Gauged Supergravity and Hidden Symmetries

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Abstract

The relation between (gauged) supergravity and possible hidden symmetries given by Kac-Moody algebras is discussed. In the first part, we review the appearance of hidden symmetries in Kaluza-Klein reduction of 11-dimensional supergravity and explain the conjecture of an underlying invariance of supergravity under the Kac-Moody algebras E_{10} or E_{11}. The second part deals with the extension of the Supergravity/Kac-Moody correspondence to gauged supergravity. We present an action principle that realizes a truncation of E_{11} and briefly discuss the possibility of encoding the dynamics of gauged supergravity in a one-dimensional E_{10} coset model.

1 Introduction

One of the most surprising features of extended supergravity theories is their intimate relation with exceptional and even infinite-dimensional Lie algebras. For instance, the maximal supergravity in $D = 4$, which has originally been constructed through dimensional reduction of 11-dimensional supergravity, exhibits a non-linearly realized E_7 symmetry. Reducing maximal supergravity even further to $D = 3$ and $D = 2$ shows more symmetry enhancement to E_8 and E_9, the latter representing an affine, that is, infinite-dimensional algebra. This appearance of symmetries that are at first sight unrelated to structures present in the original theory, has led to the conjecture that 11-dimensional supergravity or even M-theory might express in a hidden way a much bigger symmetry group, possibly containing the Kac-Moody algebras E_{10} or E_{11} [1, 2]. The basic idea is that maximal supergravity in various dimensions can be obtained from an as yet undiscovered 'universal' E_{11} (or E_{10}) covariant theory by decomposing the algebra with respect to the $SL(D) \times G_D$ subgroup, where $SL(D)$ accounts for for the space-time symmetries in D dimensions and G_D is the duality group (as E_7 in $D = 4$ or E_8 in $D = 3$). Evidence for this proposal consists of the fact that performing such a level–decomposition yields, at low levels, precisely the field content of maximal supergravity in the required dimension [4].

Here, we are going to discuss recent advances of extending the dictionary between Kac-Moody algebras and supergravity by taking *gauged* supergravities into account [5, 6, 7, 8]. It turns out that the parameters of supergravity expressing the gauging are encoded in $(D-1)$– and D–form potentials through duality. While the 'low-level match' between Kac-Moody algebras and supergravity can be viewed as a simple 'covariantization' of the hidden duality symmetries appearing in lower dimensions – and as such not representing independent evidence for the conjecture –, the results for gauged supergravity are truly confirmative and push the Kac-Moody/supergravity correspondence beyond a regime where agreement was to be expected.

The organization is as follows. We first briefly review the appearance of hidden symmetries in dimensional reductions of maximal supergravity. Then we turn to gauged supergravity in the embedding tensor formalism and discuss their corresponding $(D - 1)$– and D–forms within E_{11} for the example of three dimensions. Finally, we sketch the connection between gauged supergravity and the E_{10} coset model of [2].

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2 Supergravity, hidden symmetries and Kac-Moody algebras

We start with a review of maximal supergravity and its Kaluza-Klein reduction. In any dimension, the maximal supergravity multiplet consists of 128 bosonic and 128 fermionic degrees of freedom. The corresponding field content reads in $D = 11$:

$$128_{\rm B} + 128_{\rm F} \;=\; (g_{MN} \times 44, A_{MNP} \times 84)_{\rm B} + (\psi_M \times 128)_{\rm F} \;, \tag{1}$$

where $M, N, \ldots$ are $D = 11$ space-time indices and A denotes a 3-form. The corresponding action is given by

$$\begin{aligned} S_{11} \;=\; \int d^{11}x \big(& - \tfrac{1}{4\kappa^2}\sqrt{-g}R - \tfrac{1}{48}\sqrt{-g}F_{MNKL}F^{MNKL} \\ & + \tfrac{2\kappa}{144^2}\varepsilon^{M_1\ldots M_{11}}F_{M_1\ldots M_4}F_{M_5\ldots M_8}A_{M_9\ldots M_{11}} + \mathcal{L}_{\rm fermions}\big) \;. \end{aligned} \tag{2}$$

In order to perform a Kaluza-Klein reduction to, say, four dimensions, we have to decompose the fields according to

$$g_{MN} = (g_{\mu\nu}, B_{\mu\, m}, \phi_{mn}) \;, \qquad A_{MNP} = (A_{\mu\nu\rho}, A_{\mu\nu\, m}, A_{\mu\, mn}, A_{mnk}) \;. \tag{3}$$

Here, $\mu, \nu, \ldots$ are four-dimensional space-time indices and $m, n, \ldots = 1, \ldots, 7$ internal indices. Thus, besides the four-dimensional metric $g_{\mu\nu}$, one obtains a set of scalars and vectors, as well as 2–forms and a 3–form. The 3–form does not carry propagating degrees of freedom in $D = 4$ and will therefore be discarded in the following. However, the 2–forms do carry degrees of freedom, as they are dual to scalars in four dimensions. It turns out that in order to exhibit the hidden E_7 symmetry this dualization is inevitable.[1] Imposing a duality relation, which is roughly of the form

$$\varepsilon^{\mu\nu\rho\sigma}F_{\nu\rho\sigma\, m} \;\equiv\; \varepsilon^{\mu\nu\rho\sigma}\partial_\nu A_{\rho\sigma\, m} \;=\; \partial^\mu \tilde{A}_m \;, \tag{4}$$

yields seven additional scalar fields $\tilde{A}_m$, such that in total we find the four-dimensional bosonic field content

$$g_{\mu\nu} \;, \qquad \underbrace{B_{\mu\, m} + A_{\mu\, mn}}_{7+21=28} \;, \qquad \underbrace{\phi_{mn} + A_{mnk} + \tilde{A}_m}_{28+35+7=70} \;. \tag{5}$$

In particular, the total number of scalar fields is precisely such that it can be accommodated in the coset space $E_7/SU(8)$, whose dimensions is $133 - 63 = 70$. Also, the 28 vector fields combine together with their magnetic duals such that they can live in the fundamental **56** representation of E_7. And in fact, working out the dimensional reduction in detail, reveals that the effective theory indeed has these symmetries. Moreover, this is a pattern which persists for dimensional reductions to all dimensions $D > 1$,

$$\begin{array}{ll} D = 5: & E_{6(6)}/USp(8) \\ D = 4: & E_{7(7)}/SU(8) \\ D = 3: & E_{8(8)}/SO(16) \\ D = 2: & E_{9(9)}/K(E_9) \;, \end{array} \tag{6}$$

provided all p–form fields are dualized to their lowest possible rank. In other words, the fields of maximal supergravity and the possible dualities conspire in such a way that exceptional (and for $D = 2$ even infinite-dimensional) symmetry groups appear, a feature that one could have hardly guessed from a pure supersymmetry analysis.

[1] The fact that those dualizations are special for the given dimension and cannot just be performed in any dimension shows already that a realization of the hidden symmetries in the higher-dimensional context requires some unconventional elements.

This striking pattern of exceptional symmetry algebras E_d appearing for reductions on d-tori has led to the conjecture that the original 11-dimensional theory possesses a hidden unifying symmetry, which could be the Kac-Moody algebras E_{10} or E_{11}. In fact, the magic numerology expressed in (6) can be concisely summarized by saying that the structure of maximal D–dimensional supergravity can be obtained from, say, E_{11} by decomposing it with respect to the subgroup $SL(D) \times G_D$, where G_D denotes the duality group. For instance, decomposing E_{11} with respect to $SL(11)$ (since there is no duality group in $D = 11$), results at low levels in the following representations

$$\begin{aligned} \ell = 0 : & \qquad K^a{}_b \ , \\ \ell = 1 : & \qquad R^{a_1 \cdots a_3} \ , \\ \ell = 2 : & \qquad R^{a_1 \cdots a_6} \ , \\ \ell = 3 : & \qquad R^{a_1 \cdots a_8, b} \ , \end{aligned} \tag{7}$$

where at level $\ell = 0$ we find the $SL(11)$ subgroup associated to the metric, spanned by $K^a{}_b$, $a, b, \ldots = 1, \ldots, 11$, and at higher level Young-tableaux representations of $SL(11)$. In particular, at level $\ell = 1$ one obtains a 3-form, in agreement with the 3–form potential of 11-dimensional supergravity. That the bosonic field content is correctly reproduced in this way is to be expected, since it was the 3–form to begin with that gave rise to, say, E_7, which is now realized as a subgroup. At higher levels one encounters a 6–form, which is interpreted as the dual of the 3–form as well as a mixed Young–tableau field, which is interpreted as the dual of the graviton. This scheme extends to level decompositions for other dimensions as well in that it reproduces the supergravity spectra in a democratic formulation, in which each field appears together with its dual. However, we should note that the question of how to make this connection between infinite-dimensional Kac-Moody algebras and supergravity precise is not uncontroversial. On the one hand, it has been suggested that 11-dimensional supergravity can be interpreted in a covariant way as a non-linear realization of E_{11} (in a suitable sense) [1], while on the other hand a one-dimensional σ–model based on E_{10} has been discussed, which aims to reproduce the supergravity dynamics in a non-covariant, gauge-fixed formulation [2]. (We will have to say more about this latter proposal in sec. 3.3.) However, it is probably fair to say that so far a conclusive picture of how to implement the Kac-Moody algebra in supergravity or extensions thereof is lacking. One way to enrich our understanding of this correspondence has been the study of gauged supergravity, to which we will turn now. First it uncovers a connection between gaugings of supergravity and E_{11} beyond the regime reviewed above and, second, pushes the interpretation of the (infinite) tables of Kac-Moody level decompositions a bit further.

3 Gauged supergravity

3.1 Gauged supergravity and the embedding tensor

To begin with, we briefly review the embedding tensor formalism, which allows to relate possible gaugings of supergravity to E_{11}. We will specialize to $D = 3$ [9, 10], as this is the case we will analyze in detail below.

In three dimensions, vectors are dual to scalars, and the metric does not carry degrees of freedom. Therefore, in order to exhibit the E_8 symmetry, all bosonic degrees of freedom have to reside in scalar fields. This in turn seems to rule out the possibility of deforming theories in this 'duality symmetric' formulation by introducing gauge groups, since there are simply no gauge fields left to perform the gauging. However, in [9, 10] it has been shown how to avoid this problem: One needs to introduce 248 additional vector fields transforming in the adjoint of E_8 by hand, which enter, however, not via a Yang-Mills term, but only via a topological Chern-Simons term. Consequently, they do not change the counting of degrees of freedom, as required by supersymmetry.

As usual, the gauging requires the introduction of covariant derivatives according to

$$\partial_\mu \longrightarrow D_\mu \;=\; \partial_\mu - g\Theta_{\mathcal{MN}} A_\mu{}^{\mathcal{N}} t^{\mathcal{M}} \;,$$

where $\mathcal{M},\mathcal{N},\ldots = 1,\ldots,248$ are adjoint E_8 indices, whose generators are denoted by $t^{\mathcal{M}}$. Moreover, $A_\mu{}^{\mathcal{M}}$ are the corresponding gauge vectors (with coupling constant g), and $\Theta_{\mathcal{MN}}$ denotes the so-called embedding tensor. The latter is a convenient tool in order to treat all possible gaugings on the same footing. Technically, it encodes the subgroup of the global symmetry E_8 which is gauged in that the symmetry generators $X_{\mathcal{M}}$ of the gauge group are given by

$$X_{\mathcal{M}} \;=\; \Theta_{\mathcal{MN}} t^{\mathcal{N}} \;. \tag{8}$$

In particular, the rank of $\Theta_{\mathcal{MN}}$ determines the dimension of the gauge group. In total, the gauged supergravity Lagrangian is completely determined by the embedding tensor $\Theta_{\mathcal{MN}}$ and is given by

$$\begin{aligned} \mathcal{L}_g \;=\; & -\tfrac{1}{4}eR + \tfrac{1}{4}eP^{\mu A}P_{\mu A} - eV \\ & -\tfrac{1}{4}g\varepsilon^{\mu\nu\rho} A_\mu{}^{\mathcal{M}}\Theta_{\mathcal{MN}}(\partial_\nu A_\rho{}^{\mathcal{N}} - \tfrac{1}{3}g\Theta_{\mathcal{KS}} f^{\mathcal{NS}}{}_{\mathcal{L}} A_\nu{}^{\mathcal{K}} A_\rho{}^{\mathcal{L}}) + \mathcal{L}_{\text{fermions}} \;, \end{aligned} \tag{9}$$

where the $P_\mu{}^A$ are the non-compact part of the Maurer-Cartan forms of E_8, which encodes the dynamics of the scalar fields. The scalar potential V is given by

$$V = \tfrac{1}{32} g^2 G^{\mathcal{MN},\mathcal{KL}} \Theta_{\mathcal{MN}} \Theta_{\mathcal{KL}} \;, \tag{10}$$

where

$$G^{\mathcal{MN},\mathcal{KL}} \;=\; \tfrac{1}{14} G^{\mathcal{MK}} G^{\mathcal{NL}} + G^{\mathcal{MK}} \eta^{\mathcal{NL}} - \tfrac{3}{14}\eta^{\mathcal{MK}}\eta^{\mathcal{NL}} - \tfrac{4}{6727}\eta^{\mathcal{MN}}\eta^{\mathcal{KL}} \;, \tag{11}$$

with $G = \mathcal{V}\cdot\mathcal{V}^T$ denoting the 'E_8 metric' obtained from an scalar dependent E_8 representative $\mathcal{V}$. This action is invariant under the gauge symmetries determined by $\Theta_{\mathcal{MN}}$ and under (a deformation of) local supersymmetry, provided the embedding tensor satisfies the following constraints. First, gauge invariance requires invariance of $\Theta_{\mathcal{MN}}$ under the adjoint action of the gauge group generators (8), which takes the form of a quadratic constraint,

$$\mathcal{Q}_{\mathcal{MN},\mathcal{P}} \;\equiv\; \Theta_{\mathcal{KP}} \Theta_{\mathcal{L}(\mathcal{M}} f^{\mathcal{KL}}{}_{\mathcal{N})} \;=\; 0 \;. \tag{12}$$

Second, supersymmetry requires absence of certain irreducible E_8 representations of $\Theta_{\mathcal{MN}}$. Due to the symmetry of $\Theta_{\mathcal{MN}}$, it lives in the symmetric tensor product

$$(\mathbf{248}\otimes\mathbf{248})_{\text{sym}} \;=\; \underline{\mathbf{1}} \oplus \underline{\mathbf{3875}} \oplus \mathbf{27000} \;, \tag{13}$$

but only the underlined representations are consistent with supersymmetry. To summarize, any embedding tensor satisfying the E_8 covariant constraints (12) and (13) gives rise to a consistent gauged supergravity. However, we should stress that in spite of the 'E_8 covariant' form of the action and the constraints, E_8 is no longer a symmetry of gauged supergravity, simply due to the fact, that as a set of constants $\Theta_{\mathcal{MN}}$ cannot transform under the duality group. To put it differently, E_8 rotates different embedding tensors into each other, or in other words, it acts on the 'space of gauged supergravities'. In general, this formal symmetry can be extended to a genuine symmetry at the level of the action by introducing $(D-1)$– and D–forms as Lagrange multiplier. Precisely these fields provide the link to E_{11} and so we will turn to them now.

3.2 Deformation potentials and top-form potentials

In order to promote E_8 to a true physical symmetry, the embedding tensor $\Theta_{\mathcal{MN}}$ has to be promoted to a physical (scalar) field, such that it can transform under E_8 according to its index structure. However, the replacement of $\Theta_{\mathcal{MN}}$ by scalar fields $\Theta_{\mathcal{MN}}(x)$ will violate the gauge invariance and supersymmetry by terms proportional to $\partial_\mu\Theta_{\mathcal{MN}}$. Moreover, in order to deal with *unconstrained* fields, the quadratic constraint should be relaxed and only imposed by means of Lagrange multipliers. In total, we extend the action to [11, 12]

$$\mathcal{L}_{\rm tot} \;=\; \mathcal{L}_g + \tfrac{1}{4}g\varepsilon^{\mu\nu\rho}D_\mu\Theta_{\mathcal{MN}}B_{\nu\rho}{}^{\mathcal{MN}} - \tfrac{1}{6}g^2\Theta_{\mathcal{KP}}\Theta_{\mathcal{L}(\mathcal{M}}f^{\mathcal{KL}}{}_{\mathcal{N})}\varepsilon^{\mu\nu\rho}C_{\mu\nu\rho}{}^{\mathcal{MN},\mathcal{P}}\;, \qquad (14)$$

where we introduced 2–forms $B_{\mu\nu}{}^{\mathcal{MN}}$ and 3–forms $C_{\mu\nu\rho}{}^{\mathcal{MN},\mathcal{K}}$. Here, B satisfies the linear constraint (13), i.e., does not contain the **27000**, while C also carries only a subset of the possible irreducible representations (see below). This construction is possible in any dimension, where they would correspond to $(D-1)$– and D–forms (the deformation and top-form potentials in the nomenclature of [7]). The equations of motion in turn imply that $\Theta_{\mathcal{MN}}$ is constant together with the quadratic constraint. Thus, one recovers the original gauged supergravity, and the invariance of (14) under *global* E_8 rotations is spontaneously broken. In order for (14) to be invariant under gauge transformations and supersymmetry, certain gauge and supersymmetry variations have to be assigned to the 2– and 3–form. That this is always possible follows from the fact that the violation of these symmetries has to be proportional to either the quadratic constraint or $\partial_\mu\Theta_{\mathcal{MN}}$.

Explicitly, one finds that the bosonic gauge symmetries are given by [11]

$$\begin{aligned}
\delta A_\mu{}^{\mathcal{M}} &= D_\mu\Lambda^{\mathcal{M}} - g\Theta_{\mathcal{NK}}f^{\mathcal{MN}}{}_{\mathcal{L}}\Lambda_\mu{}^{\mathcal{KL}}\;, \qquad (15)\\
\delta B_{\mu\nu}{}^{\mathcal{MN}} &= D_{[\mu}\Lambda_{\nu]}{}^{\mathcal{MN}} + \delta A_{[\mu}{}^{\langle\mathcal{M}}A_{\nu]}{}^{\mathcal{N}\rangle} - \Lambda^{\langle\mathcal{M}}\tilde{J}_{\mu\nu}{}^{\mathcal{N}\rangle}\\
&\quad + \tfrac{2}{3}g\Theta_{\mathcal{KL}}f^{\mathcal{K}\langle\mathcal{M}}{}_{\mathcal{P}}\big(\Lambda_{\mu\nu}{}^{|\mathcal{LP}|,\mathcal{N}\rangle} - \Lambda_{\mu\nu}{}^{\mathcal{N}\rangle\mathcal{P},\mathcal{L}}\big)\;,\\
\delta C_{\mu\nu\rho}{}^{\mathcal{MN},\mathcal{P}} &= D_{[\mu}\Lambda_{\nu\rho]}{}^{\mathcal{MN},\mathcal{P}} - 3\,\delta A_{[\mu}{}^{\langle\mathcal{P}}B_{\nu\rho]}{}^{\mathcal{MN}\rangle} + A_{[\mu}{}^{\langle\mathcal{P}}A_\nu{}^{\mathcal{M}}\delta A_{\rho]}{}^{\mathcal{N}\rangle}\\
&\quad + \tfrac{3}{2}\Lambda_{[\mu}{}^{\langle\mathcal{MN}}\tilde{J}_{\nu\rho]}{}^{\mathcal{P}\rangle} + \tfrac{1}{16}ge\varepsilon_{\mu\nu\rho}\Lambda^{\langle\mathcal{P}}\big(-\tfrac{1}{7}G^{\mathcal{M}|\mathcal{K}|}G^{\mathcal{N}\rangle\mathcal{L}} - G^{\mathcal{M}|\mathcal{K}|}\eta^{\mathcal{N}\rangle\mathcal{L}}\big)\Theta_{\mathcal{KL}}\;,
\end{aligned}$$

where $\tilde{J}_{\mu\nu}{}^{\mathcal{M}}$ is the dual of the E_8 'Noether' current, and the brackets $\langle\;\rangle$ project onto the representations required by the left-hand sides. Here we assume that the embedding tensor is still invariant under *local* transformations, $\delta_\Lambda\Theta_{\mathcal{MN}} = 0$.

We would like to stress that these gauge transformations represent a rather non-trivial structure as their closure requires substantial input from the equations of motion. In fact, closure of the gauge algebra is not only up to the first–order duality relations, but requires also the second–order scalar field equations. Moreover, the field equations of $\Theta_{\mathcal{MN}}$ imply a generalized duality relation between the 'field strength' of $B_{\mu\nu}{}^{\mathcal{MN}}$ and the embedding tensor,

$$e^{-1}\varepsilon^{\mu\nu\rho}G_{\mu\nu\rho}{}^{\mathcal{MN}} + 2A_\mu{}^{\langle\mathcal{M}}J^{\mu\mathcal{N}\rangle} \;=\; \tfrac{1}{4}gG^{\mathcal{MN},\mathcal{KL}}\Theta_{\mathcal{KL}}\;, \qquad (16)$$

where the first terms of $G_{\mu\nu\rho}{}^{\mathcal{MN}}$ are given by

$$G_{\mu\nu\rho}{}^{\mathcal{MN}} \;=\; D_{[\mu}B_{\nu\rho]}{}^{\mathcal{MN}} + A_{[\mu}{}^{\langle\mathcal{M}}\partial_\nu A_{\rho]}{}^{\mathcal{N}\rangle} - 2g\Theta_{\mathcal{KL}}f^{\mathcal{K}\langle\mathcal{M}}{}_{\mathcal{P}}A_{[\mu}{}^{\mathcal{N}\rangle}B_{\nu\rho]}{}^{\mathcal{LP}} + \cdots\;. \qquad (17)$$

However, as is already manifest from the explicit presence of the gauge vectors in (16), this duality relation is not manifestly gauge–covariant. Moreover, the tensor defined in (17) is *not* a covariant field strength. In fact, the full equation (16) transforms in a highly non-trivial manner into the other equations of motion, including the second–order scalar equations [11]!

3.3 E_{11} and gauged supergravity

Let us now turn to the relation between gauged supergravity and E_{11}. Performing a level decomposition of E_{11} with respect to $SL(3)\times E_8$, encodes by definition at level $\ell = 0$ the E_8–valued scalar fields and the (topological) metric. We will see that beyond this 'easy' part of the dictionary, E_{11} contains at higher levels information about possible gaugings. The level

Level	$SL(3)\times E_8$ representation	Generator
1	$(\mathbf{3},\mathbf{248})$	$X^{\mu}{}_{\mathcal{M}}$
2	$(\bar{\mathbf{3}},\mathbf{1}\oplus\mathbf{3875})$	$Y^{\mu\nu}{}_{\mathcal{M}\mathcal{N}}$
3	$(\mathbf{1},\mathbf{248}\oplus\mathbf{3875}\oplus\mathbf{147250})$	$Z^{\mu\nu\rho}{}_{\mathcal{M}\mathcal{N},\mathcal{P}}$

Table 1: $SL(3)\times E_8$ representations within E_{11} up to level 3, of which the $SL(3)$ part is totally antisymmetric.

decomposition up to and including $\ell=3$ can be found in the table, in which we performed a p-form truncation, restricting only to those generators which are completely antisymmetric in their $SL(3)$ indices. We observe that the field content is as required from gauged supergravity after introducing the deformation and top-form potentials. And moreover, the representations of the 2–forms are in precise agreement, thus reproducing the linear constraint of gauged supergravity! Finally, the representations of the 3–form are consistent with the quadratic constraint, though within E_{11} they contain an additional **248**, whose interpretation is unknown. This pattern extends to higher dimensions and massive deformations as well. Thus, E_{11} precisely encodes also the possible deformations of supergravity into gauged or massive supergravity!

Next we are going to compare also the symmetry transformations on both sides of the correspondence. For E_{11} we assume a non-linear realization of the symmetry. Explicitly, this can be realized by introducing the coset representative

$$\mathcal{V} \;=\; \exp\left(A_\mu{}^{\mathcal{M}}X^{\mu}{}_{\mathcal{M}}+B_{\mu\nu}{}^{\mathcal{M}\mathcal{N}}Y^{\mu\nu}{}_{\mathcal{M}\mathcal{N}}+C_{\mu\nu\rho}{}^{\mathcal{M}\mathcal{N},\mathcal{P}}Z^{\mu\nu\rho}{}_{\mathcal{M}\mathcal{N},\mathcal{P}}\right)\,, \tag{18}$$

and acting with a group element,

$$g \;=\; \exp\left(\Lambda_\mu{}^{\mathcal{M}}X^{\mu}{}_{\mathcal{M}}+\Lambda_{\mu\nu}{}^{\mathcal{M}\mathcal{N}}Y^{\mu\nu}{}_{\mathcal{M}\mathcal{N}}+\Lambda_{\mu\nu\rho}{}^{\mathcal{M}\mathcal{N},\mathcal{P}}Z^{\mu\nu\rho}{}_{\mathcal{M}\mathcal{N},\mathcal{P}}\right)\,. \tag{19}$$

We should note that here we restricted to fields corresponding to the positive-level generators in the table (usually referred to as Borel gauge-fixing) and similarly for the group element. The latter choice corresponds to a truncation of the symmetry algebra, which turns out to be the one which can be identified within supergravity. Computing the transformed fields according to $\mathcal{V}'=g\mathcal{V}$ by use of the E_{11} algebra, one finds

$$\begin{aligned}
\delta A_\mu{}^{\mathcal{M}} &= \Lambda_\mu{}^{\mathcal{M}}\,,\\
\delta B_{\mu\nu}{}^{\mathcal{M}\mathcal{N}} &= \Lambda_{\mu\nu}{}^{\mathcal{M}\mathcal{N}}+\Lambda_{[\mu}{}^{\langle\mathcal{M}}A_{\nu]}{}^{\mathcal{N}\rangle}\,,\\
\delta C_{\mu\nu\rho}{}^{\mathcal{M}\mathcal{N},\mathcal{P}} &= \Lambda_{\mu\nu\rho}{}^{\mathcal{M}\mathcal{N},\mathcal{P}}-\tfrac{3}{2}B_{[\mu\nu}{}^{\langle\mathcal{M}\mathcal{N}}\Lambda_{\rho]}{}^{\mathcal{P}\rangle}+\tfrac{3}{2}\Lambda_{[\mu\nu}{}^{\langle\mathcal{M}\mathcal{N}}A_{\rho]}{}^{\mathcal{P}\rangle}-\tfrac{1}{2}A_{[\mu}{}^{\langle\mathcal{M}}\Lambda_{\nu}{}^{\mathcal{N}}A_{\rho]}{}^{\mathcal{P}\rangle}\,,
\end{aligned} \tag{20}$$

which represents a *rigid* symmetry.

In order to make contact with the symmetries of gauged supergravity, we first note that the full bosonic gauge transformations cannot be obtained, since they are first *local* and, second, contain the embedding tensor, which appears on the E_{11} side only through their dual 2–forms. Thus, at best we can expect a certain truncation of supergravity to be in agreement with E_{11}. To see that this is indeed the case, we perform a certain limit within supergravity, leaving us with an extended version of ungauged supergravity. We rescale the fields according to

$$A_\mu{}^{\mathcal{M}}\to g^{1/2}A_\mu{}^{\mathcal{M}}\,,\qquad B_{\mu\nu}{}^{\mathcal{M}\mathcal{N}}\to gB_{\mu\nu}{}^{\mathcal{M}\mathcal{N}}\,,\qquad C_{\mu\nu\rho}{}^{\mathcal{M}\mathcal{N},\mathcal{P}}\to g^{3/2}C_{\mu\nu\rho}{}^{\mathcal{M}\mathcal{N},\mathcal{P}}\,, \tag{21}$$

and then take the limit $g\to 0$. This yields the Lagrangian,

$$\mathcal{L} \;=\; \mathcal{L}_0-\tfrac{1}{4}\varepsilon^{\mu\nu\rho}\Theta_{\mathcal{M}\mathcal{N}}G^{(0)}_{\mu\nu\rho}{}^{\mathcal{M}\mathcal{N}}\,, \tag{22}$$

where $\mathcal{L}_0$ denotes the standard Lagrangian of ungauged supergravity. Here, $G^{(0)}_{\mu\nu\rho}{}^{\mathcal{MN}}$ is the $g \to 0$ limit of $G_{\mu\nu\rho}{}^{\mathcal{MN}}$, given by

$$G^{(0)}_{\mu\nu\rho}{}^{\mathcal{MN}} \;=\; \partial_{[\mu}B_{\nu\rho]}{}^{\mathcal{MN}} + A_{[\mu}{}^{\langle\mathcal{M}}\partial_{\nu}A_{\rho]}{}^{\mathcal{N}\rangle} \;. \tag{23}$$

This Lagrangian is equivalent to ungauged supergravity, since it merely represents an extension by topological 1– and 2–forms with zero-curvatures. In this limit the gauge transformations reduce to

$$\begin{aligned}
\delta_\Lambda A_\mu{}^{\mathcal{M}} &= \partial_\mu \Lambda^{\mathcal{M}} \;, \\
\delta_\Lambda B_{\mu\nu}{}^{\mathcal{MN}} &= \partial_{[\mu}\Lambda_{\nu]}{}^{\mathcal{MN}} + \partial_{[\mu}\Lambda^{\mathcal{M}} A_{\nu]}{}^{\mathcal{N}} \;, \\
\delta_\Lambda \hat{C}_{\mu\nu\rho}{}^{\mathcal{MN},\mathcal{P}} &= \partial_{[\mu}\Lambda_{\nu\rho]}{}^{\mathcal{MN},\mathcal{P}} - \tfrac{3}{2}\partial_{[\mu}\Lambda^{\mathcal{P}}B_{\nu\rho]}{}^{\mathcal{MN}} + \tfrac{3}{2}\partial_{[\mu}\Lambda_{\nu}{}^{\mathcal{MN}}A_{\rho]}{}^{\mathcal{P}} - \tfrac{1}{2}A_{[\mu}{}^{\mathcal{P}}A_{\nu}{}^{\mathcal{M}}\partial_{\rho]}\Lambda^{\mathcal{N}} \;,
\end{aligned} \tag{24}$$

such that the dependence on the embedding tensor completely disappears. Here, we performed a field redefinition, shifting $C_{\mu\nu\rho}{}^{\mathcal{MN},\mathcal{K}}$ by $\frac{3}{2}A_{[\mu}{}^{\langle\mathcal{P}}B_{\nu\rho]}{}^{\mathcal{MN}\rangle}$. Restricting the gauge parameter now to linear space-time dependence according to

$$\Lambda^{\mathcal{M}} \;=\; x^\rho \Lambda_\rho{}^{\mathcal{M}} \;, \qquad \Lambda_\mu{}^{\mathcal{MN}} \;=\; x^\rho \Lambda_{\rho\mu}{}^{\mathcal{MN}} \;, \qquad \Lambda_{\mu\nu}{}^{\mathcal{MN},\mathcal{P}} \;=\; x^\rho \Lambda_{\rho\mu\nu}{}^{\mathcal{MN},\mathcal{P}} \;, \tag{25}$$

gives exactly the global symmetry in (20) predicted by E_{11}. Thus, an extended version of ungauged supergravity can be matched to a (truncation of) E_{11}. However, the higher-order terms in g and Θ in the gauge variations cannot be obtained in this way. One might speculate that a further extension of E_{11} can reproduce these missing terms. It is, however, clear that simply extending the Lie algebra to a larger algebra and computing the non-linear realization with respect to this algebra cannot achieve this, due to the fact that these symmetries would still close off-shell, while the true symmetries contain intriguing information about the equations of motion, as we saw above. Therefore, one feels that only a somewhat unconventional approach can cure these problems.

3.4 E_{10} coset dynamics

Here we are going to briefly discuss a possible realization of gauged supergravity within the E_{10} model of [2]. We will see that it is able to circumvent some of the problems encountered above.

The model is a *one-dimensional* σ–model based on the infinite-dimensional coset space $E_{10}/K(E_{10})$, where $K(E_{10})$ is the compact subgroup of E_{10}. Thus, the action reads

$$S_{E_{10}/K(E_{10})} \;=\; \tfrac{1}{4}\int dt\, n(t)^{-1}\,(\mathcal{P}(t)|\mathcal{P}(t)) \;, \tag{26}$$

where we introduced the Maurer-Cartan forms:

$$\mathcal{V}^{-1}\partial_t\mathcal{V} \;=\; \mathcal{P}(t) + \mathcal{Q}(t) \;, \quad \mathcal{P} \in \mathfrak{e}_{10} \ominus \mathfrak{k}(\mathfrak{e}_{10}) \;, \quad \mathcal{Q} \in \mathfrak{k}(\mathfrak{e}_{10}) \;, \tag{27}$$

and $n(t)$ is a lapse function establishing one-dimensional diffeomorphism invariance. It has been shown in a number of cases that this model can account for the dynamics of supergravity in a certain truncation and after performing some gauge-fixing (see, e.g., [3]). For instance, gravity truncated to only a time-like system and in which the shift-functions N^i appearing in the ADM decomposition have been gauge-fixed to zero, only contains the fields $e_i{}^\alpha(t)$ and $n(t)$, which are the spatial part of the vielbein and the usual lapse function. The corresponding truncation of the Einstein-Hilbert action is equivalent to the (one-dimensional) non-linear σ–model based on $GL(d)/SO(d)$, where $d = D-1$. As $GL(d)$ appears as a subgroup of E_{10} (instead of the larger $GL(D)$ for E_{11}), the action (26) manifestly reproduces the correct gravitational dynamics at low levels. Moreover, this also holds for maximal supergravity, when the duality subgroups are properly taken into account in the level decomposition. Finally let us note that the conjecture is that the spatial derivatives are encoded in the higher-level representations.

Let us now turn to the question whether the E_{10} model is also capable of reproducing gauged supergravity. As in E_{11}, the level decomposition — here with respect to the subgroup $GL(2)\times E_8$ — contains a 2-form in the correct representation. However, E_{10} cannot accommodate the 3–form and thus the information about quadratic constraints. The Maurer-Cartan forms at level $\ell = 2$ are in turn given by

$$\mathcal{V}^{-1}\partial_t\mathcal{V}|_{\ell=2} \;=\; \left(\partial_t B_{ij}{}^{\mathcal{M}\mathcal{N}} - A_{[i}{}^{\mathcal{M}}\partial_t A_{j]}{}^{\mathcal{N}}\right) Y^{ij}{}_{\mathcal{M}\mathcal{N}} \;\equiv\; D_t B_{ij}{}^{\mathcal{M}\mathcal{N}} Y^{ij}{}_{\mathcal{M}\mathcal{N}}\;, \tag{28}$$

where we used the same nomenclature for the generators as in the E_{11} table above. Comparing this with (17) one infers that, as in the case of E_{11}, this does not reproduce the terms proportional to $\Theta_{\mathcal{M}\mathcal{N}}$. However, in the context of E_{10} this can be interpreted as another gauge-fixing: Due to the non-covariant formulation, we may simply impose the gauge

$$A_0{}^{\mathcal{M}} \;=\; 0\;, \tag{29}$$

which is also required by the fact that the spectrum contains at level $\ell = 1$ only 1-form potentials with respect to $GL(2)$, i.e., with indices $i, j, \ldots = 1, 2$. After this gauge fixing, the problematic terms disappear, and so the E_{10} model is in principle able to be in agreement with gauged supergravity. Moreover, the coset equations of motion for $B_{ij}{}^{\mathcal{M}\mathcal{N}}$ are such that they can be solved in closed form to yield

$$n^{-1}g^{ik}g^{jl}G_{\mathcal{M}\mathcal{K}}G_{\mathcal{N}\mathcal{L}}D_t B_{kl}{}^{\mathcal{K}\mathcal{L}} \;=\; \tfrac{1}{4}g\epsilon^{ij}\Theta_{\mathcal{M}\mathcal{N}}\;, \tag{30}$$

which naturally introduces the embedding tensor into the game. This has the same structure as the duality relation (16) between the deformation potential and the embedding tensor. Also the other equations of motion take a corresponding form on both sides of the correspondence. However, by comparing (30) and (16) in more detail one concludes that only the positive-definite term in $G^{\mathcal{M}\mathcal{N},\mathcal{K}\mathcal{L}}$ proportional to $G^{\mathcal{M}\mathcal{K}}G^{\mathcal{N}\mathcal{L}}$ is reproduced by E_{10}. In other words, the indefinite potential of gauged supergravity is not predicted by E_{10}. Similar discrepancies appear for the match between ungauged supergravity and E_{10} once spatial derivatives are taken into account in that also a certain indefiniteness of the Einstein-Hilbert term does seem to be derivable from this simple coset model. It is, however, striking that so many aspects of supergravity and even its massive deformations are contained in this naive coset space ansatz given in (26). A more exhaustive analysis of the correspondence in case of three-dimensional gauged supergravity will appear elsewhere [13].

4 Conclusions

We reviewed the appearance of hidden symmetries and the associated conjecture of Kac-Moody symmetries in higher dimensions in the context of gauged supergravity. Specifically, we reviewed the structure of deformation and top–form potentials as predicted by E_{11} and compared with their presence in gauged supergravity. While the linear constraints (and to some extend also the quadratic constraints) are correctly encoded in E_{11}, the precise form of the symmetry transformations can only be obtained from the Kac-Moody algebras in a certain limit to (extended) ungauged supergravity. Moreover, the symmetries of the $(D-1)$– and D–forms express a highly non-trivial on-shell structure, which in turn poses serious obstructions to any purely kinematical attempt. Finally, we compared the E_{10} coset dynamics with gauged supergravity and found that due to its non-covariant formulation it is better suited for dealing with the mentioned discrepancies. However, also here there are certain disagreements. We conclude that while a precise understanding of a Kac-Moody/supergravity correspondence is still not in sight, the surprising connection even to gauged supergravity strongly advocates further research.

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