2)^{\frac{n}{2}-3} \frac{\Gamma(3-\frac{n}{2}) \Gamma(\frac{n}{2}-1)}{\Gamma(\frac{n}{2})} \frac{1}{Q_1^2 - Q_2^2} \\ &\times \left\{ {}_3F_2 \left( \begin{matrix} 3-\frac{n}{2}, 1, 1 \\ \frac{n}{2}, 2 \end{matrix} \middle| \frac{Q_1^2}{m^2} \right) Q_1^2 - {}_3F_2 \left( \begin{matrix} 3-\frac{n}{2}, 1, 1 \\ \frac{n}{2}, 2 \end{matrix} \middle| \frac{Q_2^2}{m^2} \right) Q_2^2 \right\}. \end{aligned}$$

When  $Q_1^2 = Q_2^2$ , the Gram determinant is equal to zero.

- For diagram  $C_{10}$ , the Cayley determinant vanishes, so that the diagram can be reduced to a linear combination of one-loop propagator diagrams (see Ref. [37]). The hypergeometric function representation is

$$\begin{aligned} \frac{C_{10}}{i\pi^{n/2}} &= -\frac{\Gamma(3-\frac{n}{2})}{2Q^2(n-4)} \\ &\times \left\{ (Q^2 + m_1^2 - m_2^2)(m_1^2)^{\frac{n}{2}-3} {}_2F_1 \left( \begin{matrix} 1, 3-\frac{n}{2} \\ \frac{3}{2} \end{matrix} \middle| \frac{(Q^2 + m_1^2 - m_2^2)^2}{4m_1^2 Q^2} \right) \right. \\ &\quad \left. + (Q^2 - m_1^2 + m_2^2)(m_2^2)^{\frac{n}{2}-3} {}_2F_1 \left( \begin{matrix} 1, 3-\frac{n}{2} \\ \frac{3}{2} \end{matrix} \middle| \frac{(Q^2 - m_1^2 + m_2^2)^2}{4m_2^2 Q^2} \right) \right\}. \end{aligned}$$

- For this diagram, both the Gram and Cayley determinants are nonzero. The master integral is

$$\frac{C_{11}}{i\pi^{n/2}} = -\frac{1}{2} (m^2)^{\frac{n}{2}-3} \Gamma\left(3-\frac{n}{2}\right) {}_3F_2 \left( \begin{matrix} 3-\frac{n}{2}, 1, 1 \\ \frac{3}{2}, 2 \end{matrix} \middle| \frac{Q^2}{4m^2} \right).$$

The all-order  $\varepsilon$  expansions of  $C_{11}$  is expressible in terms of multiple polylogarithm of a square root of unity.

- The master integral for diagram  $C_{12}$  was evaluated in Ref. [67] in terms of a linear combination of two  ${}_3F_2$  functions of the same type, as for the diagram  $C_8$ :

$$\begin{aligned} \frac{C_{12}}{i\pi^{\frac{n}{2}}} &= -(m^2)^{\frac{n}{2}-3} \Gamma\left(3-\frac{n}{2}\right) \frac{1}{2(Q_1^2 - Q_2^2)} \\ &\times \left\{ {}_3F_2 \left( \begin{matrix} 3-\frac{n}{2}, 1, 1 \\ \frac{3}{2}, 2 \end{matrix} \middle| \frac{Q_1^2}{4m^2} \right) Q_1^2 - {}_3F_2 \left( \begin{matrix} 3-\frac{n}{2}, 1, 1 \\ \frac{3}{2}, 2 \end{matrix} \middle| \frac{Q_2^2}{m^2} \right) Q_2^2 \right\}. \end{aligned}$$

**Acknowledgments.** M.Yu.K. is grateful to the Organizers of ‘‘Quark-2008’’ for their hospitality and to all participants, but especially to K. Chetyrkin, A. Isaev, A. Kataev, S. Larin and

A. Pivovarov, for useful discussion. We are indebted to A. Davydychev and O. Tarasov for a careful reading of manuscript. M.Yu.K. is thankful to A. Kotikov, T. Huber and D. Maître for useful comments. M.Yu.K.'s research was supported in part by BMBF Grant No. 05 HT6GUA and DFG Grant No. KN 365/3-1. B.F.L.W.'s research was partly supported by US DOE grant DE-FG02-05ER41399 and by NATO grant PST.CLG.980342.

## References

- [1] T. Regge, *Algebraic Topology Methods in the Theory of Feynman Relativistic Amplitudes*, Battelle Rencontres, 1967. *Lectures in Mathematics and Physics*, ed. C. M. DeWitt, J. A. Wheeler. New York: W. A. Benjamin 1968.
- [2] L.D. Landau, Nucl. Phys. **13** (1959) 181;  
N. Nakanishi, Prog. Theor. Phys. **22** (1959) 128; *ibid* **23** (1960) 284.
- [3] R.J. Eden, P.V. Landshoff, D.I. Olive, J.C. Polkinghorne, *The Analytic S-Matrix*, Cambridge, Cambridge University Press 1966;  
R. Hwa, V. Teplitz, *Homology and Feynman Integrals*, W.A.Benjamin, New York, 1966;  
J.D. Bjorken, Doctoral dissertation, Stanford University, 1959.
- [4] D.S. Kershaw, Phys. Rev. D **8** (1973) 2708; A.C.T. Wu, Phys. Rev. D **9** (1974) 370;  
K. Mano, Phys. Rev. D **11** (1975) 452.
- [5] J.A. Lappo-Danilevsky, *Theory of Functions on Matrices and Systems of Linear Differential Equations* (Leningrad, 1934).
- [6] G. Barucchi, G. Ponzano, J. Math. Phys. **14** (1973) 396.
- [7] G. Ponzano, T. Regge, E.R. Speer, M.J. Westwater, Commun. Math. Phys. **15** (1969) 83;  
*ibid* **18** (1970) 1; T. Regge, E.R. Speer, M.J. Westwater, Fortsch. Phys. **20** (1972) 365.
- [8] V.A. Golubeva, Russ. Math. Surv. **31** (1976) 139.
- [9] M. Kashiwara, T. Kawai, Publ. Res. Inst. Math. Sci. Kyoto **12** (1977) 131; Commun. Math. Phys. **54** (1977) 121;  
T. Kawai, H.P. Stapp, Commun. Math. Phys. **83** (1982) 213.
- [10] N.N. Bogoliubov, D.V. Shirkov, *Introduction to the Theory of Quantized Fields*, (Wiley & Sons, New York, 1980);  
C. Itzykson, J.B. Zuber, *Quantum Field Theory* (McGraw-Hill, New York, 1980).
- [11] G. 'tHooft, M. Veltman, Nucl. Phys. B **44** (1972) 189.
- [12] F.V. Tkachov, Phys. Lett. B **100** (1981) 65;  
K.G. Chetyrkin, F.V. Tkachov, Nucl. Phys. B **192** (1981) 159.
- [13] O.V. Tarasov, Phys. Rev. D **54** (1996) 6479.
- [14] A.I. Davydychev, Phys. Lett. B **263** (1991) 107.
- [15] E.D. Rainville, *Special Functions* (MacMillan Co., New York, 1960).

- [16] M. Saito, B. Sturmfels, N. Takayama, *Gröbner Deformations of Hypergeometric Differential Equations*, (Springer-Verlag, Berlin, 2000).
- [17] M.Yu. Kalmykov, JHEP **0604** (2006) 056.
- [18] O.V. Tarasov, Acta Phys. Polon. B **29** (1998) 2655.
- [19] A.V. Kotikov, Phys. Lett. B **254** (1991) 158; *ibid* **259** (1991) 314; *ibid* **267** (1991) 123; E. Remiddi, Nuovo Cim. A **110** (1997) 1435.
- [20] G. 't Hooft, M.J.G. Veltman, Nucl. Phys. B **44** (1972) 189; G. Rufa, Annalen Phys. **47** (1990) 6.
- [21] M. Argeri, P. Mastrolia, Int. J. Mod. Phys. A **22** (2007) 4375.
- [22] V.V Golubev, *Lectures on the Analytic Theory of Differential Equations*, 2<sup>nd</sup> ed. (Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1950).
- [23] O.V. Tarasov, Phys. Lett. B **638** (2006) 195; U. Aglietti, R. Bonciani, L. Grassi, E. Remiddi, Nucl. Phys. B **789** (2008) 45.
- [24] F.V. Tkachov, Nucl. Instrum. Meth. A **389** (1997) 309; G. Passarino, Nucl. Phys. B **619** (2001) 257; G. Passarino, S. Uccirati, Nucl. Phys. B **629** (2002) 97; F. Jegerlehner, M.Yu. Kalmykov, O. Veretin, Nucl. Phys. B **641** (2002) 285; A. Ferroglia, M. Passera, G. Passarino, S. Uccirati, Nucl. Phys. B **650** (2003) 162; C. Anastasiou, A. Daleo, JHEP **0610** (2006) 031; C. Anastasiou, S. Beerli, A. Daleo, JHEP **0705** (2007) 071; S. Actis, G. Passarino, C. Sturm, S. Uccirati, arXiv:0809.3667; G. Heinrich, Int. J. Mod. Phys. A **23** (2008) 1457.
- [25] A.I. Davydychev, J. Math. Phys. **32** (1991) 1052; *ibid* **33** (1992) 358.
- [26] E.E. Boos, A.I. Davydychev, Theor. Math. Phys. **89** (1991) 1052.
- [27] O.V. Tarasov, Nucl. Phys. Proc. Suppl. **89** (2000) 237.
- [28] J. Fleischer, F. Jegerlehner, O.V. Tarasov, Nucl. Phys. B **672** (2003) 303.
- [29] J. Fleischer, F. Jegerlehner, O.V. Tarasov, Nucl. Phys. B **566** (2000) 423; T. Binoth, J. P. Guillet, G. Heinrich, Nucl. Phys. B **572** (2000) 361; T. Binoth, J.P. Guillet, G. Heinrich, E. Pilon, C. Schubert, JHEP **0510** (2005) 015.
- [30] G. Passarino, M.J.G. Veltman, Nucl. Phys. B **160** (1979) 151; A.V. Kotikov, Mod. Phys. Lett. A **6** (1991) 3133.
- [31] F.A. Berends, M. Buza, M. Böhm, R. Scharf, Z. Phys. C **63** (1994) 227.
- [32] A.I. Davydychev, “*Loop calculations in QCD with massive quarks*”, talk at Int. Conf. “Relativistic Nuclear Dynamics” (Vladivostok, Russia, September 1991), D.J. Broadhurst, J. Fleischer, O.V. Tarasov, Z. Phys. C **60** (1993) 287; A.I. Davydychev, A.G. Grozin, Phys. Rev. D **59** (1999) 054023; F. Jegerlehner, M.Yu. Kalmykov, Nucl. Phys. B **676** (2004) 365.

- [33] M.Yu. Kalmykov, B. Kniehl, doi: 10.1016/j.nuclphysb.2008.08.022 (arXiv:0807.0567).
- [34] A. Erdelyi (Ed.), *Higher Transcendental Functions*, vol.1 (McGraw-Hill, New York, 1953); L.J. Slater, *Generalized Hypergeometric Functions* (Cambridge University Press, Cambridge 1966).
- [35] G. 't Hooft, M.J.G. Veltman, Nucl. Phys. B **153** (1979) 365; A. Denner, U. Nierste, R. Scharf, Nucl. Phys. B **367** (1991) 637.
- [36] L. Lewin, *Polylogarithms and associated functions* (North-Holland, Amsterdam, 1981).
- [37] A.I. Davydychev, Phys. Rev. D **61** (2000) 087701; A.I. Davydychev, M.Yu. Kalmykov, Nucl. Phys. Proc. Suppl. **89** (2000) 283; Nucl. Phys. B **605** (2001) 266; arXiv:hep-th/0203212.
- [38] U. Nierste, D. Müller, M. Böhm, Z. Phys. C **57** (1993) 605.
- [39] A.I. Davydychev, Nucl. Instrum. Meth. A **559** (2006) 293; O.V. Tarasov, arXiv:0809.3028.
- [40] A.B. Goncharov, *Proceedings of the International Congress of Mathematicians, Zurich, 1994* (Birkhäuser, Basel, 1995) Vol. 1, 2, p. 374; Math. Res. Lett. **4** (1997) 617; *ibid* **5** (1998) 497; arXiv:math/0103059.
- [41] J. Fleischer, T. Riemann, O.V. Tarasov, Acta Phys. Polon. B **34** (2003) 5345.
- [42] J.G. Körner, Z. Merebashvili, M. Rogal, Phys. Rev. D **71** (2005) 054028; J. Math. Phys. **47** (2006) 072302.
- [43] D.J. Broadhurst, D. Kreimer, Int. J. Mod. Phys. C **6** (1995) 519; Phys. Lett. B **393** (1997) 403; I. Bierenbaum, S. Weinzierl, Eur. Phys. J. C **32** (2003) 67; F. Brown, arXiv:0804.1660.
- [44] S. Bauberger, F. A. Berends, M. Böhm, M. Buza, Nucl. Phys. B **434** (1995) 383; A.I. Davydychev, R. Delbourgo, J. Phys. A **37** (2004) 4871; G. Passarino, Nucl. Phys. Proc. Suppl. **135** (2004) 265; B.A. Kniehl et al., Nucl. Phys. B **738** (2006) 306; D.H. Bailey et al., J. Phys. A **41** (2008) 20520; S. Laporta, Phys. Lett. B **549** (2002) 115; arXiv:0803.1007; P. Aluffi, M. Marcolli, arXiv:0807.1690
- [45] I.M. Gelfand, M.M. Kapranov, A.V. Zelevinsky, Adv. Math. **84** (1990) 255; I.M. Gel'fand, M.I. Graev, V.S. Retakh, Russian Math. Surveys **47** (1992) 1; I.M. Gelfand, M.I. Graev, Russian Math. Surveys **52** (1997) 639; *ibid* **56** (2001) 615.
- [46] O. Ore, J. Math. Pure Appl. **9** (1930) 311; M. Sato, Nagoya Math. J. **120** (1990) 1.
- [47] S. Moch, P. Uwer, S. Weinzierl, J. Math. Phys. **43** (2002) 3363.
- [48] S. Weinzierl, J. Math. Phys. **45** (2004) 2656.
- [49] Shu Oi, math.NT/0405162.
- [50] M.Yu. Kalmykov, B.F.L. Ward, S. Yost, JHEP **0702** (2007) 040.
- [51] M.Yu. Kalmykov, B.F.L. Ward, S.A. Yost, JHEP **0710** (2007) 048.

- [52] M.Yu. Kalmykov, B.F.L. Ward, S.A. Yost, JHEP **0711** (2007) 009.
- [53] A.I. Davydychev, J.B. Tausk, Nucl. Phys. B **397** (1993) 123; Phys. Rev. D **53** (1996) 7381.
- [54] N. Gray, D.J. Broadhurst, W. Grafe, K. Schilcher, Z. Phys. C **48** (1990) 673;  
D.J. Broadhurst, Z. Phys. C **47** (1990) 115; *ibid* **54** (1992) 599; arXiv:hep-th/9604128;  
J.M. Borwein, D.M. Bradley, D.J. Broadhurst, Electron. J. Combin. **4** (1997) #R5;  
J.A.M. Vermaseren, Int. J. Mod. Phys. A **14** (1999) 2037;  
M. Bigotte, G. Jacob, N.E. Oussous, M. Petitot, Theoret. Comput. Sci. **273** (2002) 271.
- [55] S. Weinzierl, Comput. Phys. Commun. **145** (2002) 357;  
S. Moch, P. Uwer, Comput. Phys. Commun. **174** (2006) 759;  
T. Huber, D. Maître, Comput. Phys. Commun. **175** (2006) 122.
- [56] T. Huber, D. Maître, Comput. Phys. Commun. **178** (2008) 755.
- [57] D.J. Broadhurst, Eur. Phys. J. C **8** (1999) 311;  
J. Fleischer, M.Yu. Kalmykov, A.V. Kotikov, Phys. Lett. B **462** (1999) 169;  
J. Fleischer, M.Yu. Kalmykov, Phys. Lett. B **470** (1999) 168;  
J.M. Borwein, D.J. Broadhurst, J. Kamnitzer, Exper. Math. **10** (2001) 25;  
M.Yu. Kalmykov, O. Veretin, Phys. Lett. B **483** (2000) 315;  
M.Yu. Kalmykov, A. Sheplyakov, Comput. Phys. Commun. **172** (2005) 45;  
M.Yu. Kalmykov, Nucl. Phys. B **718** (2005) 276;  
Jianqiang Zhao, arXiv:math/0302055.
- [58] F. Jegerlehner, M.Yu. Kalmykov, O. Veretin, Nucl. Phys. B **658** (2003) 49.
- [59] A.I. Davydychev, M.Yu. Kalmykov, Nucl. Phys. B **699** (2004) 3;  
M.Yu. Kalmykov, Nucl. Phys. Proc. Suppl. **135** (2004) 280.
- [60] S.A. Yost, M.Yu. Kalmykov, B.F.L. Ward, ICHEP 2008, Philadelphia, arXiv:0808.2605.
- [61] J.M. Borwein et al., Trans. Am. Math. Soc. **353** (2001) 907;  
M. Waldschmidt, "Multiple polylogarithms: an introduction," in *Number theory and discrete mathematics* (Chandigarh, 2000), 1–12, (Trends Math., Birkhäuser, Basel, 2002).
- [62] E. Remiddi, J.A.M. Vermaseren, Int. J. Mod. Phys. A **15** (2000) 725.
- [63] T. Gehrmann, E. Remiddi, Nucl. Phys. B **601** (2001) 248.
- [64] J. Vollinga, S. Weinzierl, Comput. Phys. Commun. **167** (2005) 177.
- [65] D. Maître, Comput. Phys. Commun. **174** (2006) 222; arXiv:hep-ph/0703052.
- [66] J. Fleischer, A.V. Kotikov, O.L. Veretin, Nucl. Phys. B **547** (1999) 343
- [67] A.I. Davydychev, P. Osland, L. Saks, Phys. Rev. D **63** (2001) 014022; JHEP **0108** (2001) 050.
- [68] C. Anastasiou, E.W.N. Glover, C. Oleari, Nucl. Phys. B **572** (2000) 307.
- [69] R.K. Ellis, G. Zanderighi, JHEP **0802** (2008) 002;  
J.R. Andersen, T. Binoth, G. Heinrich, J.M. Smillie, JHEP **0802** (2008) 057.
- [70] M.Yu. Kalmykov, *Hypergeometric functions: reduction and epsilon-expansion*,  
<http://theor.jinr.ru/~kalmykov/hypergeom/hyper.html>