Fermionic Green function in the 3D-ball with chirally invariant boundary conditions

A. A. Abrikosov, jr.[a*]
[a] *Institute of Theoretical and Experimental Physics*
B. Cheremushkinskaya 25, 117218 Moscow, Russia

Abstract

We calculate the Green function of massless fermions inside the 3-dimensional ball with Atiah —Patodi —Singer spectral boundary conditions.

1 Motivation

The two principal phenomena in QCD, *i. e.* confinement and spontaneous breaking of chiral invariance (S χ B) both take place at large scales where the interaction is strong and the perturbation theory is unreliable. As a rule the problems of confinement and S χ B were addressed separately from each other. The question is whether these phenomena are interrelated and if so, how do they affect each other?

A way to approach the influence exerted by confinement onto S χ B is to study the behaviour of massless quarks in finite volume. Locking of quarks inside a hadron results into a discrete energy spectrum and it would be interesting to see what is the effect of the discreteness onto the chiral properties of fermions.

The idea of limiting the volume accessible to quarks was widely exploited by so called Bag models, such as the famous MIT-bag, cloudy bag, chiral bag *etc.* [1, 2, 3] They were really successful in predicting the spectrum and some other properties of hadrons. However from our point of view all of them suffered from a common drawback: they used manifestly chirally noninvariant boundary conditions. Hence the chiral invariance was broken explicitly by the boundaries and the the question of S χ B had no sense.

Obviously in order to to study the S χ B -phenomenon in limited space one needs chirally invariant boundary conditions. Fortunately those exist. They are the spectral boundary conditions introduced by Atiyah, Patodi and Singer (APS) who investigated the spectral asymmetry for manifolds with boundaries [4, 5]. Later these boundary conditions were widely applied in studies of anomalies on manifolds with boundaries [6]. Recently it was shown that in special cases the spectral boundary conditions make sense not only in even but also in odd space-time dimensions [7]

The aim of the present work is to construct the fermionic Green function in the ball with spectral boundary conditions. However at first we shall expound on chirality and APS boundary conditions. We shall describe the traditional constructive definition and the formulation in terms of the integral projector operator. After making clear the setting of the problem we will sketch out the calculation of the Green function and present the result.

[*] **e-mail**: persik@itep.ru

2 Chirality and Boundary Conditions

2.1 Chirality

Chirality is a specific symmetry peculiar to massless fermions in even space-time dimensions. It arises due to existence of the additional γ^{2n+1}-matrix:

$$\gamma^{2n+1} \propto \gamma^1 \dots \gamma^{2n}. \tag{1}$$

This matrix does not appear in the Lagrangian and one may define right and left fermion fields as follows:

$$\psi_L = \gamma^{2n+1}\psi_L; \qquad \psi_R = -\gamma^{2n+1}\psi_R. \tag{2}$$

In Weyl representation

$$\gamma^{2n+1} = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \qquad \text{and} \qquad \psi_L = \begin{pmatrix} f \\ 0 \end{pmatrix}, \qquad \psi_R = \begin{pmatrix} 0 \\ g \end{pmatrix}. \tag{3}$$

For massless fermions γ^{2n+1} anticommutes with the Lagrangian, therefore the left and right fields are independent:

$$L = -i\bar{\psi}\nabla\!\!\!\!/\,\psi = -i\bar{\psi}_L\nabla\!\!\!\!/\,\psi_L - i\bar{\psi}_R\nabla\!\!\!\!/\,\psi_R \tag{4}$$

This property is called *chiral invariance.* However the mass term couples the left and right fields

$$L_m = -m\bar{\psi}\psi = -m\bar{\psi}_R\psi_L - m\bar{\psi}_L\psi_R \tag{5}$$

The *spontaneous breaking of chiral invariance* (S χ B) means generation of mass term and formation of chiral condensate $\langle\bar{\psi},\,\psi\rangle$ due to gauge interaction between fermions:

$$m \neq 0; \qquad \langle\bar{\psi},\,\psi\rangle \neq 0. \tag{6}$$

In order to study S χ B - effects in limited space one needs not only the Lagrangian to be chiral invariant but also the boundary conditions that respect chirality, *i. e.* there must be no mass term in the Lagrangian and no left-right mixing on the boundaries.

2.2 APS: constructive definition

Now we turn to the Atiah — Patodi — Singer spectral boundary conditions. Initially they were formulated in terms of spectral harmonics and this is where the name comes from. We shall start from the traditional *"constructive"* definition of the APS boundary conditions. In essence it is extremely simple and claims that **Fourier components of fermion fields must have definite chirality on the boundary.** This ensures that on the boundary left and right modes do not mix (although nothing forbids that in the bulk). The core of the matter is how to choose what (left or right) chirality to ascribe to a particular mode. We shall see that this is done so that there exists a possibility to extend the wave functions out of the bag. More information on the spectral boundary conditions and associated subtleties may be found in [8]

In order to formulate the selection rule we need to specify the coordinate frame, the gauge and the choice of γ-matrices near the boundary. The special role is assigned to the normal coordinate.

Let us consider massless fermions ψ interacting with gauge field $\hat{A}$ in $4d$-Euclidean domain B with boundary ∂B. The coordinates near the boundary are chosen as follows: ξ points along the normal ($\xi = 0$ corresponds to points on the boundary) while q's span the surface ∂B. For simplicity let the metric depend only on q's:

$$ds^2 = d\xi^2 + g_{ik}(q)\,dq^i\,dq^k. \tag{7}$$

The second point is to fix the gauge that eliminates the normal component of the gauge field on the boundary:

$$\hat{A}_\xi\Big|_{\xi=0} = 0. \tag{8}$$

Finally we take a set of Dirac matrices that for $d = 4$ somewhat reminds the Weyl set. Suppose that I is the unity matrix and $\{\sigma^1, \sigma^2, \sigma^2\}$ are Pauli matrices. Then

$$\gamma^\xi = \begin{pmatrix} 0 & iI \\ -iI & 0 \end{pmatrix}; \qquad \gamma^q = \begin{pmatrix} 0 & \sigma^q \\ \sigma^q & 0 \end{pmatrix}. \tag{9}$$

This completes the procedure and separates the normal from the surface coordinates.

After the normal has been separated the full d-dimensional Dirac operator takes the form:

$$-i\nabla\!\!\!\!/\,|_{\partial B_4} = -i\gamma^\alpha \nabla_\alpha = \begin{pmatrix} 0 & I\,\partial_\xi + \hat{\mathcal{B}} \\ -I\,\partial_\xi + \hat{\mathcal{B}} & 0 \end{pmatrix}, \tag{10}$$

Where $\hat{\mathcal{B}}$ stands for the *boundary* operator. It is the Dirac operator that acts on $d-1$ surface coordinates and has the spinor dimension twice less than the full one:

$$\hat{\mathcal{B}} = -i\hat{\nabla\!\!\!\!/} = -i\sigma^q\,\nabla_q. \tag{11}$$

One may find the eigenfunctions of the boundary operator and classify them according to the sign of the eigenvalues:

$$\hat{\mathcal{B}}\,e^\pm_\lambda(q) = -i\hat{\nabla\!\!\!\!/}\,e^\pm_\lambda(q) = \pm\lambda\,e^\pm_\lambda(q), \qquad \lambda > 0. \tag{12}$$

We shall call $e^\pm_\lambda(q)$ positive (or, respectively, negative) boundary harmonics.

In the vicinity of the boundary one may expand fermionic wave functions in terms of the eigenfunctions of the boundary operator.

$$\psi(\xi,\,q) = \sum_{\lambda>0} \psi^+_\lambda(\xi)\,e^+_\lambda(q) + \sum_{\lambda>0} \psi^-_\lambda(\xi)\,e^-_\lambda(q). \tag{13}$$

According to the APS spectral prescription the two terms of the decomposition must differ in surface chirality. Namely, on the boundary the positive modes are right spinors and negative modes are left spinors:

$$\psi^+_\lambda(0) = \begin{pmatrix} 0 \\ g^+_\lambda(0) \end{pmatrix} \qquad \text{and} \qquad \psi^-_\lambda(0) = \begin{pmatrix} f^+_\lambda(0) \\ 0 \end{pmatrix}. \tag{14}$$

The reason for this choice will be explained in a moment.

2.3 The APS-physics

In order to make clear the grounds for demanding $\psi^\pm(0) \sim \psi_{R/L}$ and not the opposite let us turn to the spectral problem for the complete Dirac operator.

$$-i\nabla\!\!\!\!/\,\psi_\Lambda = \Lambda\,\psi_\Lambda. \tag{15}$$

It is convenient to write it in terms of the upper and lower spinor components. Suppose that the wave function looks as follows: $\psi^\pm_\lambda = \begin{pmatrix} f^\pm_\lambda(\xi) \\ g^\pm_\lambda(\xi) \end{pmatrix} e^\pm_\lambda(q)$. Then the eigenfunction equation (15) breaks up into the pair of component equations:

$$(\partial_\xi \pm \lambda)\,g^\pm_\lambda(\xi) = \Lambda\,f^\pm_\lambda(\xi) \tag{16a}$$
$$-(\partial_\xi \mp \lambda)\,f^\pm_\lambda(\xi) = \Lambda\,g^\pm_\lambda(\xi) \tag{16b}$$

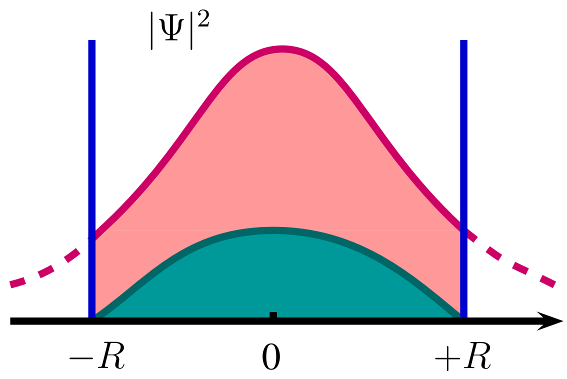

Figure 1: Continuation of a wave function out of the bag

The APS-requirements (14) are equivalent to the following boundary conditions:

$$f^+_\lambda(0) = 0; \tag{17a}$$
$$g^-_\lambda(0) = 0. \tag{17b}$$

Substituting those into the spectral equation (16) at $\xi = 0$ we get the conditions for the remaining functions f^-_λ and g^+_λ:

$$\partial_\xi g^+_\lambda(0) + \lambda g^+_\lambda(0) = f^+_\lambda(0) = 0 \tag{18a}$$
$$\partial_\xi f^-_\lambda(0) + \lambda f^-_\lambda(0) = -g^-_\lambda(0) = 0; \tag{18b}$$

Note that we have introduced the boundary harmonics (12) so that the eigenvalues λ are positive by definition. Therefore on the boundary the components of ψ_Λ either vanish like f^+, g^- or fall down (*i. e.* have negative logarithmic derivatives).

$$\frac{\partial_\xi f^-_\lambda}{f^-_\lambda} = \frac{\partial_\xi g^+_\lambda}{g^+_\lambda} = -\lambda < 0. \tag{19}$$

We may conclude that the APS requirements ensure that the eigenfunctions of the full Dirac operator may be continued out of the bag in a square-integrable way, Fig. 1

2.4 Integral form of APS-conditions

An alternative formulation of spectral boundary conditions is to to project out the positive and negative surface harmonics using an integral operator [10]. The integral form of APS boundary conditions is equivalent to the constructive definition. According to it the fermionic wave function must satisfy the following integral equation:

$$\mathcal{P}\,\psi(q) = \oint_{\partial B} \mathcal{P}(q,\,q')\,\psi(q)\,dS' = 0; \tag{20}$$

The projection operator $\mathcal{P}$ distinguishes left and right surface modes

$$\mathcal{P}(q,\,q') = \begin{pmatrix} \mathcal{P}^+(q,\,q') & 0 \\ 0 & \mathcal{P}^-(q,\,q') \end{pmatrix} \tag{21}$$

Operators $\mathcal{P}^\pm$ serve to select the positive and negative surface modes

$$\mathcal{P}^\pm(q,\,q') = \sum_{\lambda>0} e^\pm_\lambda(q) \otimes [e^\pm_\lambda(q')]^\dagger; \tag{22}$$

Therefore equation (20) is equivalent to (17) that claims that positive surface harmonics must be of right chirality and *v. v.*

The integral form of spectral boundary conditions is explicitly chiral invariant since $\mathcal{P}$ commutes with γ^{d+1}:

$$\left[\mathcal{P},\, \gamma^{d+1}\right] = 0. \tag{23}$$

We refer the reader to [10] for details and explicit expression for $\mathcal{P}$.

3 The fermion propagator

3.1 Preparations

Now we may turn to the declared topic of our talk, *i. e.* to the Green function of massless fermions in the spectral bag. The equation is quite the standard one but the problem is complicated by the boundary conditions.

$$\nabla\!\!\!\!/\, \mathcal{S}(x,\, y) = \delta(x-y), \qquad \mathcal{P}\mathcal{S}(x,\, y)|_{x\in\partial B} = \mathcal{S}\mathcal{P}(x,\, y)|_{y\in\partial B} = 0. \tag{24}$$

We are going to solve the problem for the 3-dimensional spherical B bag of unit radius. Thus ∂B is a Riemann sphere. (Here I would like to stress once more that the spectral boundary conditions are sensible in odd dimensions as well [7]. Besides the calculation is, to my mind, quite enlightening for higher dimensions.)

We shall apply the following procedure. First of all let us note that due to the spherical symmetry the boundary operator does not depend on radius and acts only onto the angular and spin variables. Therefore it is convenient to expand the Green function in terms of its eigenfunctions and solve the radial equations obtained for every harmonic separately.

Now we are going to specify the set of coordinates and the γ-matrices. Thanks to the absence of the gauge field there is no need to fix the gauge.

- We shall work in the standard spherical frame $(r,\, \theta,\, \phi)$ with $\theta \in [0,\, \pi]$. The polar axis is directed along the 3rd coordinate z. Obviously the radius takes the part of the normal to the boundary whereas the boundary operator $\hat{\mathcal{B}}$ depends only on θ, ϕ.

- The starting point in the definition of γ-matrices is the Cartesian set:

$$\gamma_a = \begin{pmatrix} 0 & \sigma_a \\ \sigma_a & 0 \end{pmatrix}, \qquad a = 1,\, 2,\, 3. \tag{25}$$

  In this setting the radial matrix is $\gamma_r = \gamma\vec{n}$ ($\vec{n} = \vec{r}/r$). In order to convert it to the prescribed form we perform the nonuniform unitary rotation V_W. This, so called, "work" representation greatly simplifies the calculations.[1]

$$V_W = \begin{pmatrix} I & 0 \\ 0 & i\vec{\sigma}\vec{n} \end{pmatrix}; \qquad \gamma_r = V_W^\dagger\, \vec{\gamma}\vec{n}\, V_W = \begin{pmatrix} 0 & iI \\ -iI & 0 \end{pmatrix}. \tag{26}$$

  The remaining two matrices γ_θ and γ_ϕ undergo the same rotation.

The Dirac operator in the work frame takes the form:

$$\begin{aligned} -i\nabla\!\!\!\!/\,_W &= -iV_W^\dagger\, \nabla\!\!\!\!/\,_{Cart}\, V_W \\ &= \begin{pmatrix} 0 & \left(\partial_r + \frac{1}{r}\right) + \frac{\hat{\mathcal{B}}_W}{r} \\ -\left(\partial_r + \frac{1}{r}\right) + \frac{\hat{\mathcal{B}}_W}{r} & 0 \end{pmatrix} \end{aligned} \tag{27}$$

Up to the term $\frac{1}{r}$ that accompanies the ∂_r-derivative (it may be eliminated by redefining the wave function $\psi \to \psi/r$) this operator has exactly the required form. Details of the procedure and explicit form of $\hat{\mathcal{B}}_W$ may be found in [10].

[1] The subscript$_W$ honours A. Wipf who invented the trick

3.2 Surface harmonics

The advantage of the work representation is that it converts the eigenfunctions of boundary operator $\hat{\mathcal{B}}$ into the ordinary spherical spinors $\Omega_{j,\,l,\,m}$:

$$
\begin{aligned}
e^{+}_{l+1} &= \Omega_{l+\frac{1}{2},\,l,\,k} = \begin{pmatrix} \sqrt{\frac{j+k}{2j}}\,Y_{l,\,k-\frac{1}{2}} \\ \sqrt{\frac{j-k}{2j}}\,Y_{l,\,k+\frac{1}{2}} \end{pmatrix}, \qquad \lambda = l+1; && \text{(28a)}\\
e^{-}_{l} &= \Omega_{l-\frac{1}{2},\,l,\,k} = \begin{pmatrix} -\sqrt{\frac{j-k+1}{2j+2}}\,Y_{l,\,k-\frac{1}{2}} \\ \sqrt{\frac{j+k+1}{2j+2}}\,Y_{l,\,k+\frac{1}{2}} \end{pmatrix}, \qquad \lambda = -l. && \text{(28b)}
\end{aligned}
$$

Here $Y_{l,\,m}$ are the Legendre polynomials. Note that due to the rotation performed on spinors the quantum number j can no longer be identified with the total angular momentum.

The next step is to expand the propagator in terms of spherical spinors. The Dirac operator consists of two off-diagonal 2×2 blocks. The Green function must have the same matrix structure:

$$
\mathcal{S}(\vec{x},\,\vec{y}) = \sum_{\lambda\neq 0} \begin{pmatrix} 0 & S_\lambda(x,\,y)\,\Omega_\lambda(\hat{x})\,\Omega^\dagger_\lambda(\hat{y}) \\ S^\dagger_\lambda(x,\,y)\,\Omega_\lambda(\hat{x})\,\Omega^\dagger_\lambda(\hat{y}) & 0 \end{pmatrix}. \qquad (29)
$$

Here $r = |\vec{r}|$ and $\hat{r} = \vec{r}/r$. Functions $S_\lambda(x,\,y)$, $S^\dagger_\lambda(x,\,y)$ depend only on radial variables x, y. The summation runs over all possible values of j, l and m and covers both (positive and negative) branches of $\hat{\mathcal{B}}$ spectrum.

The Green function equation splits into a set of separate equations for spectral components:

$$
\begin{aligned}
-\frac{\partial S_\lambda(x,\,y)}{\partial x} - \frac{1-\lambda}{x} S_\lambda(x,\,y) &= \frac{1}{xy}\delta(x-y); && \text{(30a)}\\
\frac{\partial S_\lambda(x,\,y)}{\partial y} + \frac{1+\lambda}{y} S_\lambda(x,\,y) &= \frac{1}{xy}\delta(x-y); && \text{(30b)}
\end{aligned}
$$

and the two similar ones for $S^\dagger_\lambda(x,\,y)$. The arguments $x,\,y \in [0,\,1]$. Obviously the components S_λ must be finite at $x,\,y = 0$. The boundary conditions at $x,\,y = 1$ follow from (24):

$$
\begin{aligned}
\mathcal{P}^{+}\,S\;(1 = x > y) = \mathcal{P}^{-}\,S^\dagger(1 = x > y) &= 0; && \text{(31a)}\\
S\;(x < y = 1)\,\mathcal{P}^{-} = S^\dagger(x < y = 1)\,\mathcal{P}^{+} &= 0. && \text{(31b)}
\end{aligned}
$$

Fortunately the conditions are compatible and for any positive λ the problem (30), (31) possesses the following solution:

$$
\begin{aligned}
&S\;(\vec{x},\,\vec{y}) = \\
&\frac{1}{xy}\sum_{\lambda=1}^{\infty}\theta(y-x)\left(\frac{x}{y}\right)^{\lambda}\,\Omega_\lambda(\hat{x})\,\Omega^\dagger_\lambda(\hat{y}) - \theta(x-y)\left(\frac{y}{x}\right)^{\lambda}\,\Omega_{-\lambda}(\hat{x})\,\Omega^\dagger_{-\lambda}(\hat{y}); \qquad (32a)
\end{aligned}
$$

and for the Hermitean conjugated part

$$
\begin{aligned}
&S^\dagger(\vec{x},\,\vec{y}) = \\
&\frac{1}{xy}\sum_{\lambda=1}^{\infty}\theta(x-y)\left(\frac{y}{x}\right)^{\lambda}\,\Omega_\lambda(\hat{x})\,\Omega^\dagger_\lambda(\hat{y}) - \theta(y-x)\left(\frac{x}{y}\right)^{\lambda}\,\Omega_{-\lambda}(\hat{x})\,\Omega^\dagger_{-\lambda}(\hat{y}). \qquad (32b)
\end{aligned}
$$

The two remaining steps of the calculation are to sum up over λ and to rotate γ-matrices back to their everyday form. We skip the calculations and pass directly to the very simple although startling result:

$$
\mathcal{S}_{Cart}(\vec{x},\,\vec{y}) = \frac{i\,(\not{x} - \not{y})}{4\pi\,|\vec{x}-\vec{y}|^3}. \qquad (33)
$$

This obviously coincides with the **free fermionic Green function** in 3-dimensional space without boundaries. Thus we may conclude that the APS boundary conditions are, in a sense, extremely mild and do not deform the fermionic Green function.

4 Conclusion

We have calculated the Green function of massless fermions in the 3-dimensional spherical cavity with spectral boundary conditions. Surprisingly, despite the constraints imposed at the boundaries, the obtained function is the same as in the infinite space. Thus the APS boundary conditions are in fact "minimal" by nature. They restrict the volume accessible to particles but do not distort the fermion propagator. As a consequence they do not affect technical aspects and algebra of perturbation theory and leave all integrands intact. Nevertheless the presence of boundaries inevitably cuts off long scale divergences and makes many problems at least partly treatable. This makes the spectral boundary conditions a promising tool in studies of the infrared limit of gauge theories. Certainly more work is required in order to find out whether our result admits generalization to higher dimensions and nonspherical domains.

In conclusion I want to acknowledge the partial support of my participation in the Seminar by the Organizers and thank the administration of ITEP for the help within their powers. I would like to express my warmest gratitude to Andreas Wipf whose intuition and discussions strongly inspired the present work. The work was partly supported by the RFBR grant 06–02–16905.

References

[1] A. Chodos, R. L. Jaffe, C. B. Thorn, and V. F. Weisskopf, Phys. Rev. **D9** (1974) 3471.

[2] S. Duerr, A. Wipf, Nucl. Phys. **B443** (1995) 201.

[3] S. Théberge, A. W. Thomas, G. A. Miller, Phys. Rev. **D22** (1980) 2838; Phys. Rev. **D23** (1981) 2106(E).

[4] M. F. Atiah, V. K. Patodi, I. M. Singer, Math. Proc. Camb. Phil. Soc. **77** (1975) 43.

[5] T. Eguchi, P. B. Gilkey, A. J. Hanson, Phys. Rep. **C 66** (1980) 213; H. Roemer, P.B. Schroer, Phys. Lett. **B21** (1977) 182; A. J. Niemi, G .W. Semenoff, Phys. Rep. **135** (1986) 99; Nucl. Phys. **B293** (1987) 559.

[6] A. J. Niemi, G. W. Semenoff, Phys. Rev. **D32** (1985) 471; R. Musto, L. O'Raifeartaigh, Phys. Lett. **B175** (1986) 433.

[7] A. A. Abrikosov jr., Journ. of Phys. **A39** (2006) 6109, hep-th/0512311.

[8] M. Ninomiya, C. I. Tan, Nucl. Phys. **B245** (1985) 199; Z. Q. Ma, Journ. of Phys. **A19** (1986) L317; P. Forgacs, L. O'Raifeartaigh, A. Wipf, Nucl. Phys. **B293** (1987) 559.

[9] A. A. Abrikosov jr., Int. Journ. of Mod. Phys. **A17** (2002) 885; hep-th/0212134.

[10] A. A. Abrikosov jr., A. Wipf, Journ. of Phys. **A40** (2007) 5163.