Ultra-hard fluid and scalar field in the Kerr-Newman metric

E. Babichev[ab]*, S. Chernov[bc]†, V. Dokuchaev[b]‡ and Yu. Eroshenko[b]§
[a] *APC, Universite Paris 7, rue Alice Domon Duquet, 75205 Paris Cedex 13, France*
[b] *Institute for Nuclear Research of the Russian Academy of Sciences*
Prospekt 60-letiya Oktyabrya 7a, Moscow 117312, Russia
[c] *P. N. Lebedev Physical Institute of the Russian Academy of Sciences*
Leninsky Prospekt 53, 119991 Moscow, Russia

Abstract

It is found the generalization of Petrich, Teukolsky and Shapiro solution for stationary accretion of perfect fluid with the ultra-hard equation of state onto the moving Kerr black hole to the Kerr-Newman case. In the case of extreme black hole the energy density of accreting ultra hard fluid is diverging at the event horizon and, thus, the test fluid approximation is violated. Additionally, we find solution for the stationary atmosphere around the Kerr-Newman naked singularity without the influx. There is also solution for a static scalar field with a finite total mass around the Kerr-Newman naked singularity which does not have a perfect fluid analogue.

1 Introduction

The only known three dimensional exact solution for an accretion flow onto the Kerr black hole is the analytical solution by Petrich, Shapiro and Teukolsky [1]. This solution describes the stationary accretion of ultra-hard perfect fluid with the equation of state $p = \rho$, where p is a pressure and ρ is an energy density, onto the moving Kerr black hole [1, 2, 3, 4]. Here we generalize this solution to the Kerr-Newman case. Additionally, we find the stationary distribution of ultra-hard fluid with a zero influx around the Kerr-Newman naked singularity.

Velocity of relativistic perfect fluid in General Relativity in the absence of vorticity is expressed as the gradient of a potential [5]:

$$hu_\mu = \psi_{,\mu}, \tag{1}$$

where n is a particle number-density and $h = d\rho/dn = (p + \rho)/n$ is a fluid enthalpy with normalization $h = (\psi^{,\alpha}\psi_{,\alpha})^{1/2}$. In the unique case of the ultra-hard equation of state with $p = \rho$, it can be deduced that $|p + \rho| \propto n^2$. As a result, as was demonstrated at first in [7], the continuity equation $(nu^\alpha)_{;\alpha} = 0$ is reduced to the linear Klein-Gordon equation for a massless scalar field $\psi^{;\alpha}{}_{;\alpha} = 0$. This reveals a way for an exact solution of the stationary accretion problem in the Kerr-Newman metric by separation of variables and decomposition in spherical harmonic series. Our consideration of the Kerr-Newman case follows in general to the logic of paper [1].

* **e-mail**: babichev@apc.univ-paris7.fr
† **e-mail**: chernov@lpi.ru
‡ **e-mail**: dokuchaev@ms2.inr.ac.ru
§ **e-mail**: eroshenko@ms2.inr.ac.ru

The Kerr-Newman metric in the standard form is

$$ds^2 = \frac{\Sigma\Delta}{\mathcal{A}}dt^2 - \frac{\mathcal{A}\sin^2\theta}{\Sigma}(d\phi - \omega dt)^2 - \frac{\Sigma}{\Delta}dr^2 - \Sigma d\theta^2, \qquad (2)$$

where

$$\Delta = r^2 - 2Mr + a^2 + Q^2; \qquad (3)$$
$$\Sigma = r^2 + a^2\cos^2\theta \qquad (4)$$
$$\mathcal{A} = (r^2 + a^2)^2 - a^2\Delta\sin^2\theta \qquad (5)$$
$$\omega = \frac{2Mr - Q^2}{\mathcal{A}}a. \qquad (6)$$

Here M is a mass of the black hole or naked singularity, a is a specific angular momentum, Q is an electric charge and ω is an angular dragging velocity. The event horizons of the Kerr-Newman black hole $r_\pm = M \pm \sqrt{M^2 - a^2 - Q^2}$ are a roots of the equation $\Delta = 0$. The event horizon exists if $M^2 \geq a^2 + Q^2$. In the case $M^2 < a^2 + Q^2$ the event horizon is absent and metric (2) describes the naked singularity.

The corresponding Klein-Gordon equation for a massless scalar field in the background gravitational field $g_{\alpha\beta}$ is

$$\psi^{;\alpha}_{\ ;\alpha} = \frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\alpha}\left(\sqrt{-g}g^{\alpha\beta}\frac{\partial\psi}{\partial x^\beta}\right) = 0. \qquad (7)$$

In the Kerr-Newman metric (2) this wave equation can be expanded as

$$\left\{\frac{1}{\Delta}\left[(r^2+a^2)\partial_t + a\partial_\phi\right]^2 - \frac{1}{\sin^2\theta}\left(\partial_\phi + a\sin^2\theta\partial_t\right)^2 - \partial_r(\Delta\partial_r) - \frac{1}{\sin\theta}\partial_\theta(\sin\theta\,\partial_\theta)\right\}\psi = 0. \qquad (8)$$

2 Ultra-hard fluid in the Kerr-Newman metric

2.1 Accretion onto moving black hole

We consider at first the stationary accretion of ultra-hard perfect fluid for the black hole case, i.e., when $M^2 \geq a^2 + Q^2$. The first boundary condition for equation (8) is a relation at the space infinity [1]:

$$\psi = -u^0_\infty t + u_\infty r[\cos\theta\cos\theta_0 + \sin\theta\sin\theta_0\cos(\phi - \phi_0)], \quad r \to \infty, \qquad (9)$$

where u^0_∞ is a 0-component of the black hole 4-velocity at infinity

$$u^\mu = (u^0, \mathbf{u}_\infty) = (1 - v^2)^{-1/2}(1, \mathbf{v}_\infty) \qquad (10)$$

and $\mathbf{v}_\infty$ is an ordinary 3-velocity vector of the black hole at infinity with the space orientation specified by the two arbitrary angles, θ_0 and ϕ_0. The second boundary condition for the equation (8) is a requirement of finiteness of the influx at a critical sound surface $r_* = r_+$, where the infall velocity of accreting fluid reaches the sound velocity c_s, which is the velocity of light for the considered fluid. We will show below in the Section 2.4 that there are two sound surfaces in general, which are the roots of equation $\Delta = 0$.

Following to [1], we are looking for the solution of equation (8) by separation of variables and decomposition in spherical harmonic series:

$$\psi = -u^0_\infty t + \sum_{l,m} A_{l,m}R_l(r)Y_{lm}(\theta, \phi), \qquad (11)$$

where the radial part $R_l(r)$ satisfies the equation

$$\frac{d}{dr}\left[\Delta\frac{dR_l(r)}{dr}\right]+\left[-l(l+1)+\frac{m^2a^2}{\Delta}\right]R_l(r)=0. \tag{12}$$

Defining a new variable ξ be relation $r = M + \xi\sqrt{M^2 - a^2 - Q^2}$, we rewrite the last equation in the form

$$(1-\xi^2)R''_{\xi\xi}-2\xi R'_{\xi}+\left[l(l+1)-\frac{m^2(i\alpha)^2}{1-\xi^2}\right]R=0, \tag{13}$$

where $\alpha = a/\sqrt{M^2 - a^2 - Q^2}$. This is the Legendre equation with an *imaginary* second index. The general solution of equation (11) is

$$\begin{aligned}\psi &= -u^0_\infty t+\sum_l[A_lP_l(\xi)+B_lQ_l(\xi)]Y_{l0}(\theta,\phi)\\ &\quad+\sum_{lm}{}'[A^+_{lm}P^{im\alpha}_l(\xi)+A^-_{lm}P^{-im\alpha}_l(\xi)]Y_{lm}(\theta,\phi).\end{aligned} \tag{14}$$

Here the prime ($'$) in the second sum denotes omission of the term with $m = 0$, respectively, P_l and Q_l are the Legendre functions of the first and second kind and $P^{im\alpha}_l$ is the associated Legendre function.

Using the first and second boundary conditions (9), we obtain finally (for details see [6])

$$\begin{aligned}\psi &= -u^0_\infty t+\frac{(r^2_++a^2)u^0_\infty}{2\sqrt{M^2-a^2-Q^2}}\ln\frac{r-r_-}{r-r_+}\\ &\quad+u_\infty(r-M)\cos\theta\cos\theta_0+u_\infty\mathrm{Re}[(r-M+ia)\sin\theta\sin\theta_0e^{i(\phi-\phi_0-\chi)}].\end{aligned} \tag{15}$$

This solution for the Kerr-Newman black hole is very similar to the Petrich, Shapiro and Teukolsky solution [1] for the Kerr case.

A stationary flux of the fluid onto the black hole (an accretion rate) is

$$\dot{N}=-\int_S nu^i\sqrt{-g}\,dS_I=-\int_S\psi_{,r}g^{rr}\sqrt{-g}\,d\Omega=4\pi(r^2_++a^2)n_\infty u^0_\infty. \tag{16}$$

While, the corresponding rate of the energy flux onto the black hole is

$$\dot{M}=4\pi(r^2_++a^2)(\rho_\infty+p_\infty)u^0_\infty. \tag{17}$$

As as well as in the Kerr case, the flux (16) is independent on the direction of the black hole motion.

We find the radial density distribution of the stationary accreting ultra-hard fluid around the non-moving Kerr-Newman black hole

$$n^2=\frac{\mathcal{A}-(r^2_++a^2)^2}{\Sigma\Delta}. \tag{18}$$

This radial distribution is shown in Fig. 1.

2.2 Accretion onto extreme black hole

Now we consider the case of the extreme Kerr-Newman black hole, when $M^2 = a^2 + Q^2$. From (15) it follows that a potential ψ is diverging in the extreme case. We will try to find the stationary solution of the wave equation (8) in the extreme case in the form (11) as in a general case. A corresponding radial part $R_l(r)$ satisfies now to the equation (12) with $\Delta = (r - M)^2$. By using variable $\xi = r/M - 1$ we transform (12) to a simple equation

$$\xi^2R''_{\xi\xi}+2\xi R'_{\xi}+\left[\frac{m^2a^2}{M^2\xi^2}-l(l+1)\right]R=0. \tag{19}$$

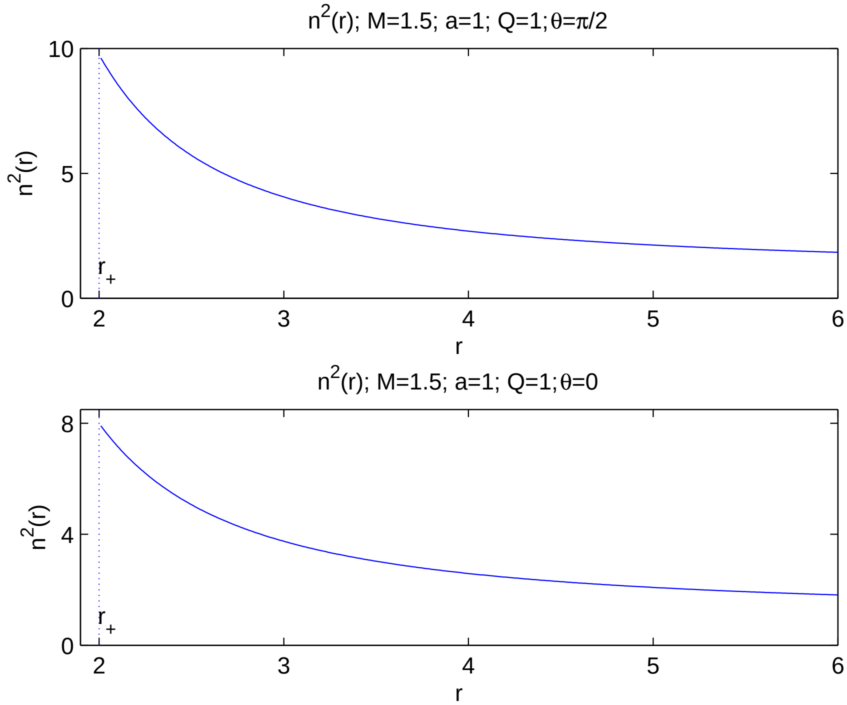

Figure 1: Radial density distribution of the stationary accreting ultra-hard fluid around the Kerr-Newman black hole with mass M=1.5, electric charge $Q = 1$ and specific angular momentum $a = 1$ (in arbitrary units) in the equatorial plane ($\theta = \pi/2$) and along the polar axis ($\theta = 0$).

The general solution of equation (19) in the extreme case is

$$\begin{aligned}\psi &= -u^0_\infty t + \sum_l \left[C^1_{l0} \left(\frac{r}{M}-1\right)^l + C^2_{l0}\left(\frac{r}{M}-1\right)^{-l-1}\right] Y_{l0} \\ &\quad + {\sum_{l,m}}' \sqrt{\frac{ma}{M\xi}} \left[C^{1+}_{lm} J_{l+1/2}\left(\frac{ma}{M\xi}\right) + C^{2-}_{lm} \Upsilon_{l+1/2}\left(\frac{ma}{M\xi}\right)\right] Y_{lm}. \end{aligned} \tag{20}$$

We will use for Bessel functions the following formulas:

$$\begin{aligned} J_\nu(x) &= \sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{\nu+2k}}{k!\Gamma(\nu+k+1)}, \quad \Upsilon_\nu(x) = \frac{J_\nu(x)\cos\pi\nu - J_{-\nu}(x)}{\sin\pi\nu}, \\ J_{-3/2}(x) &= \sqrt{\frac{2}{\pi x}} \left(-\frac{\cos x}{x} - \sin x\right). \end{aligned} \tag{21}$$

From the boundary condition at infinity it follows that $C^1_{l0} = 0$, $C^1_{10} = M u_\infty \cos\theta_0$, $C^{2-}_{1m} = 0$ and $C^{2-}_{11} = -\sqrt{\pi/2} u_\infty a \sin\theta_0$. Respectively, from the boundary condition at the sound surface we obtain $C^{1+}_{lm} = 0$, $C^2_{l0} = 0$ and $C^2_{00} = (M^2 + a^2)u^0_\infty/M$. With these boundary conditions we have

$$\begin{aligned} \psi &= -u^0_\infty t + u_\infty (r-M)\cos\theta\cos\theta_0 + \frac{(M^2+a^2)u^0_\infty}{r-M} \\ &\quad + u_\infty a \sin\theta \sin\theta_0 \cos(\phi-\phi_0)\left(\frac{r-M}{a}\cos\frac{a}{r-M} + \sin\frac{a}{r-M}\right). \end{aligned} \tag{22}$$

Using this solution, we calculate the accretion rate

$$\dot{N} = 4\pi (M^2 + a^2) u^0_\infty n_\infty \tag{23}$$

and radial density distribution (for $u_\infty = 0$)

$$n^2 = \frac{(r^2+a^2)^2 - (M^2+a^2)^2}{\Sigma(r-M)^2} - \frac{a^2\sin^2\theta}{\Sigma}. \tag{24}$$

The second boundary condition fixes the rate of inflow at the sound surface (23), which is stationary and independent on the radius of the sphere around the central source. However, in the extreme case at the sound surface both a radial component of 4-velocity u^r and density of matter $n(r)$ both behave unnaturally: $u^r \to 0$ and $n \to \infty$ at $r \to r_+ = M$, as it can be seen from (24). The divergence of density is indication of the violation of the test fluid approximation. For this reason the stationary accretion solution in the extreme case, (24) is not self-consistent, and back reaction of the accreting fluid must be taken into account.

2.3 Atmosphere without influx around naked singularity

In this Section we consider the stationary distribution of ultra-hard fluid around the Kerr-Newman naked singularity with $M^2 < a^2 + Q^2$. In this case it is useful to use a new variable $r = M + \xi\sqrt{a^2+Q^2-M^2}$ in the equation (12) for the radial part R of an effective potential ψ with $\Delta = (\xi^2+1)(a^2+Q^2-M^2)$. By using this variable, the equation (12) takes the form

$$(1-z^2)R''_{zz} - 2zR'_z + \left[l(l+1) - \frac{m^2\alpha^2}{1-z^2}\right]R = 0, \tag{25}$$

where $\xi = iz$ and $\alpha = a/\sqrt{Q^2+a^2-M^2}$. The general solution of this equation is

$$\psi = -u^0_\infty t + \sum_l (A_l P_l(z) + B_l Q_l(z)) Y_{l0}(\theta)$$
$$+ \sum_{lm}{}' (A_{lm} P_l^{m\alpha}(z) + B_{lm} P_l^{-m\alpha}(z)) Y_{lm}(\theta,\phi). \tag{26}$$

The boundary conditions in the case of naked singularity:

$$\psi|_{r\to\infty} = -u^0_\infty t;$$
$$\psi_{,r} = 0 \quad \text{or} \quad u_r = 0. \tag{27}$$

The second boundary condition in (27) corresponds to a stationary atmosphere with a zero influx of fluid. With these boundary conditions we obtain general solution

$$\psi = -u^0_\infty t \tag{28}$$

and

$$nu_t = -u^0_\infty = -1, \quad nu_r = nu_\theta = nu_\phi = 0. \tag{29}$$

For the radial density distribution we find

$$n^2 = \frac{\rho - \rho_0/2}{\rho_\infty - \rho_0/2} = \frac{\mathcal{A}}{\Sigma\Delta}. \tag{30}$$

How we can see from equation there are points when density distribution equals zero. So a physically reasonable that radial density distribution in the stationary atmosphere around the naked singularity is a sum of two solutions: (i) the solution (2.3) in the region, where $\mathcal{A} \geq 0$ and (ii) the trivial solution with $n^2 = 0$ inside the region confined by the surface $\mathcal{A} = 0$ (for more details about this region see [8]).

From equation (2.3) by putting $\mathcal{A} = 0$ we find the form of empty cavity around the central singularity $\theta = \theta_0(r)$, $0 \leq \phi \leq 2\pi$:

$$\sin\theta_0(r) = \frac{r^2+a^2}{a\Delta^{1/2}}. \tag{31}$$

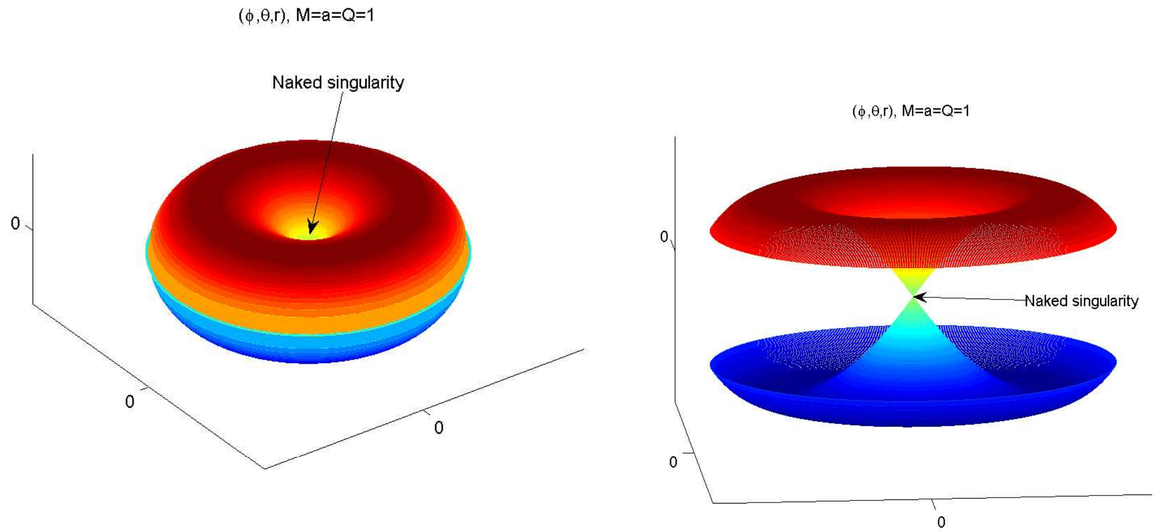

Figure 2: Three-dimensional view of a central empty region around the Kerr-Newman naked singularity with $a = Q = M$ (left panel). At the right panel is shown the same empty region with a removed part of outer boundary.

This empty cavity around the naked singularity exists only in the case when both $Q \neq 0$ and $a \neq 0$. See in Fig. 2 the three-dimensional view of central empty cavity around the Kerr-Newman naked singularity with $a = Q = M$. In the case of $Q \neq 0$ there is a central region around the naked singularity with an inverse behavior of the density distribution. This is a manifestation of repulsive properties of the charged naked singularity. The radial dependance of density along the polar axis ($\theta = 0$) is

$$n^2(r, \theta = 0) = \frac{r^2 + a^2}{\Delta}. \tag{32}$$

At the central singularity $r = 0$ this density is nonzero if $a \neq 0$. In particular, in the case of Reissner-Nordström naked singularity (a=0), the density of fluid at the central singularity is zero.

2.4 Accretion onto unmoving black hole

In a specific case of unmoving black hole ($u^0_\infty = 1$, $u_\infty = 0$), the equations for stationary distribution of ultra-hard fluid with the equation of state $p = \rho$ in the Kerr-Newman metric is integrated directly, similar to the analogous problem in the Schwarzschild [9, 10, 11] and Reissner-Noreström metrics [12]. From above equations it follows that the specific azimuthal and longitudinal angular momentum are both zero:

$$L_\phi = u_\phi = 0, \quad L_\theta = u_\theta = u^\theta = 0. \tag{33}$$

This means, in particular, that $u^\theta = 0$, and so $\theta = const$ along the lines of flow. By using this property of the stationary ultra-hard fluid, we find the first integrals of the energy momentum conservation

$$T^\alpha_{\beta;\alpha} = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\alpha} (T^\alpha_\beta \sqrt{-g}) - \frac{1}{2} g_{\alpha\gamma,\beta} T^{\alpha\gamma} = 0. \tag{34}$$

Integration of this equation at $\beta = 0$ gives the first integral of motion (the relativistic Bernoulli energy conservation equation):

$$(p + \rho) u_0 u \sqrt{-g} = C_1(\theta) M^2, \tag{35}$$

where $u = u^r$ is a radial 4-velocity component and $C_1(\theta)$ is a function of θ. To find the second integral of motion we write a projection equation

$$u_\mu T^{\mu\nu}_{;\nu} = \rho_{,\nu} u^\nu + (p+\rho)\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\alpha}\left(\sqrt{-g}u^\alpha\right) = 0, \quad (36)$$

which can be expressed as

$$\frac{\rho_{,r}}{p+\rho}u + \frac{1}{\sqrt{-g}}\frac{\partial}{\partial r}(\sqrt{-g}u) = 0. \quad (37)$$

Integration of (37) gives the second integral of motion (the energy flux conservation):

$$\sqrt{-g}u \exp\left[\int\limits_{\rho_\infty}^{\rho}\frac{d\rho'}{\rho' + p(\rho')}\right] = -A(\theta)M^2, \quad (38)$$

where $A(\theta)$ is a function of θ. Using normalization condition $u^\alpha u_\alpha = 1$, from integrals of motion (35) and (38) we find

$$(p+\rho)\exp\left[\int\limits_{\rho_\infty}^{\rho}\frac{d\rho'}{\rho' + p(\rho')}\right]\sqrt{\frac{\Sigma(\Delta + \Sigma u^2)}{\mathcal{A}}} = -\frac{C_1(\theta)}{A(\theta)} = p_\infty + \rho_\infty, \quad (39)$$

where function $\mathcal{A}$ is from (5).

From the limiting forms of (38) and (39) at $a \to 0$ and $Q \to 0$, we find functions $C_1(\theta) = -A(\theta)(p_\infty + \rho_\infty)$ and $A(\theta) = \sin\theta A_0$, where $A_0 = const$ is a dimensionless constant. Following to Michel [9], we find relations at the critical sound point $r = r_*$:

$$\Delta(r_*) = 0, \quad u_*^2 = \frac{(r_*^2 + a^2)^2(r_* - M)}{\Sigma[2r_*(r_*^2 + a^2) - a^2\sin^2\theta(r_* - M)]}, \quad (40)$$

where $u_* = u(r_*)$. From the first equation in (40) it follows that in the black hole case, when $M^2 \geq a^2 + Q^2$, there are two critical points $r_1 = r_-$ and $r_2 = r_+$ which coincides with the event horizons. The smaller critical point, $r_1 = r_-$, appears to be inside the event horizon at $M^2 \geq a^2 + Q^2$. The smaller critical point r_1 coincides with the larger critical point $r_2 = r_+$ only in the extreme black hole case $M^2 = a^2 + Q^2$. It means that for the extreme black hole the boundary conditions must be different from ones in the non-extreme case. Using the general relation $b = 2 + i - s$, derived in [13, 14, 15], between the number of boundary conditions b, the number of invariants i and the number of critical surfaces s, we can see that in the extreme case we must put only 3 boundary conditions, very similar to the case of a stationary atmosphere. Thus, a stationary atmosphere without influx is more physical for the extreme black hole and not the stationary accretion.

Using relations (40) for critical point $r = r_+$, we calculate the constant A_0, which defines the value of the accretion flow:

$$A_0 = \frac{r_+^2 + a^2}{M^2}. \quad (41)$$

Now from (38) and (39) it is determined a radial dependence of the density and radial 4-velocity of the ultra-hard fluid

$$\frac{\rho}{\rho_\infty} = \frac{(r + r_+)(r^2 + r_+^2 + 2a^2) - a^2(r - r_-)\sin^2\theta}{\Sigma(r - r_-)}; \quad (42)$$

$$u^2 = \frac{A_0^2 M^4}{\Sigma^2}\frac{\rho_\infty}{\rho} = \frac{(r_+^2 + a^2)^2(r - r_-)}{\Sigma[(r + r_+)(r^2 + r_+^2 + 2a^2) - a^2(r - r_-)\sin^2\theta]}. \quad (43)$$

This solution coincides with the one found in the preceding Sections. Note again, that a test fluid approximation is violated in the extreme black hole case, when $r_+ = r_-$, and so $\rho \to \infty$

at $r \to r_+$. We suppose, however, that back reaction of the accreting fluid onto the background metric prevents the transformation of the black hole into the naked singularity in accordance with the cosmic censorship.

In the case of a naked singularity, when $M^2 < a^2 + Q^2$, there are no stationary solution with the finite influx of fluid. Instead of, a stationary atmosphere with a zero influx of fluid is established. By putting $u = A_0 = 0$ in (38) and (39), we obtain the radial density distribution of the stationary ultra-hard fluid without influx around the naked singularity, which coincides with (2.3).

3 Scalar field in the Kerr-Newman metric

A perfect fluid is identical to the nonstatic scalar field [5, 7, 18] with an energy-momentum tensor of

$$T_{\mu\nu} = \psi_{,\mu}\psi_{,\nu} - \frac{1}{2} g_{\mu\nu} g^{\rho\sigma} \psi_{,\rho}\psi_{,\sigma}. \tag{44}$$

Here we describe some solutions for the stationary distribution of scalar field in the Kerr-Newman geometry which are analogous to the previously found solutions for the ultra-hard fluid. The only exception is solution in the Section 3.3, which does not have a perfect fluid analogue and describes the static atmosphere of the scalar field with a finite total mass around the naked singularity.

3.1 Scalar field around black hole

Using the tortoise radial coordinate

$$r^* = \int \frac{r^2 + a^2}{\Delta} dr = r + \frac{r_+^2 + a^2}{r_+ - r_-} \ln(r - r_+) - \frac{r_-^2 + a^2}{r_+ - r_-} \ln(r - r_-) \tag{45}$$

and the advanced time coordinate $v = t + r^*$, one can generalize the solution by Jacobson [16] (see also [17]) for the accretion of scalar field onto the Kerr black hole to the Kerr-Newman case. To specify the problem, we choose the first boundary condition at space infinity

$$\psi(r \to \infty) = \dot{\psi}_\infty t + \psi_\infty. \tag{46}$$

With this boundary condition the general solution (14) reduces to

$$\psi = \dot{\psi}_\infty t + \psi_\infty + \frac{B_1}{2} \ln \left(\frac{r - r_+}{r - r_-} \right), \tag{47}$$

where B_1 is the integration constant. From the requirement of finiteness of the potential ψ at the event horizon $r = r_+$ we find

$$B_1 = 2\dot{\psi}_\infty \frac{r_+^2 + a^2}{r_+ - r_-}. \tag{48}$$

The resulting solution for the stationary accreting scalar field is

$$\psi = \psi_\infty + \dot{\psi}_\infty [v - r - (r_+ + r_-) \ln(r - r_-)]. \tag{49}$$

we find a stationary flux of the scalar field onto black hole

$$\dot{M} = -\oint \sqrt{-g}\; T_t^r d\theta\, d\phi = -\oint_{r_+} \dot{\psi}_\infty^2 (r_+^2 + a^2) \sin\theta\, d\theta\, d\phi = 4\pi \psi_\infty^2 (r_+^2 + a^2), \tag{50}$$

which coincides with (17) at $\rho_\infty = \dot{\phi}_\infty^2/2$.

3.2 Scalar field around extreme black hole

In the extreme case $M^2 = a^2 + Q^2$ we use a tortoise radial coordinate

$$r^* = \int \frac{r^2 + a^2}{\Delta}\, dr = r + 2M \ln(r - M) - \frac{M^2 + a^2}{r - M} \tag{51}$$

and an advanced time coordinate $v = t + r^*$. Now the solution of wave equation (8) with the same boundary condition at infinity as in (46) is

$$\psi = \psi_\infty + \dot{\psi}_\infty t + \frac{C_2 M}{r - M}, \tag{52}$$

where integration constant is fixed by the second boundary condition at the horizon, $C_2 = -(M^2 + a^2)\dot{\psi}_\infty / M$. Resulting solution in the extreme case is

$$\psi = \psi_\infty + \dot{\psi}_\infty [v - r - 2M \ln(r - M)] \tag{53}$$

We see that a scalar field is diverging at the event horizon of the extreme black hole $r = M$ as in the case of ultra-hard fluid.

3.3 Scalar field around naked singularity

Solution of the wave equation (8) in the case of the Kerr-Newman naked singularity is expressed in terms of the Legendre functions (26). we choose the limiting behavior of the scalar field at space infinity in the Jacobson form $\psi \to \dot{\psi}_\infty t + \psi_\infty$. The scalar field ψ is real and so the Legendre function with a nonzero coefficient in the general solution (14) is

$$Q_0(z) = \mathrm{arctanh} z = \arctan \xi. \tag{54}$$

The resulting solution, satisfying the boundary condition at infinity $\psi \to \dot{\psi}_\infty t + \psi_\infty$, is

$$\psi = \dot{\psi}_\infty t + \psi_\infty + B \left[\arctan\left(\frac{r - M}{\sqrt{a^2 + Q^2 - M^2}} \right) - \frac{\pi}{2} \right], \tag{55}$$

where B is the integration constant. The corresponding nonzero components of energy-momentum tensor are

$$\sqrt{-g}\, T^r_t = -\dot{\psi}_\infty B \sin\theta \sqrt{a^2 + Q^2 - M^2}; \tag{56}$$

$$T^t_t = -T^r_r = \frac{B^2(a^2 + Q^2 - M^2) + \mathcal{A}\dot{\psi}^2_\infty}{2\Sigma\Delta}; \tag{57}$$

$$T^\theta_\theta = T^\phi_\phi = \frac{B^2(a^2 + Q^2 - M^2) - \mathcal{A}\dot{\psi}^2_\infty}{2\Sigma\Delta}. \tag{58}$$

The energy flux onto the central singularity is nonzero only when the product $B\dot{\psi}_\infty \neq 0$. A corresponding radial 4-velocity of the inflowing fluid equals

$$u^r = -\frac{B\sqrt{a^2 + Q^2 - M^2}\,\Delta}{\dot{\psi}^2_\infty \mathcal{A} - B^2(a^2 + Q^2 - M^2)}. \tag{59}$$

To fix the value of the influx it is needed some additional boundary condition at the singularity, i.e. at $r = 0$. This requires some additional knowledge on the physics of the central singularity, which is unknown yet.

Meanwhile, the energy flux onto the central singularity being zero in the particular cases when $B = 0$ or $\dot{\psi}_\infty = 0$. For example, the particular solution (55) with $B = 0$ is identical to the corresponding solution for the ultra-hard fluid, as can be directly verified by comparison of the respective energy density distributions in (2.3) and (57). The other particular case of the solution with (55) is rather nontrivial. It describes the static distribution of the scalar field with a finite total mass M_f around the naked singularity. This static solution does not have a perfect fluid analogue.

We write the integral mass formula for the total mass of the scalar field M_f the Kerr-Newman metric by using the quasi-static coordinate frame $(t, r, \theta, \tilde{\phi})$ with azimuthal coordinate $\tilde{\phi}$ defined by the relation

$$\tilde{\phi} = d\phi - \omega dt. \tag{60}$$

This coordinate frame is "rotating with the geometry" [19, 20] with an angular velocity ω from (6). In other words, the observers in this frame is dragging by the gravitational filed and rotate with the angular velocity ω. The the integral mass formula in this frame is to reduced to a very simple expression:

$$M_f = \int \xi^\alpha T^\beta_\alpha \sqrt{-g}\, d\Sigma_\beta = \int \tilde{T}^t_t \sqrt{-\tilde{g}}\, d^3\tilde{x}, \tag{61}$$

$d^3\tilde{x} = drd\theta d\tilde{\phi}$ $\sqrt{-\tilde{g}} = \Sigma \sin\theta$. Using expression for the energy -momentum tensor (44), we calculate the scalar energy density $\tilde{T}^t_t$ for the solution (55) in the quasi-static frame $(t, r, \theta, \tilde{\phi})$:

$$\tilde{T}^t_t = -\frac{1}{2}\tilde{g}^{rr}(\psi_{,r})^2 = \frac{1}{2}B^2\frac{\epsilon^2}{\Sigma\Delta}, \tag{62}$$

where we define a super-extreme parameter

$$\epsilon = \sqrt{\frac{a^2 + Q^2 - M^2}{M^2}}, \quad a^2 + Q^2 - m^2 > 0. \tag{63}$$

Putting $\tilde{T}^t_t$ from (62) to (61), we obtain

$$M_f = 2\pi B^2\epsilon^2 M^2 \int\limits_0^\infty \frac{dr}{\Delta} = \pi B^2(\pi + 2\,\text{arccot}\epsilon)\,\epsilon M. \tag{64}$$

Now we express the constant B in (55) through a total mass of the scalar field M_f:

$$B = \sqrt{\frac{1}{\pi(\pi + 2\text{arccot}\epsilon)\epsilon}\frac{M_f}{M}}. \tag{65}$$

Note that in the black hole case ($\epsilon^2 \leq 0$), a corresponding analogue of the solution (55) without the influx results in the divergence of energy density at the event horizon $r = r_+$.

4 Discussion

We found the generalization of Petrich, Teukolsky and Shapiro solution for the accretion of perfect fluid with ultra-hard equation of state onto the moving Kerr black hole to the Kerr-Newman case. The fluid distributions around the black hole and naked singularity are entirely different. Due to the one-way membrane origin of the black hole horizon the fluid is stationary flowing into the black hole. Contrarily, the fluid forms the atmosphere around the naked singularity with a zero influx. This atmosphere is static in the case of the Reissner-Nordström naked singularity and is stationary in the case of the Kerr-Newman naked singularity due to the azimuthal frame-dragging. The central singularity is surrounded by the empty toroidal cavity in the case when both angular momentum and electric charge of the naked singularity are nonzero. In the central region around a charged naked singularity a fluid has an inverse density distribution, which is a manifestation of repulsive properties of charged naked singularity. In the case of extreme black hole the energy density of ultra hard fluid is diverging at the event horizon and, thus, the test fluid approximation is violated.

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