Large curvature perturbations in single-scalar-field models of inflation

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Abstract

We consider several examples of single-scalar-field inflation models which predict large amplitudes of the curvature perturbation power spectrum at relatively small scales while not contradicting with currently available experimental data on large (cosmological) scales. It is shown that in models with an inflationary potential of double-well type the peaks in the power spectrum, having, in maximum, the amplitude as large as $\mathcal{P}_\mathcal{R} \sim 0.1$, can exist (if parameters of the potential are chosen appropriately). It is shown also that the spectrum amplitude of the same magnitude (at large k values) is predicted in the model with the running mass potential, if the positive spectral index running, n', exists and is about 0.005 at cosmological scales. Because of the large value of perturbation amplitude on small scales, such models generally predict significant amount of primordial black holes produced in the early Universe. The calculations of power spectra are performed numerically, and comparison with approximate analytic formulae is made.

1 Introduction

Inflation, i.e., accelerated expansion in the early Universe, is known to solve several problems of the Big Bang cosmology [1, 2, 3]. It also provides the natural mechanism for generation of primordial curvature perturbations. Standard paradigm states that the quantum fluctuations in the inflaton field have later revealed as classical density perturbations which led to the subsequent structure formation in the Universe and can be observed also in experiments measuring the cosmic microwave background (CMB) anisotropy.

To date, a lot of inflationary models have been proposed. Simplest of these models have only one scalar field ϕ, with the potential $V(\phi)$, the form of which is highly model-dependent (it should be extracted from the underlying theory, but such a theory does not exist yet). Energy density and pressure for the spatially homogenous field are

$$\rho = \frac{\dot{\phi}^2}{2} + V(\phi) \ \ ; \ \ p = \frac{\dot{\phi}^2}{2} - V(\phi) \ , \tag{1}$$

and, if the kinetic term is small compared to the potential term $V(\phi)$, the equation of state is just $p \approx -\rho$, which is suitable for driving inflation. The evolution of $\phi(t)$ is given by the Klein-Gordon equation in the expanding background,

$$\ddot{\phi} + 3H\dot{\phi} + \frac{dV}{d\phi} = 0 \ , \tag{2}$$

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with dot meaning d/dt. The Hubble parameter $H(t) \equiv \dot{a}/a$ during inflation is related to other quantities by the Friedman equation:

$$H^2 = \frac{8\pi}{3m_{Pl}^2}\left(\frac{\dot{\phi}^2}{2} + V(\phi)\right), \tag{3}$$

where $m_{Pl} = 1/\sqrt{G_N}$ is the Planck mass (this constant should not be confused with the reduced Planck mass, which is $M_P = 1/\sqrt{8\pi G_N}$).

For the perturbed scalar field ϕ, we can write

$$\phi_p(t, \vec{x}) = \phi(t) + \delta\phi(t, \vec{x}) ; \tag{4}$$

here, the first term represents the homogenous part and the second is a small perturbation. The equations for field perturbations become much simpler in gauge invariant formalism, which introduces a new variable [4, 5, 6]

$$u = a\delta\phi\Big|_{\text{flat}} , \tag{5}$$

where "flat" means that $\delta\phi$ must be evaluated is spatially flat gauge (this quantity is gauge dependent). Variable u is, by its definition, gauge-independent. Another convenient variable, z, depends only on the background quantities:

$$z = \frac{a\dot{\phi}}{H} \tag{6}$$

(it is sometimes referred to as the "pump field" for scalar perturbations). With this variables, the equation for the perturbation is (for the Fourier mode with the comoving wave number k):

$$u_k'' + \left(k^2 - \frac{z''}{z}\right) u_k = 0. \tag{7}$$

Here, the prime means $d/d\tau$, and τ is the conformal time, defined by $d\tau = dt/a$. Eq. (7) is the equation of the oscillator with time-dependent mass. When the physical length is much smaller than the Hubble length, $aH \ll k$, the solution of Eq. (7) is

$$u_k(\tau) = \frac{1}{\sqrt{2k}}\, e^{-ik\tau} \tag{8}$$

(the Bunch-Davies vacuum [7]). The normalization is dictated by the quantum origin of fluctuations.

In the opposite case of super-horizon perturbations, $aH \gg k$, the growing mode solution of Eq. (7) is $u_k \sim z$. In this case, the comoving curvature perturbation, $\mathcal{R}_k$, becomes asymptotically constant, due to the connection

$$\mathcal{R}_k = \frac{u_k}{z}. \tag{9}$$

The power spectrum of the curvature perturbations is given by

$$\mathcal{P}_{\mathcal{R}}(k) = \frac{4\pi k^3}{(2\pi)^3}\, |\mathcal{R}_k|^2 \ . \tag{10}$$

2 Approximate formulas for $\mathcal{P}_{\mathcal{R}}(k)$ and its numerical calculation

Using Eqs. (7) and (9), we easily obtain the equation for the comoving curvature perturbation

$$\mathcal{R}_k'' + 2\frac{z'}{z}\mathcal{R}_k' + k^2\mathcal{R}_k = 0, \tag{11}$$

$$\frac{z'}{z} = aH(1 + \epsilon - \eta) \ . \quad (12)$$

In the last equality, the Hubble slow-roll parameters are defined by the relations [8]

$$\epsilon = -\frac{\dot{H}}{H^2} = \frac{4\pi}{m_{Pl}^2}\frac{\dot{\phi}^2}{H^2} \quad , \quad \delta = -\frac{\ddot{\phi}}{H\dot{\phi}} \ . \quad (13)$$

If, during inflation, this parameters are small compared to unity, the slow-roll limit applies. In this case, terms $\ddot{\phi}$ in Eq. (2) and $\dot{\phi}^2$ in Eq. (3) can be neglected, and approximate solutions of this equations can be easily obtained. Outside the slow-roll limit, ϵ and δ are not necessarily small.

The power spectrum $\mathcal{P}_{\mathcal{R}}(k)$ to the leading order in the slow-roll approximation is [9]

$$\mathcal{P}_{\mathcal{R}}^{1/2}(k) = \frac{1}{2\pi}\left.\frac{H^2}{|\dot{\phi}|}\right|_{k=aH} = \left.\frac{H}{m_{Pl}\sqrt{\pi\epsilon}}\right|_{k=aH} , \quad (14)$$

and to the first order it is given by the Stewart-Lyth formula [10]:

$$\mathcal{P}_{\mathcal{R}}^{1/2}(k) = [1 - (2C+1)\epsilon + C\delta]\left.\frac{1}{2\pi}\frac{H^2}{|\dot{\phi}|}\right|_{k=aH} \quad ; \quad C \approx -0.73 \ . \quad (15)$$

The second-order formulas are also available [11], but they are much more complicated and involve higher-order slow-roll parameters.

It is well known that in situations when there is a failure of the slow-roll evolution the perturbations on super-horizon scales can be amplified and specific features in the power spectrum can arise [12, 13, 14, 15, 16, 17]. This means that the predictions of the slow-roll approximation which are based on the assumption that perturbations reach an asymptotic regime outside the horizon cannot be trusted.

In such cases, the numerical calculation of perturbation amplitude is needed. To do this, we must solve Eq. (7) or (11) numerically, using the initial condition (8) in the sub-horizon regime. Sometimes it is even more convenient to change the independent variable to t. The equation for $\mathcal{R}_k$ then becomes

$$\ddot{\mathcal{R}}_k + H\dot{\mathcal{R}}_k(3 + 2\epsilon - 2\delta) + \frac{k^2}{a^2}\mathcal{R}_k = 0 \ . \quad (16)$$

The numerical integration starts when $k = N_{\rm und} \cdot aH$, with $N_{\rm und} \gtrsim 100$, and proceeds to the region $k \ll aH$, where $\mathcal{R}_k$ is effectively constant. We have checked, that the result for $\mathcal{P}_{\mathcal{R}}(k)$ calculated by this way does not change when we increase the value of $N_{\rm und}$. Of course, $\mathcal{P}_{\mathcal{R}}(k)$ also does not depend on the choice of the initial phase in (8), because it is proportional to $|\mathcal{R}_k|^2$ (see Eq. (10)), and all coefficients in equation for $\mathcal{R}_k$ are real functions.

3 Inflationary models and the formation of primordial black holes

Since the pioneering works of Zeldovich and Novikov [18] and Hawking [19], the possibility of black hole formation in the early Universe has been widely discussed (for reviews, see, e.g., [20] and [21]). Primordial black holes (PBHs), as they are commonly called, can contribute to dark matter, if they are massive enough ($M_{\rm PBH} \gtrsim 10^{15}$g). Evaporation of black holes, predicted by Hawking [22], causes smaller PBHs to completely evaporate by now, so such PBHs can not be observed directly, but non-observation of products of their evaporation can, in principle, be used to constrain their initial abundance and the value of power spectrum $\mathcal{P}_{\mathcal{R}}$ in the region of small k values (see, e.g., [23]). The most challenging experimental task here is to observe the

PBH on the final stage of evaporation. To date, such direct searches have only placed the upper limits on PBH number density (see recent papers [24, 25, 26]), but the work will undoubtedly be continued.

In the context of inflationary models with a single scalar field, the possibility of PBH formation was discussed in detail in [13], where a "plateau" inflaton potential (having a flat part in some region of field values) was considered. The motivation is simple: the slow-roll formula (14) gives

$$\mathcal{P}_{\mathcal{R}}(k) \sim \frac{V^3}{(dV/d\phi)^2} , \tag{17}$$

so more flat regions of the potential will give higher values of power spectrum. If, on some small scale k, the values of order $10^{-3} \div 10^{-1}$ are approached for $\mathcal{P}_{\mathcal{R}}$, the probability for the density contrast to be of order of unity at the moment of horizon re-entry for perturbations with comoving size k^{-1} can be large. The significant PBH production in such case becomes possible. (One should note that this is not the only possible mechanism of PBH production, though the most natural one - see reviews [20, 21]).

Several other "toy" potentials have been considered in [14]. In work [27], a scenario with multiple inflation stages supported by one scalar field was proposed. It was shown that PBH formation in such case is possible, but the power spectrum of perturbations was only estimated using approximate formulas at that time. The explicit potential form that was found to be compatible with PBH production is

$$V(\phi) = -\frac{1}{2}m^2\phi^2 + \frac{\lambda}{4}\phi^4 \left(\ln\left|\frac{\phi}{v}\right| - \frac{1}{4} \right) + V_0 , \tag{18}$$

which is the Coleman-Weinberg (CW) potential [28] with added negative mass term; the constant V_0 is determined from the condition $V(\phi_m) = 0$ and the local minimum of the potential is approached at $\phi = \pm\phi_m$.

Recently, several papers appeared [29, 30, 31, 32], in which single-field inflation models predicting (potentially) large amplitudes of the curvature perturbations on relatively small scales are discussed. It is shown in [29] that large class of such models exists, namely, the models with a potential of hill-top type (the idea of the hill-top inflation was proposed, to author's knowledge, in the earlier work [33]). In such models, the potential can be of concave-downward form at cosmological scales (in accordance with data) and be much flatter at the end of inflation when small scales leave horizon. Correspondingly, the amplitude of the perturbation power spectrum can be rather large. However, simplest hill-top potentials, considered by the authors of [29],

$$V(\phi) = V_0 \left(1 + \frac{1}{2}\eta_0 \frac{\phi^2}{M_P^2} \right) - \lambda \frac{\phi^p}{M_P^{p-4}} , \tag{19}$$

with $p = 4$ and $p = 6$, were shown not to satisfy observational constraint $n' < 0.01$. They require $n' \sim 0.1$ for significant production of PBHs. The spectral index in such models is typically growing monotonously from values $n < 1$ at large scales to $n > 1$ at smaller ones.

It is noticed in [29] that the running mass model [34, 35], having the potential with the similar behavior, also can predict the large spectrum amplitude.

Paper [30] discusses also more general scenarios of producing large amplitudes of perturbation spectrum. It shows the limitedness of the standard procedure of potential reconstruction which can easily miss the potentials leading to large spectrum amplitude and to noticeable PBH production.

Authors of [31] carried out the numerical calculation of the power spectrum using the CW potential and explicitly showed that PBH production is possible in single-field models of two-stage type ("chaotic + new").

In the present work we continue a study of the problems discussed in the previous papers [29, 30, 31]. In the following sections we consider several examples of single-scalar-field inflation

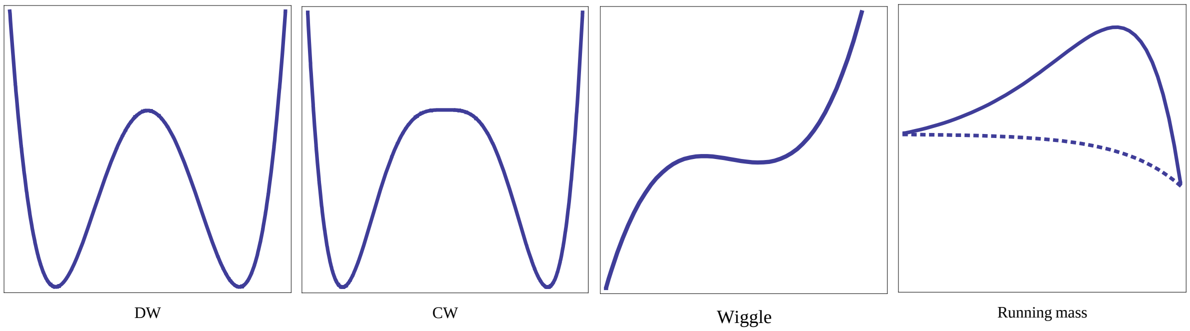

Figure 1: Forms of the potentials considered in this work. Shown from left to right are: double-well, Coleman-Weinberg, wiggle, and running mass model potentials.

models which predict large amplitudes of the curvature perturbation power spectrum at relatively small scales while not contradicting with currently available experimental data on large (cosmological) scales.

4 The double-well potential

This form of the inflaton potential having an unstable local maximum at the origin has been discussed many times in studies of eternal and new inflation. The main problem was to realize the initial condition for the new inflation when system starts from a top of the hill. Ten years ago the model of "chaotic new inflation" has been proposed [27], in which the system climbs on the top during dynamical evolution of the inflaton field with initial conditions coinciding with those of chaotic inflation models. In the approach of [27] the inflation has two stages, chaotic and new, and during transition from the first stage to the second the slow-roll conditions break down (in general).

The potential has two parameters (its form is sketched in Fig. 1):

$$V(\phi) = \frac{\lambda}{4}(\phi^2 - v^2)^2 \ . \tag{20}$$

The inflaton starts with the rather high value of ϕ (we take $\phi_{\rm in} \sim 5m_{Pl}$) and rolls down to the origin. The parameter λ is fixed by a normalization of the power spectrum on experimental data,

$$\mathcal{P}_{\mathcal{R}}(k = 0.002 \ {\rm Mpc}^{-1}) \cong 2.4 \times 10^{-9}, \tag{21}$$

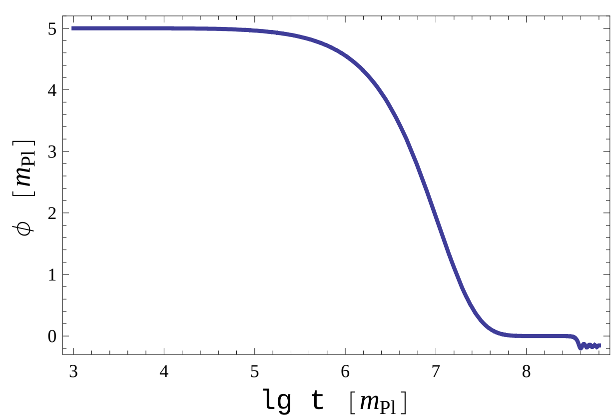

Figure 2: The solution of the background equation for inflation with the double-well potential (20). The parameters of the potential are: $v = 0.16286748 m_{Pl}, \lambda = 1.7 \times 10^{-13}$.

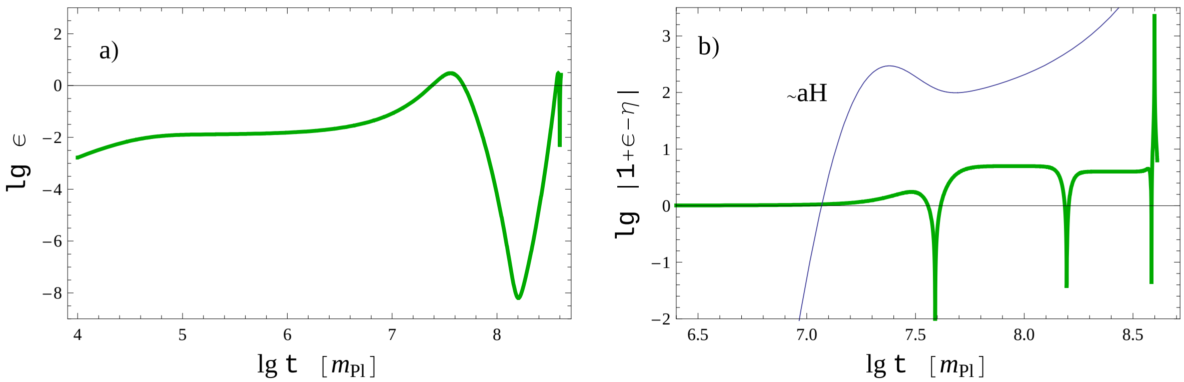

Figure 3: The time dependence of the parameter ϵ and the combination $1+\epsilon-\eta$ corresponding to the background field evolution shown in Fig. 2

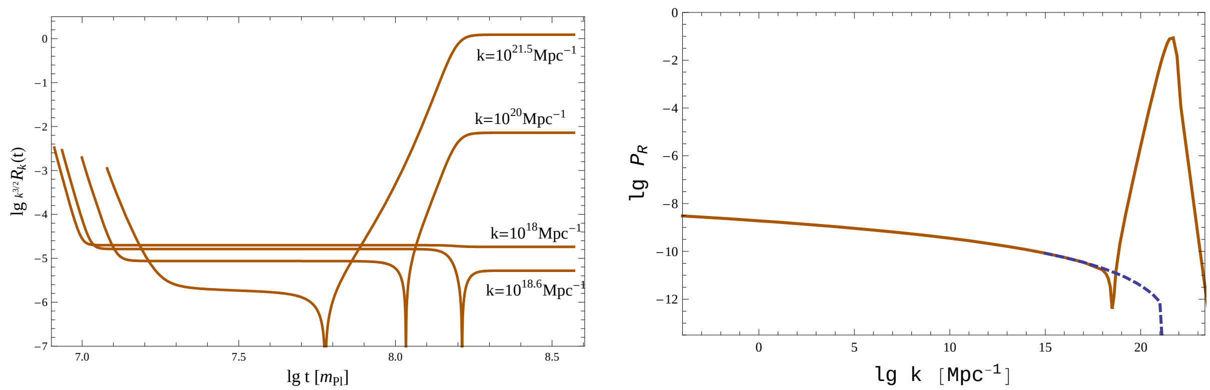

Figure 4: *(left panel)* A time evolution of the curvature perturbation $\mathcal{R}_k(t)$ for several different values of wave number k during inflation with the DW potential. *(right panel)* The numerically calculated power spectrum $\mathcal{P}_\mathcal{R}(k)$ for the model with the DW potential (solid line) and the slow-roll prediction (dashed line). For both panels, the parameters of the potential are the same as in Fig. 2.

which leads to $\lambda \sim 10^{-13}$. The evolution of the system strongly depends on the value of v: if v is finely tuned, ϕ can spend some time near the origin, i.e. on the top, and then roll down to one of the two minima. In Figs. 2 and 3a the time evolution for the inflaton and the parameter ϵ for the definite values of the parameters λ , v are shown. One can see that, really, $\phi \approx 0$ at some period of time and, what is important, the slow-roll approximation is invalid ($\epsilon \sim 1$) just at the time of the transition from a rolling to a temporary stay at the top of the potential.

It had been demonstrated in [15] that solutions of the equation (11), at $k \ll aH$, i.e., outside horizon, are well approximated by constant if the coefficient of the friction term, z'/z, doesn't change sign near the horizon crossing. In the opposite case, if z'/z changes sign at some time, the friction term becomes a negative driving term, and one can expect strong effects on modes which left horizon near that time. In the present paper we study the corresponding features of the power spectrum, following closely the analysis of [15].

According to Eq. (12), z'/z is proportional to $1+\epsilon-\eta$ and the comoving Hubble wave number aH. The time dependences of these functions are shown in Fig. 3b. One can see that the interruption of inflation correlates with the change of the sign of $1+\epsilon-\eta$.

The time evolution of curvature perturbations for several modes is shown in Fig. 4 (left panel). It is clearly seen that the perturbations $\mathcal{R}_k$ for different modes freeze out at different amplitudes. The mode which crosses horizon near the moment of time when the sign of $1+\epsilon-\eta$ changes (i.e., near $t \approx 7.5 m_{Pl}$) freezes at maximum amplitude, due to the exponentially growing

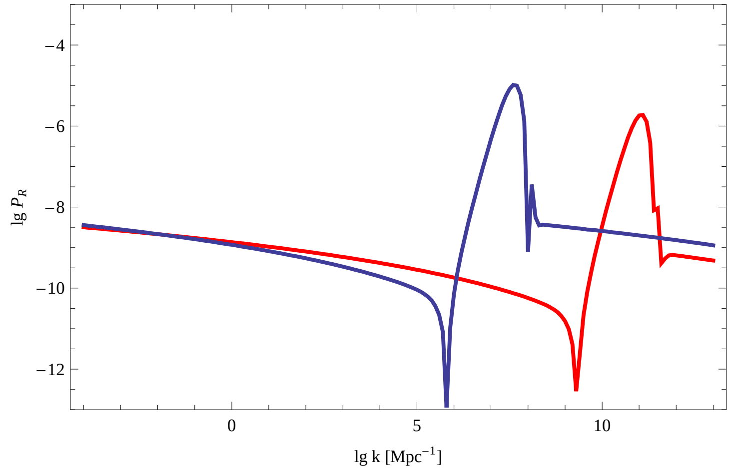

Figure 5: The result for the power spectrum $\mathcal{P}_{\mathcal{R}}(k)$ calculation for the CW potential (25), for two sets of parameters. Left peak is for $v = 1.113 M_P, \lambda = 5.5 \times 10^{-13}$. For the right peak, $v = 1.112 M_P, \lambda = 2.4 \times 10^{-13}$.

driving term in Eq. (11) (which is most effective just for this mode). It leads to the characteristic peak in the power spectrum $\mathcal{P}_{\mathcal{R}}(k)$, shown in Fig. 4 (right panel).

The calculations of $\mathcal{R}_k$ (Fig. 4) are carried out up to the end of inflation, and the calculated power spectrum also corresponds to this moment of time. We estimate approximately the reheat temperature in our case as $\sim (\lambda v^4)^{1/4} \sim 10^{14}$ GeV. The horizon mass at the beginning of radiation era is

$$M_{hi} \sim 10^{17} \mathrm{g} \left(\frac{10^7 \mathrm{GeV}}{T_{\mathrm{RH}}} \right)^2 \sim 10^3 \mathrm{g} \, , \tag{22}$$

and maximum wave number, which equals the Hubble radius at the end of inflation, is

$$k_{\mathrm{end}} = a_{\mathrm{eq}} H_{\mathrm{eq}} \left(\frac{M_{\mathrm{eq}}}{M_{hi}} \right)^{1/2} \sim 10^{23} \ \mathrm{Mpc}^{-1}. \tag{23}$$

The mass of the produced PBHs is roughly equal to the horizon mass at the moment when the scale with comoving size k_{peak}^{-1} enters horizon. So we can write

$$M_{BH} \approx M_h = M_e \left(\frac{k_e}{k_{\mathrm{peak}}} \right)^2 , \tag{24}$$

where $k_e = a_e H_e$ (quantities evaluated at the end of inflation). In our case, $k_e / k_{\mathrm{peak}} \sim 10^2$, so $M_{BH} \sim 10^7$ g (in the peak). Such light PBHs have evaporated before nucleosynthesis, but their existence could still, in principle, lead to observable consequences.

5 The Coleman-Weinberg potential

The CW potential has the form [28]:

$$V(\phi) = \frac{\lambda}{4} \phi^4 \left(\ln \left| \frac{\phi}{v} \right| - \frac{1}{4} \right) + \frac{\lambda}{16} v^4. \tag{25}$$

It looks very similar to the previous one (see Fig. 1), but the important difference is its behavior near the origin. Namely, the CW potential behaves as $A + B\phi^4 \ln(\phi/v)$ near the origin, i.e., it is more flat near zero, in comparison with the DW potential. Therefore, it has more e-folds of "new inflation" [27] and, as a consequence, the peaks of the power spectrum (arising, as in the

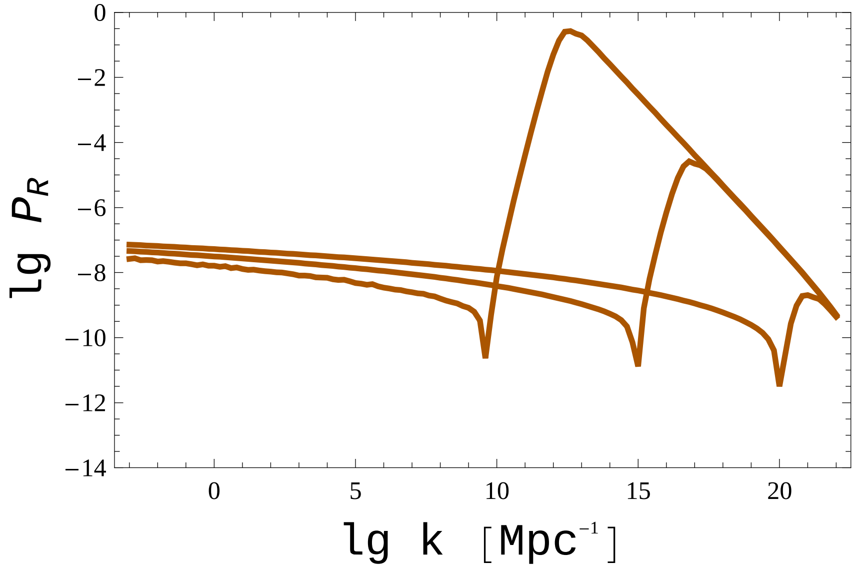

Figure 6: The result for the power spectrum $\mathcal{P}_\mathcal{R}(k)$ calculation for the wiggle potential. The parameters used are: $A = 136.717$; $\lambda = 5 \times 10^{-14}$; for peaks from top to bottom $B = 0.11155; B = 0.111; B = 0.05$.

previous case, due to the temporary interruption of inflation) correspond to relatively smaller k values.

In Fig. 5 two examples of the power spectrum calculations are shown for two different sets of parameter values. As before, the peaks are very distinct, although their amplitudes are smaller.

Recently, it has been shown [31] that inflation with CW potential is capable to produce significant number of PBHs: the parameter v can be chosen (by finest tuning) in such a way that inflaton field makes several oscillations from one minimum to another before it climbs on the top and "new inflation" starts.

6 Wiggle potential

Yet another simple example is the wiggle potential, proposed in [14]:

$$V(\phi) = \lambda(1 + A\phi^3 - B\phi) \tag{26}$$

(see Fig. 1). The bump on the path of the inflaton causes it to slow down, producing a spike in the power spectrum [14], Fig. 6. In this figure, we show the results for power spectrum calculation for several sets of model parameters. The more e-folds the inflaton spends in the wiggle region, the higher is the produced spike. The mechanism of its formation is similar to one considered in previous sections.

Again, we see that for producing a large spike or a peak in the power spectrum, rather large fine tuning of parameters of the potential is required.

7 The running mass model

We consider in more detail a case of the running mass inflation model [34, 35, 36, 37, 38, 39, 40] which predicts a spectral index with rather strong scale dependence. The potential in this case takes into account quantum corrections in the context of softly broken global supersymmetry and is given by the formula

$$V = V_0 + \frac{1}{2}m^2(\ln\phi)\phi^2. \tag{27}$$

The dependence of the inflaton mass on the renormalization scale ϕ is determined by the solution of the renormalization group equation (RGE).

1. The inflationary potential in supergravity theory is of the order of $M_{\rm inf}^4$, where $M_{\rm inf}$ is the scale of supersymmetry breaking during inflation. In turn, the mass-squared of the inflaton (and any other scalar field) in supergravity has, in general, the order of the square of Hubble expansion rate during inflation,

$$|m^2| \sim H_I^2 = \frac{V_0}{3M_P^2}. \tag{28}$$

We suppose, for simplicity (see [34, 35, 37, 38]), that $M_{\rm inf} \sim M_{\rm s}$, where $M_{\rm s}$ is the scale of supersymmetry breaking in the vacuum,

$$M_{\rm s} \sim \sqrt{\tilde{m}_s M_P} \sim 10^{11}{\rm GeV} \sim 3 \times 10^{-8} M_P \tag{29}$$

($\tilde{m}_s$ is the scale of squarks and slepton masses, $\tilde{m}_s \sim 3$ TeV). These assumptions give the scale of the inflationary potential:

$$V_0 \sim M_{\rm s}^4 \sim 10^{-30} M_P^4 \quad , \quad H_I \approx 10^{-15} M_P. \tag{30}$$

2. RGE for the inflaton mass is the following (we consider a model [37, 38] of hybrid inflation using the softly broken SUSY with gauge group $SU(N)$ and small Yukawa coupling):

$$m^2(t) = m_0^2 - A\tilde{m}_0^2 \left[1 - \frac{1}{(1+\tilde{\alpha}_0 t)^2}\right] \quad , \quad t \equiv \ln\frac{\phi}{M_P} , \tag{31}$$

m_0^2 and $\tilde{m}_0^2$ are, correspondingly, the inflaton and gaugino masses at $\phi = M_P$,

$$\tilde{\alpha}_0 = \frac{B\alpha_0}{2\pi}, \tag{32}$$

α_0 is the gauge coupling constant, $\alpha_0 = g^2/4\pi$. A and B are positive numbers of order 1, which are different for different variants of the model, even if they are based on the same supersymmetric gauge group $SU(N)$ (it depends on a form of the superpotential, particle content of supermultiplets, etc). We use in the present parer the variant of [38] and, correspondingly, put everywhere below $A = 2$ and $B = N = 2$.

3. A truncated Taylor expansion of the potential around the particular scale ϕ_0 (in our case, ϕ_0 is the inflaton value at the epoch of horizon exit for the pivot scale $k_0 \approx 0.002h$ Mpc^{-1}) is

$$V(\phi) = V_0 + \frac{\phi^2}{2}\left[m^2(\ln(\phi_0)) - c\frac{V_0}{M_P^2}\ln\frac{\phi}{\phi_0} + ...\right]. \tag{33}$$

Here, constant c is defined by the equation

$$c\frac{V_0}{M_P^2} = -\left.\frac{dm^2}{d\ln\phi}\right|_{\phi=\phi_0} . \tag{34}$$

In turn, a Taylor expansion of Eq. (31) up to linear terms gives ($t_0 = \ln\frac{\phi_0}{M_P}$):

$$m^2(t) = m^2(t_0) - 4\tilde{m}_0^2 \frac{\tilde{\alpha}_0}{(1+\tilde{\alpha}_0 t_0)^3}\ln\frac{\phi}{\phi_0} . \tag{35}$$

From eqs. (34) and (35) we obtain the expression for the constant c,

$$c\frac{V_0}{M_P^2} = 4\tilde{m}_0^2 \frac{\tilde{\alpha}_0}{(1+\tilde{\alpha}_0 t_0)^3}. \tag{36}$$

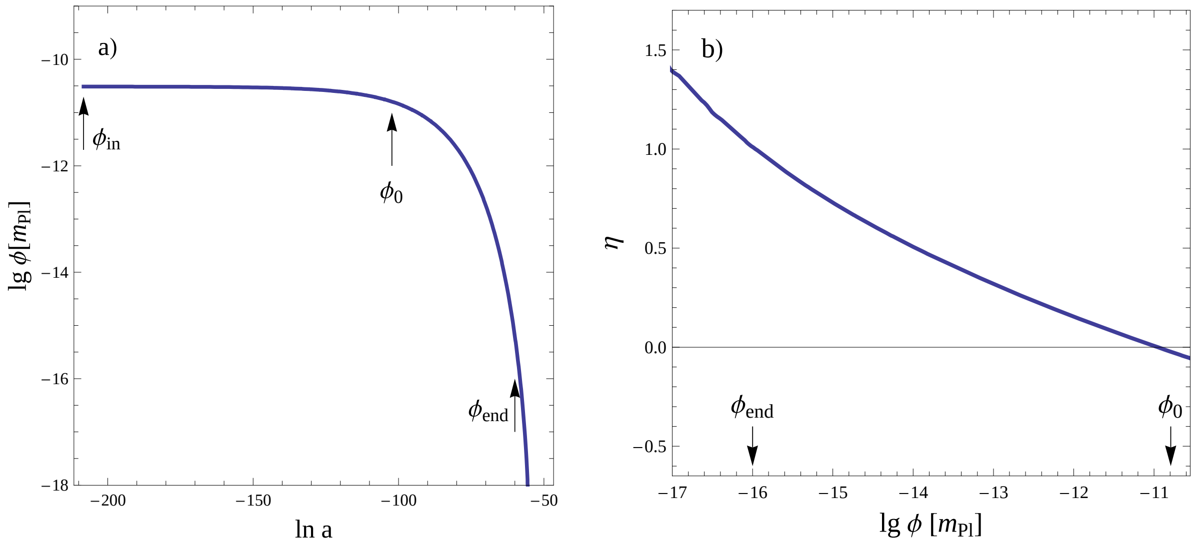

Figure 7: **a)** Evolution of the inflaton field $\phi(\ln a)$ in the running mass model. **b)** The dependence of the parameter η on a value of the field ϕ. For both plots, $H_I = 10^{-15} M_P$, $c = 0.062$, $s = 0.040$.

If $|m_0^2| \sim \tilde{m}_0^2 \approx H_I^2$, then

$$c = \frac{4}{3} \frac{\tilde{\alpha}_0}{(1+\tilde{\alpha}_0 t_0)^3}. \tag{37}$$

It appears (see Fig. 7b) that in our example $\phi_0 \sim 10^{-10} M_P$, so, $t_0 \sim \ln 10^{-10} \sim (-23)$. Assuming that $\alpha_0 \sim 1/24$ (as in SUSY-GUT models), one has $\tilde{\alpha}_0 \sim \frac{2}{2\pi}\frac{1}{24}$. In such a case, $c \sim 4\tilde{\alpha}_0 \sim 0.06$.

If we would keep terms of higher order in $t - t_0 = \ln \frac{\phi}{\phi_0}$ in the Taylor expansion of $m^2(t)$ in Eq. (35) we would see that the real expansion parameter is $\tilde{\alpha}_0 \ln \frac{\phi}{\phi_0}$ rather than $\ln \frac{\phi}{\phi_0}$. The smallest value of ϕ, $\phi_{\rm end}$, in our case is $\sim 10^{-16} M_P$ (see Fig. 7b). Even for such value of $\phi_{\rm end}$, the expansion parameter is rather small,

$$\tilde{\alpha}_0 \ln \frac{\phi_{\rm end}}{\phi_0} \sim \tilde{\alpha}_0 \ln 10^{-6} \sim (-0.1)\ . \tag{38}$$

Having this in mind, we will use the linear approximation for the inflaton mass (Eq. (35)) in the entire region of inflaton field values exploited in the present paper.

Following the previous papers, we introduce also another parameter,

$$s = c \ln\left(\frac{\phi_*}{\phi_0}\right), \tag{39}$$

where ϕ_* is the inflaton value corresponding to a maximum of the potential. This parameter connects the field value ϕ_0 with the Hubble parameter during inflation and with the normalization of the CMB power spectrum:

$$\phi_0 s = \frac{H_I}{2\pi \mathcal{P}_{\mathcal{R}}^{1/2}(k_0)}. \tag{40}$$

4. The minimum value of the inflaton field which corresponds to the end of inflation can be determined from the approximate equation [38]

$$\eta = M_P^2 \frac{V''}{V} \cong \frac{M_P^2}{V_0} m^2 = 1\ . \tag{41}$$

Using RGE, one obtains from this formula the relation

$$\frac{M_P^2}{V_0}\left(m_0^2 - A\tilde{m}_0^2 + \frac{A\tilde{m}_0^2}{(1+\tilde{\alpha}_0 t)^2}\right) = 1. \tag{42}$$

Substituting here $A = 2$, $\tilde{m}_0^2 = |m_0^2| = V_0/3M_P^2$, one has finally the approximate expression for $\phi_{\rm end}$,

$$\phi_{\rm end} = M_P \exp\left[-\frac{1}{\tilde{\alpha}_0}\left(1-\frac{1}{\sqrt{3}}\right)\right] , \tag{43}$$

which shows that the minimum field value is very sensitive to the value of the model parameter $\tilde{\alpha}_0$ and, in our case, does not depend on V_0. More exactly, the condition $\eta = 1$ means the end of the *slow-roll part* of inflation. We suppose, as usual (see, e.g. [34, 35]) that in reality inflation ends by hybrid mechanism, and the critical value of inflaton field, $\phi_{\rm cr}$, is determined by the value of the Yukawa coupling λ (in spite of the inequality $\lambda^2 \ll \alpha$). One can check [38] that the value of λ can always be chosen such that $\phi_{\rm cr} < \phi_{\rm end}$ and slow-roll ends before the reaching of $\phi_{\rm cr}$.

One should note that the accuracy of the approximate formula (43) is not very good. Luckily, in the approach based on the numerical calculation of the power spectrum there is no need to use it, because the value of $\phi_{\rm end}$ appears in a course of the calculation (Fig. 7b).

An analysis of CMB anisotropy data [41, 42], including other types of observation [43], leads to the following main conclusions:

i) the power spectrum of scalar curvature perturbations is red, i.e., the spectral index at cosmological scales is smaller than unity,

$$n = 0.963^{+0.014}_{-0.015} \ \ ; \ \ dn/d\ln k = -0.037 \pm 0.028 \tag{44}$$

(WMAP 5-year data, [42]);

ii) observations are consistent, or, at least, are not in contradiction with the small positive running of the spectral index, $n_0' < 0.01$;

iii) the contribution of tensor perturbations in the value of the spectral index is small ($\lesssim 10^{-2}$) and, as a result, $n \approx 1 + 2\eta$; it means that η is negative, and the potential must be concave-downward (i.e., of hill-top type), while cosmological scales cross horizon during inflation [29] (see, however, recent analysis in [44]: strictly speaking, the present data still admit any sign of η and V'').

These conclusions constrain the possible values of the parameters s and c. Approximately, for cosmological scale one has

$$n_0 - 1 \approx 2(s-c) \ \ , \ \ n_0' \approx 2sc\,. \tag{45}$$

From the conclusion *iii)* it follows that $c > 0$ (it is consistent with Eq. (40)), from the positivity of n_0' (the conclusion *ii)*) it follows that $s > 0$. At last, the conclusion *i)* leads to the inequality $s < c$.

We choose for the power spectrum calculation the following values:

$$c = 0.062 \ \ , \ \ s = 0.040. \tag{46}$$

These numbers correspond, at cosmological scales, to the following values of slow-roll parameters:

$$\epsilon \approx \frac{s\phi_0^2}{M_P^2} \sim 10^{-21} \ \ ; \ \ \eta \approx s - c \sim (-0.02)\ , \tag{47}$$

that seems to be consistent with the present data [32].

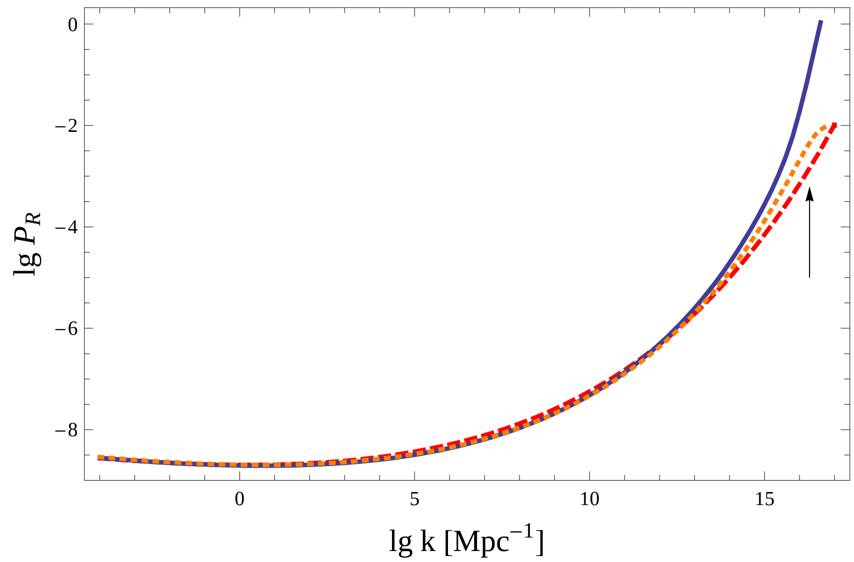

Figure 8: Power spectrum $\mathcal{P}_\mathcal{R}(k)$ in the running mass model, calculated numerically (solid line), by the approximate analytic formula (48) (long-dashed line) and using the Stewart-Lyth extended slow-roll approximation (short-dashed line). The parameters of the potential are the same as used in Fig. 7. The arrow shows the value of $k_{\rm end}$.

To check the validity of the slow-roll approximation, we calculate the spectrum by the three ways: *i)* using the approximate analytic slow-roll formula

$$\frac{\mathcal{P}_\mathcal{R}(k)}{\mathcal{P}_\mathcal{R}(k_0)} = \exp\left[\frac{2s}{c}\left(e^{c\Delta N(k)} - 1\right) - 2c\Delta N(k)\right] \quad ; \quad \Delta N(k) \equiv \ln\frac{k}{k_0} \tag{48}$$

(this expression is easily derived from the simplest slow-roll prediction (14));

ii) using the Stewart-Lyth approximation [10], which is valid to first order in the slow-roll approximation, Eq. (15);

iii) by numerical integration of the differential equation for $\mathcal{R}_k$, Eq. (11).

The results of the calculations are presented in Figs. 7 and 8. Fig. 7 shows the evolution of the inflaton field ϕ with the scale factor and a growth of the slow-roll parameter η with a decrease of ϕ from ϕ_0 to $\phi_{\rm end}$. The power spectrum is shown in Fig. 8 for a broad interval of comoving wave numbers. It is clearly seen that near the end of inflation, when

$$\phi \sim 10^{-16} M_P \quad , \quad k \sim k_{\rm end} = a_{\rm end} H_{\rm end} = 3\times 10^{16}\mathrm{Mpc}^{-1} \; , \tag{49}$$

the slow-roll formulae are inaccurate: they strongly underestimate values of $\mathcal{P}_\mathcal{R}$.

8 Conclusions

We investigated thoroughly, as a particular example, the model of two-stage inflation with a potential of the double-well (DW) form, and showed that the characteristic features of the power spectrum in models of this type (such as an amplitude and a position of the peak, a degree of tuning of parameters of the potential) are very sensitive to an exact form of the potential. We considered several other potentials (namely, CW and wiggle) and found that they are also capable of producing analogous features in $\mathcal{P}_\mathcal{R}(k)$. Further, we carried out the numerical calculation of the power spectrum in a running mass model and showed that the spectrum amplitude at small scales can be rather large. Our calculation differs from the previous one [45] in several aspects: we express the results through the values of parameters s, c, which are used nowadays and prove to be very convenient for a comparison with data; we studied, in details, the difference in predictions of slow-roll and numerical approaches at high k-values; we exactly specified the value of the positive running, n', which corresponds to our spectrum prediction.

We used numerical methods for the power spectrum calculation, comparing the results with ones obtained with approximate analytic formulas. We have shown that in many cases, when there is a failure of the slow-roll evolution, the difference can be significant.

We showed that several single-field inflationary models exist, in which large power on small scales is produced, even with the "red" ($n < 1$) spectrum on large scales. For the running mass model, this can be achieved even with $n \sim 0.95$ and $n' \sim 10^{-3}$ on large scales. In such models, the significant number of PBHs is produced in the early Universe, so their existence remains an open possibility.

We calculated the curvature perturbations in terms of the classical trajectories of a scalar field associating, in particular, points in a field space with definite numbers of e-folds from the end of inflation. This description becomes incorrect if the quantum diffusion destroys the classical evolution of the field. In this case we should use the methods of stochastic inflation. The latter approach operates with the coarse-grained field, which is defined to be spatial average of the field ϕ over a physical volume with size larger than the Hubble radius H^{-1}. Our analysis of the quantum diffusion effects for the running mass model potential can be found in [46].

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