Operators of observables for neutrino in dense matter and electromagnetic field

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Abstract

The explicit form of kinetic momentum and spin projection operators is found for neutral particle with anomalous magnetic moment interacting with dense matter and electromagnetic field. Possible applications of obtained results for neutrino physics are discussed.

1 Introduction

In mathematical apparatus of quantum field theory an elementary particle is usually identified with an irreducible unitary representation of the Poincare group. The irreducible representations are characterized by values of two invariants of the group:

$$P^2 \equiv P^\mu P_\mu = m^2, \tag{1}$$

$$W^2 \equiv W^\mu W_\mu = -m^2 s(s+1). \tag{2}$$

The translation generators P^μ are identified with the particle momentum, and the Pauli–Lubanski–Bargmann vector

$$W^\mu = -\frac{1}{2}\, e^{\mu\nu\rho\lambda} M_{\nu\rho} P_\lambda \tag{3}$$

characterizes the particle spin. The invariant m^2 is the particle mass square and s is the value of the particle spin.

The space of unitary representation is marked out by the condition called "wave equation for a particle possessing the mass m and the spin s". The wave equation for particles with spin $s = 1/2$ is the Dirac equation

$$(\hat{p} - m)\,\Psi(x) = 0. \tag{4}$$

In this case the realization of the momentum and the Pauli–Lubanski–Bargmann vector in the coordinate representation is

$$p^\mu = i\partial^\mu, \quad w^\mu = \frac{i}{2}\,\gamma^5(\gamma^\mu \hat{\partial} - \partial^\mu), \tag{5}$$

Operators p^μ and w^μ commute with the operator of the Dirac equation and can be identified with observable physical values (since only integrals of the motion can be considered as

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observables in relativistic quantum mechanics [1]) and have a self-conjugate extension on the solution set of the Dirac equation (4) with regard to the standard scalar product

$$(\Psi_f, \Psi_i) = \int d\mathbf{x}\, \Psi_f^\dagger(\mathbf{x}, t)\Psi_i(\mathbf{x}, t). \tag{6}$$

Three dimensional particle spin vector $\mathbf{S}$ is the set of coefficients of the expansion of the W^μ vector in space-like normals n_i^μ $(i = 1, 2, 3)$, $(n_i P) = 0$, $(n_i n_j) = -\delta_{ij}$:

$$S_i = -\frac{1}{\sqrt{P^2}}\,(W n_i)\,. \tag{7}$$

Obviously,

$$\left[S_i, S_j\right] = i e_{ijk} S_k. \tag{8}$$

The choice of normals is not unique and it is possible to construct spin operators determining the spin projection on any direction in an arbitrary Lorentz frame.

The above description of the particle characteristics can not be directly used in the presence of the external fields. In this case the Dirac equation has the form

$$\left(i\hat{\partial} - e\hat{A} - m\right)\Psi(x) = 0, \tag{9}$$

and operators p^μ and w^μ are not always integrals of motion. Linear combinations of operators p^μ and w^μ with coefficients depending on coordinates are used for the classification of particle states in the external field [2]. Generally it is not easy to give physical interpretation for these operators, and it often leads to logical difficulties.

Even greater difficulties arise in the consideration of the Dirac–Pauli equation with the phenomenological term describing the interaction of the anomalous magnetic moment μ_0 with the external field:

$$\left(i\hat{\partial} - e\hat{A} - \frac{i}{2}\,\mu_0 F^{\alpha\beta}\sigma_{\alpha\beta} - m\right)\Psi(x) = 0, \tag{10}$$

or with the axial-vector term describing neutrino propagation in dense matter consisting of fermions [3]:

$$\left(i\hat{\partial} - \frac{1}{2}\hat{f}(1+\gamma^5) - m\right)\Psi(x) = 0. \tag{11}$$

In this equation the interaction of neutrino with moving and polarized matter is described by the effective four-potential f^μ, which is a linear combination of currents and polarizations of the background fermions.

2 Statement of problem

Since the irreducible representation of group is defined accurately up to the equivalence transformation, it is reasonable to state a problem of finding such realization of the Lie algebra of the Poincare group for which the condition of the representation irreducibility leads to wave equation describing a particle in the given external field. To solve this problem it is necessary to find operator $U(x, x_0)$ which converts solutions of the wave equation for a free particle

$$\big(D_0(x) - m\big)\Psi_0(x) = 0 \tag{12}$$

to solutions of the equation for the particle in the external field

$$\big(D(x) - m\big)\Psi(x) = 0, \tag{13}$$

i.e.

$$U(x, x_0)\Psi_0(x) = \Psi(x). \tag{14}$$

The specified operator should satisfy the equation

$$D(x)U(x,x_0)-U(x,x_0)D_0(x)=0. \tag{15}$$

Therefore, generators

$$\tilde{P}^{\mu}=UP^{\mu}U^{-1}, \quad \tilde{M}^{\mu\nu}=UM^{\mu\nu}U^{-1} \tag{16}$$

commute with operator of the wave equation. As a consequence the Pauli–Lubanski–Bargmann vector and the three dimensional spin vector can be constructed in the same way as in the case of a free particle.

3 Neutrino in homogeneous electromagnetic field

Let us consider the Dirac–Pauli equation for a neutral particle with the anomalous magnetic moment μ_0 in a stationary homogeneous electromagnetic field:

$$\left(i\hat{\partial}-\frac{i}{2}\mu_0F^{\mu\nu}\sigma_{\mu\nu}-m\right)\Psi(x)=0. \tag{17}$$

When the second invariant of the electromagnetic field tensor $F^{\mu\nu}$ is equal to zero

$$I_2=\frac{1}{4}F^{\mu\nu}H_{\mu\nu}=0, \qquad H^{\mu\nu}=-\frac{1}{2}e^{\mu\nu\rho\lambda}F_{\rho\lambda},$$

the special type of the solutions of this equation was found in paper [4]. These solutions can be presented as a result of action of some integral operator on the solutions of the equation for a free particle. If the solutions for the free particle are chosen in the plane wave form then the action of the given operator reduces to the multiplication by the matrix function depending on the parameter q^{μ}. This parameter satisfies the condition $q^2=m^2$ and can be interpreted as a kinetic momentum of the particle in the external field.

The explicit form of the wave function system is defined by the formula

$$\Psi_{q\zeta_0}(x)=\frac{1}{2}\sum_{\zeta=\pm1}e^{-i(P_{\zeta}x)}(1-\zeta\gamma^5\hat{S}_{tp}(q))(1-\zeta_0\gamma^5\hat{S}_0(q))(\hat{q}+m)\psi_0. \tag{18}$$

Here

$$P^{\mu}_{\zeta}=q^{\mu}-\zeta H^{\mu\alpha}H_{\alpha\nu}q^{\nu}/\sqrt{\mathcal{N}}, \qquad S^{\mu}_{tp}(q)=-H^{\mu\nu}q_{\nu}/\sqrt{\mathcal{N}}, \tag{19}$$

where

$$\mathcal{N}=q_{\mu}H^{\mu\nu}H_{\nu\rho}q^{\rho}.$$

The four dimensional vector S^{μ}_0 defines the initial direction of the particle polarization; $\zeta_0=\pm1$ is the sign of the spin projection on this direction; ψ_0 is the constant bispinor normalized by the condition $\bar{\Psi}_0(x)\Psi_0(x)=m/q_0$.

The system of solutions (18) describes spin coherent states of neutrino and is the complete system of solutions of equation (17) characterized by the particle kinetic momentum q^{μ} and by the quantum number ζ_0. However this system is not stationary in the general case. The solutions are stationary when the initial polarization vector $S^{\mu}_0(q)$ is equal to the vector of the total polarization $S^{\mu}_{tp}(q)$: $S^{\mu}_0(q)=S^{\mu}_{tp}(q)$.

Let us consider the stationary case $S^{\mu}_0(q)=S^{\mu}_{tp}(q)$. In this case wave functions are the eigenfunctions of the spin projection operator to the $S^{\mu}_{tp}(q)$ direction with eigenvalues $\zeta=\pm1$ and of the canonical momentum operator $p^{\mu}=i\partial^{\mu}$ with eigenvalues P^{μ}_{ζ}. The orthonormalized system of the stationary solutions of equation (17) can be written down as

$$\Psi_{q\zeta}(x)=e^{-i(P_{\zeta}x)}\sqrt{|J|}(1-\zeta\gamma^5\hat{S}_{tp}(q))(\hat{q}+m)\psi_0. \tag{20}$$

In this equation $J = 1 - 2\zeta I_1/\sqrt{\mathcal{N}}$ is the transition Jacobian between the variables q^μ and P^μ_ζ, $I_1 = \frac{1}{4}F^{\mu\nu}F_{\mu\nu}$ is the first invariant of the $F^{\mu\nu}$ tensor. The dispersion law is determined by the relation

$$P^2_\zeta = m^2 - 2I_1 - 2\zeta\Delta\sqrt{P^\mu_\zeta H_{\mu\alpha}H^{\alpha\nu}P_{\zeta\nu}}, \quad \Delta = \mathrm{sign}(J(P)). \tag{21}$$

It follows from the dispersion law that group velocities for particles with different spin orientations are equal and defined by the equation

$$\mathbf{v}_{gr} = \frac{\partial P^0_\zeta}{\partial \mathbf{P}_\zeta} = \frac{\mathbf{q}}{q^0}. \tag{22}$$

Since wave functions (20) are the eigenfunctions of both the canonical p^μ and the kinetic $\mathfrak{Q}^\mu$ momentum operators, it is possible to express eigenvalues of the kinetic momentum operator q^μ in terms of eigenvalues of the canonical momentum operator P^μ_ζ. Replacing P^μ_ζ by the operator p^μ, we obtain the explicit form of the kinetic momentum operator $\mathfrak{Q}^\mu$:

$$\mathfrak{Q}^\mu = p^\mu + \gamma^5\frac{H^{\mu\alpha}H_{\alpha\nu}p^\nu H_{\beta\alpha}p^\alpha\gamma^\beta}{p^\beta H_{\beta\alpha}H^{\alpha\rho}p_\rho}. \tag{23}$$

On the solutions of the Dirac–Pauli equation, we have

$$\hat{\mathfrak{Q}} = m, \quad \mathfrak{Q}^2 = m^2. \tag{24}$$

Natural generalization of the Pauli–Lubanski–Bargmann vector to the case of a particle moving in an external field is given by the following expression:

$$W^\mu = \frac{1}{2}\gamma^5(\gamma^\mu\hat{\mathfrak{Q}} - \mathfrak{Q}^\mu).$$

If we normalize spin operators $\mathfrak{S}$ by the condition $\mathfrak{S}^2 = 1$, then a basis in the spin operators space has the form

$$\mathfrak{S}_i = -\gamma^5\hat{S}_i(q).$$

For our task it is convenient to choose the basis in the following way:

$$\mathfrak{S}_{tp} = -\gamma^5\hat{S}_{tp}(q), \quad \mathfrak{S}_{1\perp} = -\gamma^5\hat{S}_{1\perp}(q), \quad \mathfrak{S}_{2\perp} = -\gamma^5\hat{S}_{2\perp}(q), \tag{25}$$

where the normal $S_{tp}(q)$ is defined by equation (19) and

$$S^\mu_{1\perp}(q) = \frac{S^\mu_0(q) + S^\mu_{tp}(q)(S_0(q)S_{tp}(q))}{\sqrt{1-(S_0(q)S_{tp}(q))^2}}, \quad S^\mu_{2\perp}(q) = \frac{e^{\mu\nu\rho\lambda}q_\nu S_{0\rho}(q)S_{tp\lambda}(q)}{m\sqrt{1-(S_0(q)S_{tp}(q))^2}}.$$

Spin operator $\mathfrak{S}_{tp}$ with eigenfunctions (20) is defined by the formula

$$\mathfrak{S}_{tp} = \frac{\gamma^5\gamma_\mu H^{\mu\nu}\mathfrak{Q}_\nu}{\sqrt{\mathfrak{Q}^\beta H_{\beta\alpha}H^{\alpha\rho}\mathfrak{Q}_\rho}} = \mathrm{sign}\left(1 + \frac{2I_1\gamma^5H^{\mu\nu}p_\nu\gamma_\mu}{p^\beta H_{\beta\alpha}H^{\alpha\rho}p_\rho}\right)\tilde{\mathfrak{S}}_{tp}, \tag{26}$$

where

$$\tilde{\mathfrak{S}}_{tp} = \frac{\gamma^5\gamma_\mu H^{\mu\nu}p_\nu}{\sqrt{p^\beta H_{\beta\alpha}H^{\alpha\rho}p_\rho}}. \tag{27}$$

Operator $\mathfrak{S}_{tp}$ is the integral of motion and characterizes a particle spin projection on the magnetic field direction $\mathbf{H}_0$ in the particle rest frame.

Spin operator $\mathfrak{S}_0$ with non stationary eigenfunctions (18) is a linear combination of operators (25) with coefficients depending on coordinates:

$$\begin{aligned}\mathfrak{S}_0 &= -(S_0(q)S_{tp}(q))\mathfrak{S}_{tp} + \\ &+ [\cos 2\theta\,\mathfrak{S}_{1\perp} - \sin 2\theta\,\mathfrak{S}_{2\perp}]\sqrt{1-(S_0(q)S_{tp}(q))^2},\end{aligned} \tag{28}$$

where $\theta = x_\mu H^{\mu\nu} H_{\nu\rho} q^\rho / \sqrt{\mathcal{N}}$.

Form (28) of operator $\mathfrak{S}_0$ implies that solution (18), which is a linear combination of solutions (20), describes a state of a neutral particle moving with a constant velocity $\mathbf{q}/q^0$ and with a spin precessing around $\mathbf{H}_0$ with frequency $\omega = 2m|\mathbf{H}_0|/q^0$.

It should be emphasized that this state is a pure quantum mechanical state. The existence of plane wave solutions of the Dirac–Pauli equation (17), describing a pure state of a neutral particle with a non-conserved spin projection on the fixed space axis is possible only by choosing the kinetic momentum components as quantum numbers. This state is a spin coherent one, so solutions (18) do not form an orthogonal basis. The considered system is not overfull, since a spin operator spectrum is finite, so the system can be easily orthogonalized.

Let us consider now applications of the obtained results to the description of neutrino. In investigating of the influence of a stationary pure magnetic field on neutrino oscillations in pioneer paper [5], as well as in others papers, stationary solutions $\Psi_{p\tilde{\zeta}}(x)$ first found in [6] were used as the wave functions of a particle. These solutions are the eigenfunctions of the canonical momentum operator p^μ and of the spin operator $\tilde{\mathfrak{S}}_{tp}$. It was supposed that the mean value of neutrino helicity is equal to 1 (in the absolute value) at the fixed time moment and the further spin evolution is described by linear combinations of the above mentioned wave functions

$$\Psi(x) = \sum_{\tilde{\zeta}=\pm 1} \alpha_{\tilde{\zeta}}(p) \Psi_{p\tilde{\zeta}}(x).$$

However, such a description is not correct in the general case. Since the standard helicity operator $(\mathbf{\Sigma p})/|\mathbf{p}|$ does not commute with the operator from the Dirac–Pauli equation (17), the state of the particle with a fixed canonical momentum and with the helicity mean value equal to 1 (at the fixed time moment) can only be a mixed spin state. But in a mixed state the change of the polarization can be caused only by a distinction of group velocities of the neutrino beam components. This effect should disappear on large distances since the beam is no longer coherent. So our results enable to treat a possible effect of the neutrino polarization change in electromagnetic field as a precession of the particle spin.

4 Neutrino in dense matter

Generalization of the results for the case of the neutrino interaction with dense matter, i.e. solution of the equation

$$\left(i\hat{\partial} - \frac{1}{2}\hat{f}(1+\gamma^5) - \frac{i}{2}\mu_0 F^{\mu\nu}\sigma_{\mu\nu} - m \right) \Psi(x) = 0, \tag{29}$$

was obtained in papers [7–9].

In this case the expression for the wave function takes the form

$$\Psi_{q\zeta_0}(x) = \frac{1}{2} \sum_{\zeta=\pm 1} e^{-i(P_\zeta x)} (1 - \zeta\gamma^5 \hat{S}_{tp})(1 - \zeta_0 \gamma^5 \hat{S}_0)(\hat{q} + m)\psi_0. \tag{30}$$

Here

$$\begin{aligned} P^\mu_\zeta = q^\mu \left(1 + \zeta \frac{(f\varphi)}{2\sqrt{(\varphi q)^2 - m^2\varphi^2}} \right) \\ + \frac{1}{2} f^\mu \left(1 - \frac{\zeta\sqrt{(\varphi q)^2 - m^2\varphi^2}}{(\varphi q)} \right) - \varphi^\mu \frac{\zeta (f\varphi) m^2}{2(\varphi q)\sqrt{(\varphi q)^2 - m^2\varphi^2}}, \end{aligned} \tag{31}$$

$$S^\mu_{tp} = \frac{q^\mu (\varphi q)/m - \varphi^\mu m}{\sqrt{(\varphi q)^2 - \varphi^2 m^2}}, \tag{32}$$

where

$$\varphi^\mu = f^\mu/2 + H^{\mu\nu}q_\nu/m.$$

It is obvious that the system of solutions (30) is a complete system of solutions of equation (29), which is characterized by the kinetic momentum of the particle q^μ and the quantum number $\zeta_0 = \pm 1$. In the general case this system is not stationary. The obtained solutions are stationary only when $S_0^\mu = S_{tp}^\mu$. In this case the wave functions are the eigenfunctions of the spin projection operator to the direction S_{tp}^μ with the eigenvalues $\zeta = \pm 1$ and of the canonical momentum operator $i\partial^\mu$ with eigenvalues P_ζ^μ. The orthonormalized system of the stationary solutions of equation (29) can be written down in the following way:

$$\Psi_{q\zeta}(x) = e^{-i(P_\zeta x)}\sqrt{|J|}(1 - \zeta\gamma^5\hat{S}_{tp})(\hat{q} + m)\psi_0, \tag{33}$$

where J is the transition Jacobian between the variables q^μ and P_ζ^μ:

$$J = \left(1 + \zeta\frac{(f\varphi)}{2\sqrt{(\varphi q)^2 - m^2\varphi^2}}\right)^2\left(1 + \zeta\frac{f_\mu H^{\mu\nu}q_\nu/2m - 2I_1}{\sqrt{(\varphi q)^2 - m^2\varphi^2}}\right). \tag{34}$$

The dispersion law is determined by the relation

$$\tilde{P}^2 = m^2 - f^2/4 - 2I_1 - 2\zeta\Delta\sqrt{(\tilde{\Phi}\tilde{P})^2 - \tilde{\Phi}^2 m^2}, \tag{35}$$

where

$$\tilde{P}^\mu = P_\zeta^\mu - f^\mu/2, \quad \tilde{\Phi}^\mu = f^\mu/2 + H^{\mu\nu}\tilde{P}_\nu/m, \quad \Delta = \text{sign}(J(P)). \tag{36}$$

Eigenvalues of the kinetic momentum operator q^μ are expressed in terms of eigenvalues of the canonical momentum operator P^μ in the following way:

$$q^\mu = \tilde{P}^\mu + \left[\tilde{P}^\mu(f\tilde{\Phi}) - f^\mu(f\tilde{P})/2 - 2mH^{\mu\nu}\tilde{\Phi}_\nu\right]\left[\tilde{P}^2 - m^2 + f^2/4 + 2I_1 - (f\tilde{\Phi})\right]^{-1}. \tag{37}$$

The total polarization vector is connected with P^μ by the relation

$$S_{tp}^\mu = \text{sign}\left(1 + \frac{f_\mu H^{\mu\nu}\tilde{P}_\nu/m - 4I_1}{\tilde{P}^2 - m^2 + f^2/4 + 2I_1 - (f\tilde{\Phi})}\right)\frac{q^\mu(\tilde{\Phi}\tilde{P})/m - \tilde{\Phi}^\mu m}{\sqrt{(\tilde{\Phi}\tilde{P})^2 - m^2\tilde{\Phi}^2}}. \tag{38}$$

If we replace in equations (37), (38) P^μ by operator p^μ (like it was done in Section 3) then we can find the kinetic momentum operator $\mathfrak{Q}^\mu$ and the operator $\mathfrak{S}_{tp} = -\gamma^5\hat{S}_{tp}(q)$. However, the explicit expressions of these operators can not be written down as simple formulae, so we do not present them here.

The obtained solutions possess the same properties as the solutions of equation (17) considered above. Hence, they describe a neutrino moving with a constant velocity and with a spin rotating due to the interaction with dense matter and electromagnetic field.

5 Conclusions

We obtained the exact solutions of the Dirac–Pauli equation for neutrino in dense matter and electromagnetic field. We found the explicit forms of the kinetic momentum and spin projection operators. We demonstrated that if the neutrino production occurs in the presence of an external field and a dense matter, then its spin orientation is characterized by the vector S_{tp}^μ. Using both the stationary and the nonstationary solutions obtained in this paper, it is possible to calculate, in the framework of the Furry picture, the probabilities of various processes with neutrino.

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