Entanglement entropy in $SU(2)$ lattice gauge theory

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Abstract

The entropy of entanglement between a three-dimensional slab of thickness l and its complement is studied numerically for four-dimensional $SU(2)$ lattice gauge theory. We find a signature of a nonanalytic behavior of the entanglement entropy, which was predicted recently for large N_c confining gauge theories in the framework of AdS/CFT correspondence. The derivative of the entanglement entropy over l is likely to have a discontinuity at some $l = l_c \approx 0.5\, fm$.

The application of the concepts of quantum information theory to quantum field theories in continuous space-times and on the lattices has led recently to many important advances in our understanding of their quantum behavior [1–3]. One such concept is the quantum entanglement of states of systems with many degrees of freedom, which turned out to be a very useful model-independent characteristic of the structure of the ground state of quantum fields. In particular, for quantum fields on the lattice in the vicinity of a quantum phase transition (i.e. a phase transition which occurs at zero temperature when some parameters of the system, such as the coupling constants, are varied) the ground state is a strongly entangled superposition of the states of all elementary lattice degrees of freedom (such as spins in Heisenberg model, or link variables in lattice gauge theory), and different phases of lattice theories can be characterized by different patterns of entanglement [1–3]. Quantum entanglement thus appears to be an adequate concept for the description of the emergence of collective degrees of freedom in quantum field theories [1–3].

A commonly used measure of quantum entanglement of the ground state of quantum fields in $(D-1)+1$-dimensional space-time is the entropy of entanglement $S[A]$ between some $(D-1)$-dimensional region A and its $(D-1)$-dimensional complement B, which characterizes the amount of information shared between A and B [1, 2, 4]. Entanglement entropy is defined as the usual von Neumann entropy for the reduced density matrix $\hat{\rho}_A$ associated with the region A:

$$S[A] = -\mathrm{Tr}_A \left(\hat{\rho}_A \ln \hat{\rho}_A\right) \tag{1}$$

The reduced density matrix is obtained from the density matrix of the ground state of the theory, $\hat{\rho}_{AB} = |0\rangle\langle 0|$, by tracing over all degrees of freedom which are localized outside of A, i.e. within B [2, 4]:

$$\hat{\rho}_A = \mathrm{Tr}_B \hat{\rho}_{AB} = \mathrm{Tr}_B |0\rangle\langle 0| \tag{2}$$

This density matrix describes the state of quantum fields as seen by an observer who can only perform measurements within A.

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The entropy of entanglement of confining gauge theories has recently become a subject of extensive studies in the framework of AdS/CFT correspondence, where a simple geometric expression for the entanglement entropy was conjectured [5–7]. One of the most interesting predictions of [5–7] is that for confining gauge theories the entanglement entropy (1) should be non-analytic in the size of the region A. In the limit $N_c \to \infty$ the derivative of the entropy over the size of A changes from being proportional to N_c^2 to a quantity of order N_c^0. This property can be intuitively understood without any reference to AdS/CFT. Indeed, at small distances an observer in A can only see quarks and gluons, whose number and hence the entropy scales as N_c^2, where N_c is the number of colours. At large distances the effective degrees of freedom are quarks and hadrons, whose number and the entropy are of order of N_c^0. Thus at some size of A roughly determined by a typical hadronic scale the entropy should change from N_c^2 to N_c^0. The nontrivial prediction from AdS/CFT is that this change is stepwise, and in some sense colourless and colourfull degrees of freedom never coexist at one energy scale. A similar non-analuticity has also been predicted recently using the approximate Migdal-Kadanoff decimations for $SU(2)$ lattice gauge theory [8]. Thus the behavior of entanglement entropy of confining gauge theories can be an interesting new test of Maldacena duality between string theories on $(D+1)$-dimensional curved spaces and D-dimensional gauge theories which live on their boundary. It can be also used as an alternative probe which indicates the emergence of the colourless degrees of freedom in the low-energy limit of gauge theories. In this work we report on our recent numerical measurements of entanglement entropy in $SU(2)$ lattice gauge theory. A detailed report can be found in [9]. Our main result is the observation of the non-analytic behavior of entanglement entropy, in agreement with the predictions of [5–7].

The results of [5–7] are obtained for the case when A is a slab of thickness l in $(D-1)$-dimensional space, and the entanglement entropy is normalized per unit area of the boundary ∂A of A. A typical behavior of entanglement entropy for field theories in $(D-1)+1$-dimensional space-time is [5–7]:

$$\frac{1}{|\partial A|} S(l) = \frac{1}{|\partial A|} S_{UV} + \frac{1}{|\partial A|} S_f(l) = \\ = \kappa \Lambda_{UV}^{D-2} - l^{2-D} f(l) \tag{3}$$

where κ is some constant which is different for different theories, $\Lambda_{UV} = a^{-1}$ is the UV cutoff scale, a is the lattice spacing, $|\partial A|$ is the area of the boundary of the slab A and $f(l)$ is some function which is finite as $l \to 0$. The l^{2-D} behavior of the UV finite term in (3) can be fixed by the requirement that the entropy does not diverge at $l \to 0$, which is reasonable for a local field theory. Indeed, the ultraviolet divergence in (3) can only be cancelled if $S_f(l)$ tends to $-\kappa\, l^{2-D}$ at small distances of order of Λ_{UV}^{-1}. This means that the entanglement entropy associated with a set of points of zero measure is finite. Further it is convenient to consider the derivative of the entanglement entropy over the width of the slab l, which is UV finite.

In order to measure the entanglement entropy, we have used the formula by Cardy and Calabrese [1, 2, 9], which relates the entanglement entropy with the set of free energies $F[A, s, T]$ of the theory on the spaces with an integer number s of cuts along the region A. For the sake of brevity, here we state the expression for the derivative $\frac{\partial}{\partial l} S[A]$:

$$\frac{\partial}{\partial l} S[A] = \lim_{T \to 0} \frac{\partial}{\partial l} \lim_{s \to 1} \frac{\partial}{\partial s} F[A, s, T] \tag{4}$$

where T is the temperature which should be eventually set to zero. More specifically, such spaces have the topology $\mathbb{C}_{(s)} \otimes \mathbb{T}^{D-2}$ [1, 2], where $\mathbb{C}_{(s)}$ is the s-sheeted Riemann surface with cut of length l, $\mathbb{T}^{D-2}$ is the $(D-2)$-dimensional torus and D is the dimensionality of space-time. An example of such space for $s = 2$ and $D = 2$ is shown on Fig. 1. When the spatial coordinates are within the region A, the space is periodic in time with period $2/T$, for spatial coordinates within B, the period is $1/T$. In practice the CPU time required to find $F[A, s, T]$ grows as s^2,

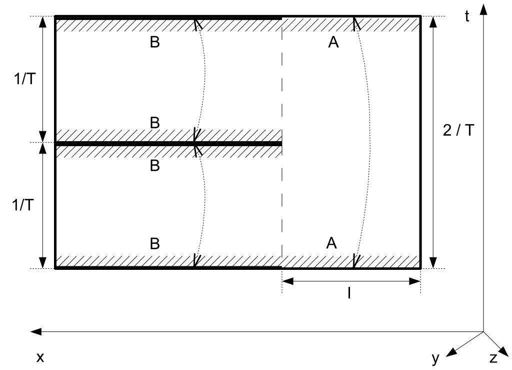

Figure 1: The topology of space on which the free energy $F[A,s,T]$ in (4) is calculated, an example for $s=2$. Dashed lines with arrows denote identification of cut sides, i.e. periodic boundary conditions in time direction.

thus with reasonable lattice sizes it is difficult to go beyond $s=O(10)$. In our measurements we have approximated the derivative $\lim_{s\to 1}\frac{\partial}{\partial s}F[A,s,T]$ by a finite difference between $s=2$ and $s=1$. Furthermore, since $F[A,1,T]$ does not depend on the size of A, in order to calculate the entropy (4) it is sufficient to find only the derivative $\frac{\partial}{\partial l}F[A,2,T]$, which is replaced by a finite difference $a^{-1}(F(l+1,2,T)-F[l,2,T])$ on the lattice. It is this finite difference that we have measured in our simulations. Since the free energies are not directly measurable in lattice simulations, we have used the method proposed in [10] which allows one to find directly the differences of free energies for close values of lattice parameters.

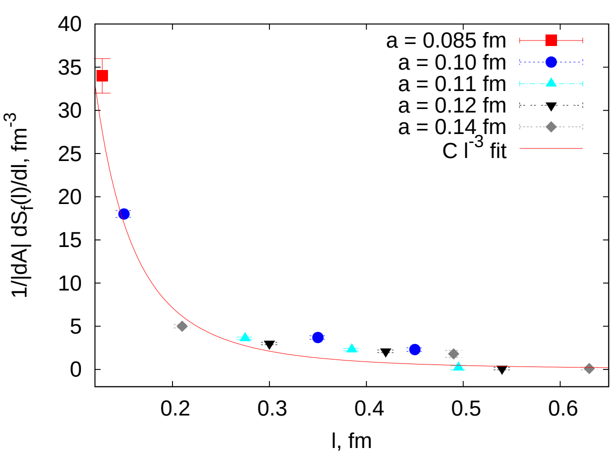

Figure 2: The dependence of the derivative of the entanglement entropy $\frac{1}{|\partial A|}\frac{S_f(l)}{\partial l}$ on l. Solid line is the fit of the data by the function $C\,l^{-3}$.

The derivative $\frac{1}{|\partial A|}\frac{\partial}{\partial l}S(l)$ estimated from these measurements is plotted on Fig. 2. It can be seen that this derivative grows rapidly at small distances. For comparison with the asymptotic behavior $\frac{\partial}{\partial l}S(l)\sim l^{-3}$ at $l\to 0$, we have fitted these results by the function Cl^{-3} (solid line on Fig. 2). For the data points with the smallest l the finite differences $a^{-1}(F(l+1,2,T)-F[l,2,T])$ were found at fixed l/a at different a, so that the finite differences $(l+a/2)^{-2}-(l-a/2)^{-2}$ still behave as $l^{-3}\sim a^{-3}$. The estimated ratio of derivatives $\frac{\partial}{\partial l}S_f(2a_2)/\frac{\partial}{\partial l}S_f(2a_1)=(2.0\pm 0.4)$ at $a_1=0.0996\,fm$ and $a_2=0.0854\,fm$ is indeed close to the ratio $(a_1/a_2)^3=1.586$, which is an indication that at small l the derivative $\frac{\partial}{\partial l}S_f(l)$ indeed

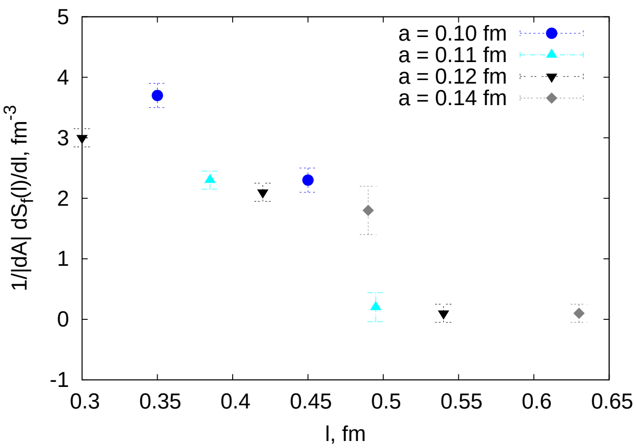

Figure 3: The discontinuity of the derivative of the entanglement entropy over l near $l_c \approx 0.5\, fm$.

scales in physical units of length. At larger l $\frac{\partial}{\partial l} S_f(l)$ goes to zero faster than l^{-3}, and seem to approach a kind of plateau for the values of l between $0.3\, fm$ and $0.5\, fm$. Here the values of $\frac{\partial}{\partial l} S_f(l)$ obtained for different values of lattice spacing differ rather significantly, which indicates that for our lattice parameters finite-volume and finite-spacing effects may still be rather strong. Nevertheless, at least qualitatively all data points for different values of a display the same behavior.

What is most interesting, however, is that at $l_c \approx 0.5\, fm$ the estimated derivative $\frac{\partial}{\partial l} S_f(l)$ rapidly goes to zero, and remains equal to zero within error range for larger values of l. Data points near this l_c are plotted with larger scale on Fig. 3. This is a clear signature of the discontinuity of the derivative of the entanglement entropy over l. Of course, it should be remembered that these results were obtained under some simplifying assumptions, and one can not completely exclude the possibility that once higher s are included in the analysis, the dependence of the entanglement entropy on l may be changed. It could be therefore interesting to perform similar measurements on larger lattices with larger number of cuts.

Thus we have confirmed numerically the basic properties of the entanglement entropy of confining gauge theories such as the l^{-2} behavior at small distance, as well as its nontrivial dependence on the size of the region A. A more detailed study of the volume-dependence and s-dependence of the entanglement entropy may be useful in order to establish more precisely the properties of this unusual deconfinement "phase transition" with respect to the size of the region [9]. Finally, we would like to explain why, in our opinion, the confirmation of the predictions of [5–7] is important methodologically. Until recently the development of the AdS/QCD phenomenology was based on the attempts to fit some known data with the results of calculations of some geometric quantities in the dual higher-dimensional geometries. In contrast, the nonanalytic behavior of the entanglement entropy was not previously known for confining gauge theories and was originally found in [6] using the dual description, which demonstrates the predictive power of the AdS/QCD approach.

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