# Operator approach to analytical evaluation of Feynman perturbative integrals

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## 1 Introduction

In this report we discuss new operator approach [1], [2] to analytical evaluation of multi-loop Feynman diagrams (FDs). We show that the known analytical methods of evaluation of massless Feynman integrals, such as the integration by parts method [3] and the method of "uniqueness" [4] (which is based on the star-triangle relation), can be drastically simplified by using this operator approach. To demonstrate the advantages of the operator method of analytical evaluation of multi-loop Feynman diagrams, we calculate ladder diagrams for the massless $\phi^3$ theory (analytical results for these diagrams are expressed in terms of multiple polylogarithms). We also show how operator formalism can be applied to calculation of certain massive Feynman diagrams.

The main idea of our algebraic method for evaluation of FDs is that we replace manipulations with multiple integrals by manipulations with the corresponding algebraic expressions. In other words, identical transformations of multiple perturbative integrals are substituted with transformations of elements of special infinite dimensional algebra. This drastically simplifies all calculations.

## 2 Feynman diagrams in configuration space

The Feynman diagrams, which will be considered in this paper, are graphs with vertices connected by edges (propagators). To each edge we assign a complex number (the index of the propagator). With each vertex we associate the point in the $D$-dimensional space $\mathbf{R}^D$ while the edges of the graph (with index $\alpha$) are associated with the propagator of massless particle

$$x \;\overset{\alpha}{\overline{\qquad\qquad}}\; y \qquad = \qquad 1/(x-y)^{2\alpha}$$

where $x, y \in \mathbf{R}^D$, $(x-y)^{2\alpha} := (\sum_{i=1}^{D}(x_i - y_i)\,(x_i - y_i))^{\alpha}$, and $\alpha \in \mathbf{C}$. Moreover, we consider the graphs with two types of vertices: boldface vertices $\bullet$ denote that the corresponding points are integrated over $\mathbf{R}^D$. These FDs are called *FDs in the configuration space*.

Consider examples of FDs in the configuration space and present the corresponding multiple integrals.

**1. Graph with 5 vertices and 5 edges (3-point function)**:

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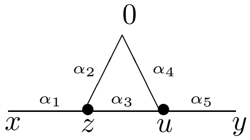
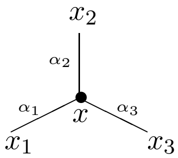



$$= \int \frac{d^D z \, d^D u}{(z-y)^{2\alpha_1} \, z^{2\alpha_2} \, y^{2\alpha_3} \, u^{2\alpha_4} \, (u-y)^{2\alpha_5}}$$

**2. "Star" graph**:




$$= \int \frac{d^D x}{(x-x_1)^{2\alpha_1} \, (x-x_2)^{2\alpha_2} \, (x-x_3)^{2\alpha_3}}$$

The problem (which we need to solve when analytically calculate multiple integrals corresponding to FDs) consists in searching for special transformations of graphs (FDs) such that the number of boldface vertices (the number of integrations) decreases at each step. In the next section, we discuss these special transformations and describe the corresponding operator formalism, which gives us a possibility to represent these transformations using a more compact algebraic language.

## 3 Algebraic manipulations with perturbative integrals

Consider the $D$-dimensional Heisenberg algebra $[\hat{q}_k, \hat{p}_j] = \mathrm{i}\, \delta_{kj}$, where $\hat{q}_i = \hat{q}_i^\dagger$, $\hat{p}_i = \hat{p}_i^\dagger$ are operators of coordinates and momenta, respectively. Introduce the eigenvectors $|x\rangle \equiv |\{x_i\}\rangle$, $|k\rangle \equiv |\{k_i\}\rangle$ of these operators: $\hat{q}_i|x\rangle = x_i\,|x\rangle$, $\hat{p}_i|k\rangle = k_i\,|k\rangle$. We normalize the states as follows

$$\langle x|k\rangle = \frac{1}{(2\pi)^{D/2}} \exp(\mathrm{i}\, k_j\, x_j) \;, \quad \int d^D k\, |k\rangle\, \langle k| = \hat{1} = \int d^D x\, |x\rangle\, \langle x| \;. \tag{3.1}$$

The heat-kernels ("matrix representations") of the operators $\hat{p}^{-2\beta}$ are:

$$\langle x|\frac{1}{\hat{p}^{2\beta}}|y\rangle = a(\beta)\, \frac{1}{(x-y)^{2\beta'}} \;, \quad \left( a(\beta) = \frac{\Gamma(\beta')}{\pi^{D/2}\, 2^{2\beta}\, \Gamma(\beta)} \right) . \tag{3.2}$$

where $\boxed{\beta' = D/2 - \beta}$ and $\Gamma(\beta)$ is the Euler gamma-function. Formula (3.2) relates the propagators for massless particles and pseudo-differential operators $\hat{p}^{-2\beta}$. For the operators $\hat{q}^{2\alpha}$ the "matrix representations" have the form:

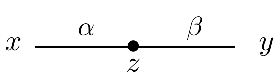
$$\langle x|\hat{q}^{2\alpha}|y\rangle = x^{2\alpha}\, \delta^D(x-y) \;. \tag{3.3}$$

Below we consider three **(a,b,c)** algebraic relations which are operator analogs of relations used for the analytical evaluation of multi-loop perturbative integrals for FDs. Recall that these relations give us a possibility to reconstruct FD in such a way that the number of integrations (the number of boldface vertices in the graph) decreases to zero (this will indicate that a given FD is calculated analytically).

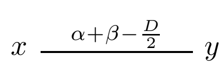
**a. Group relation**. A convolution product of two propagators:

$$\int \frac{d^D z}{(x-z)^{2\alpha}\, (z-y)^{2\beta}} = \frac{G(\alpha', \beta')}{(x-y)^{2(\alpha+\beta-D/2)}} \;, \quad \left( G(\alpha,\beta) = \frac{a(\alpha+\beta)}{a(\alpha)\, a(\beta)} \right) , \tag{3.4}$$

is graphically represented as




$$= G(\alpha', \beta') \;\cdot$$



The relation (3.4) describes a reconstruction of the graph in which the number of integrations (boldface vertices) decreases by one. The operator analog of this relation is (group relation)

$$\hat{p}^{-2\alpha'}\,\hat{p}^{-2\beta'} = \hat{p}^{-2(\alpha'+\beta')}. \tag{3.5}$$

Indeed, by using (3.2) and (3.3) we easily demonstrate that the "matrix" analog of (3.5)

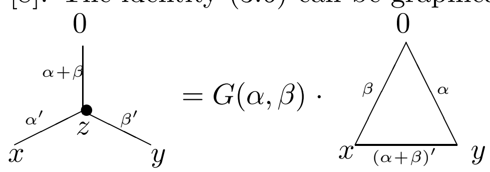
$$\int d^D z\,\langle x|\hat{p}^{-2\alpha'}\,|z\rangle\,\langle z|\,\hat{p}^{-2\beta'}\,|y\rangle = \langle x|\hat{p}^{-2(\alpha'+\beta')}|y\rangle$$

coincides with (3.4). Note that in the operator relation (3.5) the tedious coefficient $G(\alpha',\beta')$ (presented in (3.4)) is vanished.

**b. Star-triangle relation**. This relation is in the basis of the so-called "method of uniqueness" [4] (see also [5]) which is an efficient method of analytical evaluation of FDs. In fact, this relation is a special case of the Yang-Baxter equation [6], [7], [2]. The star-triangle relation (STR) has the form

$$\int \frac{d^D z}{(x-z)^{2\alpha'}\,z^{2(\alpha+\beta)}\,(z-y)^{2\beta'}} = \frac{G(\alpha,\beta)}{(x)^{2\beta}\,(x-y)^{2(\frac{D}{2}-\alpha-\beta)}\,(y)^{2\alpha}}\,, \tag{3.6}$$

and was initially used in the framework of investigations of multi-dimensional conformal field theories [8]. The identity (3.6) can be graphically represented as




Thus, STR (3.6) describes such a reconstruction of the graph for which the number of integrations (boldface vertices) decreases by one. The operator version of this relation was proposed in [1] and is written in the form

$$\hat{p}^{-2\alpha}\hat{q}^{-2(\alpha+\beta)}\hat{p}^{-2\beta} = \hat{q}^{-2\beta}\hat{p}^{-2(\alpha+\beta)}\hat{q}^{-2\alpha} \quad (\forall\alpha,\beta)\;. \tag{3.7}$$

Here again we note the absence of the coefficient $G(\alpha,\beta)$. To demonstrate the equivalence of (3.6) and (3.7) we act on (3.7) by vectors $\langle x|$ and $|y\rangle$ from the left and right, respectively, insert the unit operator $\hat{1} = \int d^D z\,|z\rangle\,\langle z|$ and use representations (3.2), (3.3).

**Remark 1.** The algebraic version of STR is equivalent to the commutativity for the infinite set of operators $H(\alpha) = \hat{p}^{2\alpha}\hat{q}^{2\alpha}$:

$$H(\alpha)\,H^{-1}(-\beta) = H^{-1}(-\beta)\,H(\alpha) \;\Rightarrow\; \hat{p}^{2\alpha}\hat{q}^{2(\alpha+\beta)}\,\hat{p}^{2\beta} = \hat{q}^{2\beta}\,\hat{p}^{2(\alpha+\beta)}\,\hat{q}^{2\alpha}\;.$$

**Remark 2.** Here we present the algebraic proof of STR (3.7). Introduce an inversion operator $\mathcal{R}$ which obeys the conditions

$$\mathcal{R}^2 = 1\,,\quad \mathcal{R}\,\hat{q}_i\,\mathcal{R} = \hat{q}_i\,/\,\hat{q}^2\,,\quad \langle x_i|\,\mathcal{R} = \langle x_i/x^2|(x^2)^{-D/2}\,, \tag{3.8}$$

$$\mathcal{R}^\dagger = \mathcal{R}\,,\;\; \mathcal{R}\,\hat{p}^{2\beta}\,\mathcal{R} = \hat{q}^{2\beta}\,\hat{p}^{2\beta}\,\hat{q}^{2\beta}\,. \tag{3.9}$$

Using (3.8), (3.9) the algebraic version of STR is proved as follows:

$$\mathcal{R}\,\hat{p}^{2\alpha}\,\mathcal{R}^2\,\hat{p}^{2\beta}\,\mathcal{R} = \mathcal{R}\,\hat{p}^{2(\alpha+\beta)}\,\mathcal{R} \;\;\Rightarrow\;\; \hat{p}^{2\alpha}\hat{q}^{2(\alpha+\beta)}\,\hat{p}^{2\beta} = \hat{q}^{2\beta}\,\hat{p}^{2(\alpha+\beta)}\hat{q}^{2\alpha}\;.$$

**Remark 3.** For propagators in $\alpha$-representation one can consider another STR [9]

$$\exp\left(\frac{\hat{q}^2}{2\alpha_1}\right)\exp\left(\frac{\alpha_2}{2}\hat{p}^2\right)\exp\left(\frac{\hat{q}^2}{2\alpha_3}\right) = \exp\left(\frac{\beta_3}{2}\hat{p}^2\right)\exp\left(\frac{\hat{q}^2}{2\beta_2}\right)\exp\left(\frac{\beta_1}{2}\hat{p}^2\right)\,,$$

where parameters $\alpha_i$ and $\beta_i$ are related by the identity $\alpha_i = \frac{\beta_1\beta_2+\beta_1\beta_3+\beta_2\beta_3}{\beta_i}$ which is well known as star-triangle transformation for resistances in electric networks.

**c. Integration by parts rule** [3].
First, we present the graphical version of this rule

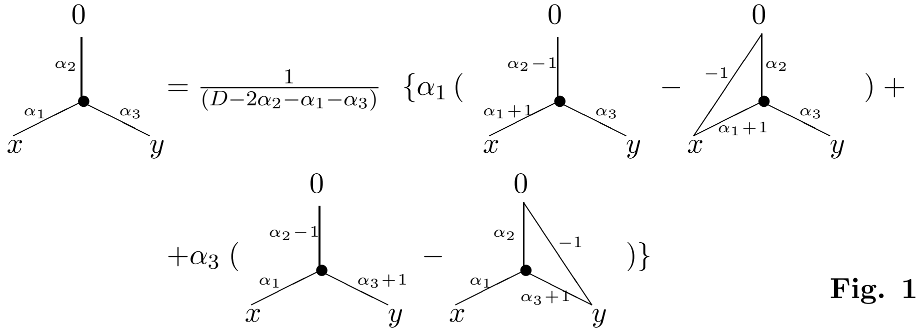



**Fig. 1**

With the help of this rule we reconstruct graphs in such a way that the number of integrations (boldface vertices) is not decreased. However, this rule is extremely useful, since the corresponding reconstruction leads to variations of the indices on the lines, which further permits to apply previous relations **a,b** and decrease the number of integrations.

The operator version of the integration by parts rule (Fig. 1) has the form

$$(2\gamma-\alpha-\beta)\,\hat{p}^{2\alpha}\hat{q}^{2\gamma}\hat{p}^{2\beta} = \frac{[\hat{q}^2,\,\hat{p}^{2(\alpha+1)}]}{4(\alpha+1)}\,\hat{q}^{2\gamma}\,\hat{p}^{2\beta} - \hat{p}^{2\alpha}\,\hat{q}^{2\gamma}\,\frac{[\hat{q}^2,\,\hat{p}^{2(\beta+1)}]}{4(\beta+1)} \qquad (3.10)$$

where $\alpha = -\alpha_1'$, $\gamma = -\alpha_2$ and $\beta = -\alpha_3'$. Identity (3.10) can be directly proved by using the relations for the Heisenberg algebra:

$$\begin{gathered}[\hat{q}^2,\,\hat{p}^{2(\alpha+1)}] = 4\,(\alpha+1)\,(H+\alpha)\,\hat{p}^{2\alpha}\ , \\ H\,\hat{q}^{2\alpha} = \hat{q}^{2\alpha}\,(H+2\alpha)\ , \quad H\,\hat{p}^{2\alpha} = \hat{p}^{2\alpha}\,(H-2\alpha)\ ,\end{gathered} \qquad (3.11)$$

where $H := \frac{i}{2}(\hat{p}_i\hat{q}_i + \hat{q}_i\hat{p}_i)$ is the dilatation operator. It follows from (3.11) that the operators $\{\hat{q}^2, \hat{p}^2, H\}$ generate the algebra $sl(2)$:

$$[\hat{q}^2,\,\hat{p}^2] = 4\,H\ , \quad [H,\,\hat{q}^2] = 2\,\hat{q}^2\ , \quad H\,\hat{p}^2 = -2\,\hat{p}^2\ . \qquad (3.12)$$

# 4 Applications

## 4.1 Ladder FDs for $\phi^3$ theory in $D = 4$; relation to conformal quantum mechanics

Consider dimensionally and analytically regularized massless perturbative integrals

$$D_L(p_0, p_{L+1}, p; \alpha, \beta, \gamma) = \left[\prod_{k=1}^{L}\int \frac{d^D p_k}{p_k^{2\alpha}\,(p_k-p)^{2\beta}}\right]\prod_{m=0}^{L}\frac{1}{(p_{m+1}-p_m)^{2\gamma}}\ , \qquad (4.13)$$

which correspond to FDs ($x_1 = p_0$, $x_2 = p_{L+1}$, $x_3 = p$)

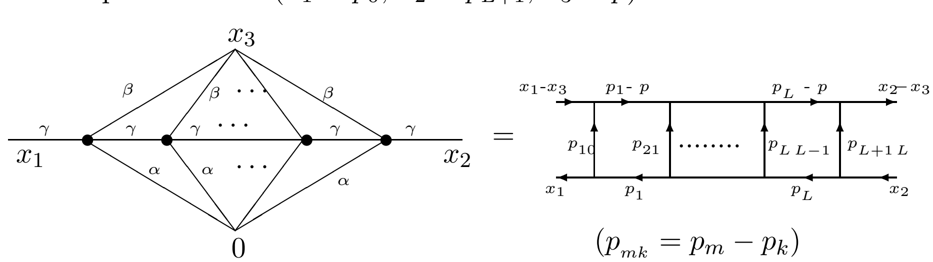



**Fig.2**

Perturbative integral (4.13) can be graphically represented in two ways – as diagrams in configuration and momentum spaces, as it is shown in Fig. 2, where $\alpha, \beta, \gamma$ are indices on the lines in the left diagram (in configuration space) , while in the right diagram (in momentum space) the indices $p_i$ indicate momenta flowing over the lines. These diagrams are dual to each other (boldface vertices in the left diagram correspond to the loops in the right diagram). The operator version of the integral (4.13) follows from the representation for the left diagram

$$D_L(x_a;\alpha,\beta,\gamma) = (a(\gamma'))^{-L-1}\,\langle x_1|\hat{p}^{-2\gamma'}\left(\prod_{k=1}^{L}\hat{q}^{-2\alpha}(\hat{q}-x_3)^{-2\beta}\hat{p}^{-2\gamma'}\right)|x_2\rangle\,.$$

It is convenient to consider the generating function for the integrals $D_L$ (4.13)

$$D_g(x_a;\alpha,\beta,\gamma) = a(\gamma')\sum_{L=0}^{\infty} g^L\, D_L(x_a;\alpha,\beta,\gamma) = \langle x_1\,|\left(\hat{p}^{2\gamma'} - \frac{g/a(\gamma')}{\hat{q}^{2\alpha}(\hat{q}-x_3)^{2\beta}}\right)^{-1}|\,x_2\rangle\,. \qquad (4.14)$$

If the indices on the lines are related by the condition $\boxed{\alpha+\beta=2\gamma'}$, then by using properties (3.8), (3.9) of the inversion operator $\mathcal{R}$ we obtain (for details see [1]):

$$D_g(x_a;\,2\gamma'-\beta,\beta,\gamma) = \frac{1}{(x_1^2x_2^2)^{\gamma}}\langle u|\,\left(\hat{p}^{2\gamma'} - \frac{g_{\gamma',\beta}}{\hat{q}^{2\beta}}\right)^{-1}|\,v\rangle\,, \qquad (4.15)$$

where $g_{\gamma,\beta} = \frac{g}{(x_3)^{2\beta}a(\gamma)}$, $u_i = \frac{(x_1)_i}{(x_1)^2} - \frac{(x_3)_i}{(x_3)^2}$, $v_i = \frac{(x_2)_i}{(x_2)^2} - \frac{(x_3)_i}{(x_3)^2}$ $(i=1,\dots,D)$. In the case when the indices on the lines are fixed as $\gamma' = \beta = 1$, the generating function $D_g$ (4.15) is reduced to the Green function for $D$-dimensional conformal mechanics

$$D_g(x_a;\,1,1,D/2-1) = \frac{1}{(x_1^2x_2^2)^{(D/2-1)}}\langle u|\,\left(\hat{p}^2 - \frac{g_{1,1}}{\hat{q}^2}\right)^{-1}|\,v\rangle = \qquad (4.16)$$

$$= a(1)\sum_{L=0}^{\infty} g^L\, D_L(x_a;1,1,D/2-1)\,.$$

Thus, we have shown that with a special choice of indices on the lines $\alpha=\beta=1$, $\gamma=\frac{D}{2}-1=1-\epsilon$ the ladder diagrams (in momentum space):

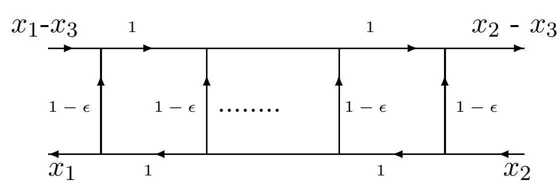



**Fig. 3**

are related to the Green function for $D$-dimensional conformal mechanics. Moreover, according to the definition of the generating function $D_g$ (4.14), the expression $D_L$ for the ladder diagram with $L$ loops (or $L$ boldface vertices for FD in the configuration space) is obtained by expanding of the Green function (4.16) over the coupling constant $g$ up to the coefficient in order $g^L$.

The operator method of evaluation of Green function (4.16) is based on the remarkable identity [1]

$$\frac{1}{\hat{p}^2 - g/\hat{q}^2} = \sum_{L=0}^{\infty}\left(-\frac{g}{4}\right)^L\left[\hat{q}^{2\alpha}\,\frac{(H-1)}{(H-1+\alpha)^{L+1}}\,\frac{1}{\hat{p}^2}\,\hat{q}^{-2\alpha}\right]_{\alpha^L}\,, \qquad (4.17)$$

where we have used the notation $[\dots]_{\alpha^L} = \frac{1}{L!}\left(\partial_\alpha^L\,[\dots]\right)_{\alpha=0}$. Taking into account the integral representation for the rational function of $H$ (in the right-hand side of (4.17))

$$\frac{(H-1)}{(H-1+\alpha)^{L+1}} = \frac{(-1)^{L+1}}{L!}\int_0^\infty dt\, t^L\, e^{t\alpha}\,\partial_t\left(e^{t\,(H-1)}\right)\,,$$

and using obvious properties of the operator $e^{t\,H}$: $e^{t(H+\frac{D}{2})}\,|x\rangle = |e^{-t}x\rangle$, we can rewrite Green function appearing in (4.16) in the form

$$\langle u|\,\frac{1}{(\hat{p}^2 - g_{1,1}/\hat{q}^2)}\,|v\rangle = \frac{a(1)}{u^{2(D/2-1)}} \sum_{L=0}^{\infty} \frac{1}{L!}\left(\frac{g_{1,1}}{4}\right)^L \Psi_L\left(\frac{v^2}{u^2}, 2\frac{(uv)}{u^2}\right)\,, \tag{4.18}$$

where $g_{1,1} = g/(a(1)x_3^2)$ and

$$\Psi_L\left(\frac{v^2}{u^2}, 2\frac{(uv)}{u^2}\right) = -\int_0^\infty dt\, t^L \left[\left(\frac{u^2}{v^2}\right)^\alpha e^{t\alpha}\right]_{\alpha^L} \partial_t \left(\frac{e^{-t}u^2}{(u-e^{-t}v)^2}\right)^{(\frac{D}{2}-1)}\,. \tag{4.19}$$

Finally the result for the evaluation of the $L$-loop ladder diagram (Fig. 3) is

$$D_L(x_1,x_2,x_3;1,1,\frac{D}{2}-1) = \left(\frac{1}{L!4^L a(1)^L}\right) \frac{x_3^{2(D/2-L-1)}}{(x_{13}^2 x_2^2)^{D/2-1}} \Psi_L(\frac{v^2}{u^2}, 2\frac{(uv)}{u^2})\,, \tag{4.20}$$

where $u^2 = \frac{x_{13}^2}{x_1^2 x_3^2}$, $v^2 = \frac{x_{23}^2}{x_2^2 x_3^2}$, $(u-v)^2 = \frac{x_{12}^2}{x_1^2 x_2^2}$ and $x_{ab} = x_a - x_b$.

For $D = 4-2\epsilon$ the function $\Psi_L(\frac{v^2}{u^2}, 2\frac{(uv)}{u^2})$ (4.19) is expanded over $\epsilon$

$$\Psi_L\left(\frac{v^2}{u^2}, 2\frac{(uv)}{u^2}\right) = \frac{1}{(z_1-z_2)} \sum_{k=0}^\infty \frac{\epsilon^k}{k!}\,\Phi_L^{(k)}(z_1,z_2)\,.$$

where $z_1+z_2 = 2(uv)/u^2$ and $z_1 z_2 = v^2/u^2$. The coefficient functions $\Phi_L^{(l)}$ are expressed in terms of multiple polylogarithms

$$\mathrm{Li}_{m_0,m_1,\dots,m_r}(w_0,w_1,\dots,w_r) = \sum_{n_0>n_1>\dots>n_r>0} \frac{w_0^{n_0} w_1^{n_1}\cdots w_r^{n_r}}{n_0^{m_0} n_1^{m_1}\dots n_r^{m_r}}\,. \tag{4.21}$$

The first coefficient (for $D=4$ or $\epsilon=0$) has the form [11], [12]

$$\Phi_L^{(0)}(z_1,z_2) = \sum_{f=0}^{L} \frac{(-)^f\,(2L-f)!}{f!\,(L-f)!}\,\ln^f(z_1z_2)\big[\mathrm{Li}_{2L-f}(z_1) - \mathrm{Li}_{2L-f}(z_2)\big]\,.$$

and is expressed via the standard polylogarithms $\mathrm{Li}_m(w) = \sum_{n=1}^\infty \frac{w^n}{n^m}$. The next coefficient was calculated in [1]:

$$\Phi_L^{(1)}(z_1,z_2) = \sum_{n=L}^{2L} \frac{n!\,\left[(n\mathrm{Li}_{n+1}(z_1) - \mathrm{Li}_{n,1}(z_1,1) - \mathrm{Li}_{n,1}(z_1,\frac{z_2}{z_1})) - (z_1\leftrightarrow z_2)\right]}{(-1)^n\,(2L-n)!\,(n-L)!\,\ln^{n-2L}(z_1z_2)}\,,$$

and is expressed via multiple polylogarithms $\mathrm{Li}_{n,1}(w_0,w_1)$ (4.21).

**Remark.** The conformal symmetry requires that the Green function (4.15) (after the special normalization) is a function which depends only on two conformal variables (cf. (4.18)):

$$u^{2(D/2-\gamma)}\,\langle u|\left(\hat{p}^{2\gamma} - \frac{g(u^2v^2)^{\frac{\beta-\gamma}{2}}}{\hat{q}^{2\beta}}\right)^{-1}|v\rangle = \Psi^{(\gamma,\beta)}\left(\frac{v^2}{u^2},\frac{2(uv)}{u^2}\right)\,, \tag{4.22}$$

where $\Psi^{(\gamma,\beta)}(u_1,u_2) = u_1^{\gamma-D/2}\Psi^{(\gamma,\beta)}(u_1^{-1},u_2u_1^{-1}) = \Psi^{(\gamma,2\gamma-\beta)}(u_1,u_2)$.

Let $u = \frac{1}{x_1} - \frac{1}{x_3}$, $v = \frac{1}{x_2} - \frac{1}{x_3}$ (see (4.15)) and $u' = \frac{1}{x_1} - \frac{1}{x_{12}}$, $v' = \frac{1}{x_{13}} - \frac{1}{x_{12}}$, where we have used concise notation $(\frac{1}{x_a})_i = \frac{(x_a)_i}{x_a^2}$. Then, one can deduce the cross-ratio relations

$$\frac{v^2}{u^2} = \frac{(v')^2}{(u')^2} = \frac{x_{23}^2 x_1^2}{x_2^2 x_{13}^2}\,, \quad \frac{(u-v)^2}{u^2} = \frac{(u'-v')^2}{(u')^2} = \frac{x_{12}^2 x_3^2}{x_{13}^2 x_2^2}\,,$$

which lead to the identities $\Psi^{(\gamma,\beta)}\left(\frac{v^2}{u^2}, \frac{2(uv)}{u^2}\right) = \Psi^{(\gamma,\beta)}\left(\frac{v'^2}{u'^2}, \frac{2(u'v')}{u'^2}\right)$. Using representation (4.22) we write these identities as

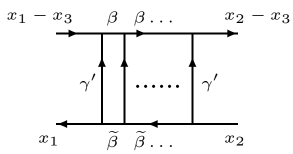
$$u^{2(\frac{D}{2}-\gamma)} \langle u| \left(\hat{p}^{2\gamma} - g\frac{(u^2v^2)^{\frac{\beta-\gamma}{2}}}{\hat{q}^{2\beta}}\right)^{-1} |v\rangle = (u')^{2(\frac{D}{2}-\gamma)} \langle u'| \left(\hat{p}^{2\gamma} - g\frac{(u'^2\, v'^2)^{\frac{\beta-\gamma}{2}}}{\hat{q}^{2\beta}}\right)^{-1} |v'\rangle \, , \quad (4.23)$$

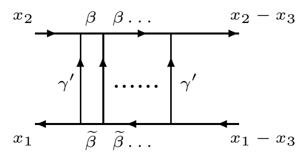
Now we use (4.15) for both sides of eq. (4.23). As a result, we write (4.23) in the form of relation on generating functions for the ladder diagrams (4.14):

$$x_3^{2(\gamma-D/2)} \langle x_1| \left(\hat{p}^{2\gamma} - \frac{g\, x_3^{2\gamma}\, \widetilde{u}^{\frac{\beta-\gamma}{2}}}{\hat{q}^{2(2\gamma-\beta)}(\hat{q}-x_3)^{2\beta}}\right)^{-1} |x_2\rangle =$$

$$= x_{12}^{2(\gamma-D/2)} \langle x_1| \left(\hat{p}^{2\gamma} - \frac{g\, x_{12}^{2\gamma}\, \widetilde{v}^{\frac{\beta-\gamma}{2}}}{\hat{q}^{2(2\gamma-\beta)}(\hat{q}-x_{12})^{2\beta}}\right)^{-1} |x_{13}\rangle \, .$$

where $\widetilde{u} = \frac{x_{13}^2 x_{23}^2}{x_1^2 x_2^2}$ and $\widetilde{v} = \frac{x_2^2 x_{23}^2}{x_1^2 x_{13}^2}$. Expanding both sides over the coupling constant $g$, we obtain $D$-dimensional identities for the $L$-loop ladder momentum diagrams in the order $g^L$:

$$\frac{\widetilde{u}^{\frac{L(\beta-\gamma)}{2}}}{x_3^{2(\frac{D}{2}-\gamma-\gamma L)}} \times$$ 

$$= \frac{\widetilde{v}^{\frac{L(\beta-\gamma)}{2}}}{x_{12}^{2(\frac{D}{2}-\gamma-\gamma L)}} \times$$ 


where $\beta$, $\widetilde{\beta} = 2\gamma - \beta$ and $\gamma' = D/2 - \gamma$ are special indices on the lines and $x_1, x_2, x_3$ parametrize external momenta. These identities, in the special case $D = 4$ ($\epsilon = 0$) and $\beta = \gamma' = \widetilde{\beta} = 1$, were obtained in [13] and used there for deriving many remarkable relations for various planar FDs.

## 4.2 Diagrams with massive propagators

In this subsection we consider an example of the operator approach to analytical evaluation of the 1-loop 3-point function with one massive propagator.

First, we use the automorphism ($\hat{p}^2 \leftrightarrow \hat{q}^2$, $H \leftrightarrow -H$) of the $sl(2)$-algebra (3.12) to write the first relation in (3.11) as $[\hat{q}^{2\beta}, \hat{p}^2] = 4\beta\,(H - \beta + 1)\hat{q}^{2(\beta-1)}$ and, then, generalize it by introducing the massive parameter $m$ as follows:

$$[(\hat{q}^2 + m^2)^\beta, \hat{p}^2] = 4\beta\,(H + m^2\partial_{m^2} - \beta + 1)(\hat{q}^2 + m^2)^{(\beta-1)} \, . \quad (4.24)$$

This identity can be converted into the integral form

$$\frac{1}{\hat{p}^2}(\hat{q}^2 + m^2)^{(\beta-1)}\frac{1}{\hat{p}^2} = \frac{1}{4\beta}\int_0^\infty dt e^{t(H-\beta-1+m^2\partial_{m^2})}[\frac{1}{\hat{p}^2}, (\hat{q}^2+m^2)^\beta]$$

from which the representation for the 3-point function is deduced

$$\langle x_1|\frac{1}{\hat{p}^2}(\hat{q}^2 + m^2)^{(\beta-1)}\frac{1}{\hat{p}^2}|x_2\rangle = \frac{a(1)}{4\beta}\int_0^\infty dt\; e^{t(D/2-1)}\frac{(e^{-t}x_2^2 + m^2)^\beta - (e^t x_1^2 + m^2)^\beta}{(e^t x_1 - x_2)^{2(D/2-1)}} \, .$$

Here the left-hand side is represented in the form of the perturbative integral and we obtain the equality

$$\int \frac{d^D k\,(k^2+m^2)^{\beta-1}}{((k-x_1)^2(k-x_2)^2)^{(D/2-1)}} = \frac{1}{4a(1)}\int_0^\infty dt \frac{((e^{-t}x_2^2 + m^2)^\beta - (x_1^2 e^t + m^2)^\beta)\beta^{-1}}{e^{t(1-D/2)}\,(x_1 e^t - x_2)^{2(D/2-1)}} \, .$$

Finally, we consider the limit $D \to 4$, $\beta \to 0$ for this relation and deduce the identity

$$\int \frac{d^4 k}{(k-x_1)^2 (k^2+m^2)(k-x_2)^2} = \pi^2 \int_0^\infty dt \, \frac{e^{-t}}{(x_1 - e^{-t} x_2)^2} \log\left( \frac{e^{-t} x_2^2 + m^2}{e^t x_1^2 + m^2} \right) ,$$

which is important for physical applications and corresponds to the evaluation of the 3-point one-loop FD (in the momentum space) with one massive line.

## Conclusion

Now let us make a few remarks about the results presented above.
**1.** It should be noted that the coefficient functions $\Psi_L(u,v)$ (4.18) appear in the calculations of the 4-point functions in the $N=4$ supersymmetric Yang-Mills theory [10].
**2.** The proposed operator relations (3.7) clarify the structure of the integrable Lipatov model (see [2]) and its certain generalizations [14].
**3.** The important problem is the search for generalizations of the described algebraic formalism in the cases of supersymmetric quantum mechanics and for massive propagators. In the last case we have succeeded in calculating the special 3-point FD with one massive propagator (see subsection 4.2). However, this calculation is particular. From this point of view it would be important to calculate the coefficients $\Phi_L(u,v;m^2)$ in the expansion over $g$ of the spectral Green function for conformal mechanics:

$$\langle u | \, \frac{1}{(\hat{p}^2 - g/\hat{q}^2 + m^2)} \, | v \rangle = \sum_{L=0}^{\infty} g^L \, \Phi_L(u,v;m^2) \; .$$

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