Infinite statistics, induced gravity and combinatorial view of the hierarchy problem

V.I.Shevchenko[a*]

[a] *ITEP, B.Cheremushkinskaya, 25 Moscow 117218 Russia*

Abstract

If one assumes that fundamental degrees of freedom at TeV scale obey the so called infinite statistics and condense, it is possible to get exponentially large multiplicity and shift gravitational scale seen by low energy observer from a few TeV range to 10^{19} GeV by induced gravity mechanism.

A truly remarkable phenomenon one often observes in Nature is coexistence of a few vastly different scales in one theoretical framework, widely known as a problem of "large numbers". The best known and physically the most interesting example of this kind is given by the Standard Model. As is well known this theory contains a set of scales starting from the electron mass $m_e = 511\text{keV}/c^2$ and going up to QCD scale $m_p = 938\text{MeV}/c^2$, weak scale $m_W = 80\text{GeV}/c^2$ and finally ending by the Planck scale $M_P = 1.2 \cdot 10^{19}$ GeV$/c^2$ which represents, as many physicists believe, the ultimate ultraviolet edge of our world.[1] Despite each large ratio in this tower calls for explanation, of particular interest is the parameter M_P in the weak scale units, which is given by the number of order of 10^{17}. The danger this large number is of for the stability of radiative corrections to Higgs boson mass has been widely discussed in the literature.

A new approach to the hierarchy problem known as TeV-scale gravity and extra-dimensions scenarios came about a decade ago. These models assume that the geometric properties of our familiar (3+1) dimensional space-time change at some scale L, which is supposed to be much larger than $L_P = 1/M_P$ and perhaps of (1-2 TeV)$^{-1}$ range. This change can be accompanied by appearance of some additional particles, for example of the Kaluza-Klein type. Among attractive features of these models is emergent nature of the Planck scale L_P. In the original proposal [1, 2] the fundamental ratio between L_P and L takes the form

$$\left(\frac{L}{L_P}\right)^2 \sim \frac{V_n}{L^n} \tag{1}$$

where L is electroweak scale, n - number of compact extra dimensions and V_n - their volume. Soft SM fields propagates only in 4 dimensions, while gravitons feel the full $4+n$ dimensional geometry. In this scenario the smallness of the ratio L/L_P follows from the large (in units of L, i.e. sub-millimeter for the most phenomenologically interesting case $n = 2$) size of extra dimensions. The picture suggested in [3, 4] is different: the model with one warped extra dimension generates the hierarchy

$$\log \frac{L}{L_P} \sim \frac{\pi r_c}{L_P^{(5)}} \tag{2}$$

where πr_c is the size of compact extra dimension and $L_P^{(5)}$ - fundamental five dimensional Planck length. The logarithmic function maps huge hierarchy between L and L_P into much weaker

*e-mail: shevchen@itep.ru

[1] We leave aside the lower part of this tower, corresponding to smaller energies relevant for condensed matter physics, chemistry and biology.

hierarchy between r_c and $L_P^{(5)}$. In other words the weakness of gravity in this approach follows from the fact that only exponential tail of the full graviton's wave function can be seen by four-dimensional observer.

In a broader prospective, both these scenarios reduce hierarchy of the mass scales to some geometrical hierarchy. However, in functional terms, (1) and (2) are quite different. While the power law (1) makes L_P small introducing some large artificial scale R - extra dimension size: $L/L_P \sim (R/L)^{n/2}$; expression (2) corresponds to exponential increase: $L/L_P \sim \exp\gamma$, where huge number L/L_P is mapped into not so huge number γ. The physical reason for appearance of the exponent in [3] is quantum tunneling of gravitons.

Recently the extra-dimensional approach to the hierarchy problem has been given a new interpretation in [5, 6] (see also earlier papers [7, 8, 9]). It is based on the following physical idea: suppose that above some scale L there is a dramatic rise of multiplicity, i.e. the number of relevant degrees of freedom N becomes huge at energies $E \gtrsim 1/L$. Then one can argue, both on perturbative and on nonperturbative grounds, that the number of stable species with typical mass M must not exceed the following bound:

$$NM^2 \lesssim M_P^2 \tag{3}$$

up to some logarithmic corrections. Another incarnation of the same idea takes a single particle species of mass M but carrying exactly conserved quantum number of periodicity N, for example, Z_N gauge symmetry charge. Then the gravitational cutoff in such theories goes down to $M_P/\sqrt{N}$ and can be lowered to a TeV scale (thus solving the hierarchy problem, or, at least, giving it a completely new prospective) if one takes N of the order of 10^{32}.

The bound (3) can be given a natural interpretation from Sakharov's induced gravity [10] point of view. Indeed, typical contribution Einstein-Hilbert gravitational Lagrangian gets from one-loop effective action of matter particles in curved space can be written as:

$$M_P^2 = \frac{1}{G} = \frac{1}{G_0} - \frac{1}{2\pi}\mathrm{Tr}_{\,s}\left[M^2 - \mu^2\log\left(\frac{M^2}{\mu^2}\right)\right] \tag{4}$$

where G_0 is "bare gravitational constant" (taken to be equal to infinity in original Sakharov's approach), trace $\mathrm{Tr}_{\,s}$ is taken over the spectrum with the corresponding numerical factors accounting for particle content of the theory and M is ultraviolet cutoff. If there are N particle species of a given type[2] in the theory, the trace scales as $-\mathrm{Tr}_{\,s}[..] \propto N$. Then, making original Sakharov's one-loop dominance assumption, one has from (4): $1/G \sim NM^2$, i.e. just eq.(3). The physical interpretation of this result is clear. The rigidity of space, i.e. its resistance against an attempt to curve it is proportional to the number of particle species living in quantum vacuum in this space because the curvature costs energy. Roughly speaking, more different particle types the theory contains, weaker is the gravity in this theory,[3] i.e. rich and non-degenerate spectrum tends to wash space-time distortions out.

As another example it is instructive to consider large N_c QCD. Let us take $SU(N_c)$ Yang-Mills gauge theory with two flavors of light fundamental quarks u and d. There are $2N_c$ spinor degrees of freedom in the theory with 2 staying for the number of flavors and N_c for the number of colors. At low energies the relevant excitations are three light pseudoscalar pions π^+, π^-, π^0 and the corresponding lowest order effective Lagrangian has the standard form:

$$L = \frac{1}{4}F^2\mathrm{Tr}\,\left(\partial_\mu U^\dagger \partial^\mu U\right) + \frac{1}{2}F^2 B\mathrm{Tr}\,\left(m\left(U + U^\dagger\right)\right) \tag{5}$$

where the $SU(2)$ matrix field U is expressed in terms of the pion fields $\vec{\pi}(x) = \left(\pi^1(x), \pi^2(x), \pi^3(x)\right)$, mass matrix $m = \mathrm{diag}(m_u, m_d)$, and F is just pion decay constant (up to $\mathcal{O}(m)$ corrections).

[2] Say, N light noninteracting scalars. Of course, if both bosons and fermions present in the theory, they partly cancel each other.

[3] Again, leaving aside supersymmetric-like cancelations.

The Lagrangian rewritten in terms of the pion fields takes the following form (neglecting $\mathcal{O}(\vec{\pi}^4)$ terms):

$$L = (m_u + m_d)F^2 B + \frac{1}{2}(\partial\vec{\pi})^2 - \frac{1}{2}(m_u + m_d)B\vec{\pi}^2 \tag{6}$$

with the obvious assignment $m_\pi^2 = (m_u + m_d)B$. If as a next step one compares the vacuum energy expression from the effective low-energy theory described by the Lagrangian (6) : $\Delta E = -V(m_u + m_d)F^2 B$ with the energy shift from the original theory (i.e. QCD): $\Delta E = V\langle 0|m_u\bar{u}u + m_d\bar{d}d|0\rangle$, the Gell-Mann - Oakes - Renner relation immediately follows:

$$(m_u + m_d)\langle\bar{q}q\rangle = -F^2 m_\pi^2 \tag{7}$$

It is worth noticing that despite the Lagrangian (6) does not contain N_c explicitly, the low energy description as such is valid up to pion momenta smaller than ultraviolet cut-off $\Lambda \sim 4\pi F$. The latter is N_c-dependent quantity (of course one can formally remove N_c-dependence from the Lagrangian (5) away by redefinition of the pion fields but we prefer not to do it). Indeed, taking into account that quark condensate scales linearly with N_c and $m_\pi \to$ const in large N_c limit, one has from (7): $F^2 \sim N_c$. Thus induced contribution of pions to gravitational constant scales as[4]

$$\delta\left(\frac{1}{G}\right)_{pions} \sim \Lambda^2 \sim N_c \tag{8}$$

The meaning of (8) is the same as has been just discussed: the ultraviolet cut-off of the low energy theory encodes information about the number of degrees of freedom in the "fundamental" theory.[5] More degrees of freedom (associated with color in the considered example) the underlying theory contains, weaker its low-energy excitations interacts with the gravity. This is exactly original idea behind [6] and it is not surprising that (8) is nothing than (3) seen from a different prospective.

If weakness of gravity is indeed a consequence of the fundamental theory huge multiplicity, a natural question to ask is what could be possible ways to get it? The main message of the scenario we are going to describe below is the following: one needs much less number of different species, if they satisfy non-conventional statistics. To be more precise, we take n degrees of freedom, which are neither bosons nor fermions, but of quon type and suppose that they are condensed below some energy scale 1/L, while low energy modes of normal statistics are excitations above this nontrivial vacuum.

Quons, i.e. objects satisfying quantum Boltzmann statistics

$$a_i a_j^\dagger = \delta_{ij} \tag{9}$$

augmented by the Fock-state representation defining relation $a_i|0\rangle = 0$ are known for a long time in mathematical (e.g. free random variables [11] and stochastic dynamics [12]) and physical (quon field theories [13], black holes statistics [14], D0 branes and matrix theory [15]) contexts.

Let us remind some basic facts about quon states. The m-particle state is constructed as

$$|m\rangle = (a_{i_1}^\dagger)^{k_1}(a_{i_2}^\dagger)^{k_2}...(a_{i_l}^\dagger)^{k_l}|0\rangle \tag{10}$$

with $k_1 + k_2 + ... + k_l = m$. All such states have positive norm and can be normalized to unity by the condition $\langle 0|0\rangle = 1$. The states created by any permutations of creation operators are orthogonal, i.e.

$$\begin{aligned}(a_i^\dagger..a_m^\dagger|0\rangle)^\dagger \cdot (a_i^\dagger..a_m^\dagger|0\rangle) &= \langle 0|a_m..a_i a_i^\dagger..a_m^\dagger|0\rangle = 1 \\ (a_i^\dagger..a_k^\dagger..a_l^\dagger..a_m^\dagger|0\rangle)^\dagger \cdot (a_i^\dagger..a_l^\dagger..a_k^\dagger..a_m^\dagger|0\rangle) &= 0 \ \text{ for any } \ k \neq l\end{aligned} \tag{11}$$

[4] At large N_c pure gluon contribution scaling as N_c^2 starts to dominate, but it is not seen at low energies until Λ becomes of the order of the lowest glueball mass.

[5] For Λ larger than masses of other, non-Goldstone hadrons, they have to be included as well.

For particles of the type i one can define the number operator N_i such that

$$N_i|m\rangle = k_i|m\rangle \;\; ; \;\; \text{and} \;\; [N_i, a_i]_- = -a_i \tag{12}$$

as

$$N_i = a_i^\dagger a_i + \sum_l a_l^\dagger a_i^\dagger a_i a_l + \sum_{l,m} a_m^\dagger a_l^\dagger a_i^\dagger a_i a_l a_m + ... \tag{13}$$

For free normal-ordered Hamiltonian one has $H_0 = \sum_{i=1}^{n} \mathcal{E}_0^i N_i$ The condition (9) automatically makes any product of quon operators normally ordered. It is a very important property of quon statistics that free Hamiltonian is given by an infinite series in creation and annihilation operators (in contrast with conventional Bose or Fermi case where free Hamiltonian is or can be made quadratic). Nevertheless there is no rich dynamics in free quon theory due to (9). Indeed, any state of the kind (10) or superposition of such states is an eigenstate of the Hamiltonian with an eigenvalue

$$\sum_{i=1}^{n} \mathcal{E}_0^i k_i \tag{14}$$

This is nothing than the standard harmonic oscillator equidistant spectrum.

Leaving the domain of free theory, suppose that we turn on the interactions in such a way that the ground state of the interacting quon Hamiltonian is given by superposition

$$|\Phi\rangle_l = c_l \sum_P a_{i_1}^\dagger a_{i_2}^\dagger ... a_{i_l}^\dagger |0\rangle \tag{15}$$

where c_l is normalization factor and $H|\Phi\rangle_l = 0$. In fact, one can think of different symmetry properties of $|\Phi\rangle_l$, but typically for $l \sim n$ one has $c_l \sim n^{-n/2}$. It is also important that the state $|\Phi\rangle_l$ is not quon vacuum anymore:

$$a_i|0\rangle = 0 \;\; , \;\; \text{but} \;\; a_i|\Phi\rangle_l \neq 0 \tag{16}$$

Now we have come to the spectrum over this nontrivial vacuum consisting of excitations of two types. The first are created by the effective low-energy operators $f_i^\dagger$ such that $f_i|\Phi\rangle_l = 0$ where these operators $f_i, f_i^\dagger$ obey conventional statistics (e.g. Fermi-Dirac one for totally symmetric choice of (15)). These are "light" excitations, i.e. their mass spectrum is unrelated to the masses of quon states. On the other hand, there is also a zoo of "heavy" excitations over the unbroken vacuum subject to the condition $F_P|0\rangle = f_i a_{i_1} a_{i_2} ... a_{i_l} |\Phi\rangle_l = 0$. The corresponding scale L_l is a dynamical scale of the interacting quon theory.

The excitation pattern described above easily allows to get exponential multiplicity of states needed to screen gravitational constant. Indeed, computing induced contribution one gets

$$\delta\left(\frac{1}{G}\right) = i\frac{\pi}{6} \int d^4x x^2 \langle \mathcal{T}(\tilde{T}(x)\tilde{T}(0))\rangle \sim \frac{n}{c_l^2 L_l^2} \tag{17}$$

where $\tilde{T}(x) = T_\mu^\mu(x) - \langle T_\mu^\mu(x)\rangle$. Thus the typical pattern we have is given by

$$M_P^2 L_n^2 \sim \exp(ng(n)) \;\; \text{with} \;\; g(n) \sim \log n \tag{18}$$

and the exponent function here has combinatorial and not tunneling origin. Of course, actual numerical values depend on the concrete choice of the vacuum (15), but for most general case n is given by number between 20 and 30 for $L_n^{-1} \sim 10$ TeV, i.e. is comparable with the number of species in the Standard Model seen at low energies.

Conclusion. It is very natural to think in induced gravity framework that weakness of gravity is a consequence of dramatic rise of multiplicity above some TeV-scale $1/L$. The only large number we have in low energy domain is the number of different particle species n. Assuming change of statistics to the infinite one at the scale $1/L$ could allow to get factorial growth in the number of dynamical degrees of freedom and hence in Planck mass as measured by low-energy observer.

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